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Rank bounds and polynomial-time PIT for $\Sigma^k\Pi\Sigma\Pi^2$ circuits

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Abstract

A depth-4 algebraic circuit with top fan-in k and bottom fan-in 2 is a circuit Φ of the form $\Phi = \sum_{i=1}^{k}\prod_{j=1}^{m_i} Q_{ij}$, where the polynomials $Q_{ij} \in \mathbb{K}[x_1,\dots,x_n]$ have degree at most 2. The class of all such circuits is denoted by $\Sigma^k\Pi\Sigma\Pi^2$. We say that the circuit Φ is an identity if it formally computes the zero polynomial. An important parameter of $\Sigma^k\Pi\Sigma\Pi^2$ circuits Φ is their (linear) rank, which is defined as the vector space dimension of the polynomials $\{Q_{ij}\}_{i\in[k],j\in[m_i]}$.

We prove that, when the base field $\mathbb{K}$ is of characteristic zero, the rank of any (simple and minimal) $\Sigma^k\Pi\Sigma\Pi^2$ identity is upper bounded by a function which depends only on the top fan-in k. This result makes progress on [BMS13, Conjecture 28], being the first work to establish a bound on the rank of such identities that depends only on the top fan-in. Moreover, when combined with [BMS13, Theorem 2], our main result yields the first deterministic, *polynomial time* PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^2$ circuits.

One of the key components of our proof of the rank bounds is the derivation of an approximate Hansen-type result, which is interesting in its own right. This result can be seen as an algebraic and higher-dimensional analogue of the approximate Sylvester-Gallai result of [ADSW14], and a distinct approximate fractional Sylvester-Gallai result than the one from [GOPS23]. Additionally, we prove a robust version of it, in the spirit of the generalization of Hansen's theorem by [BDWY13].

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1 Introduction

Polynomial Identity Testing (PIT) is the problem of checking whether a given algebraic circuit over a field $\mathbb{K}$ computes the zero polynomial. When the base field $\mathbb{K}$ is sufficiently large, the PIT problem is equivalent to asking whether the given algebraic circuit evaluates to zero on all inputs. This equivalent formulation suggests a very simple randomized algorithm: simply evaluate the circuit at a randomly chosen point. The correctness of this algorithm is given by the polynomial identity lemma.

Derandomizing PIT is a foundational problem in theoretical computer science: it is intrinsically related to proving lower bounds for algebraic circuits [HS80, KI04, Agr05, KS19, KST23] and the derandomization of PIT even for restricted classes of circuits has resulted in the derandomization of important problems in mathematics and computer science [AKS04, FS13, Mul17, FGT19, GT20, ST17]. For an overview on PIT and its connections, we refer the reader to [Sax09, SY10, Sax14, DG24].

Given the difficulty of derandomizing the (full) PIT problem, much of the focus has shifted to natural subclasses of circuits, such as sparse polynomials (equivalently, depth-2 circuits) [BT88, GKS90, KS01] and depth-3 circuits with bounded top fan-in [DS07, KS07, KS09, KS11, SS12, SS13]. Recent advances in depth reduction [AV08, Koi12, Tav15, GKKS17] have shown that a deterministic PIT algorithm for unrestricted depth-3 circuits or homogeneous depth-4 circuits would yield a quasipolynomial-time deterministic PIT algorithm for general circuits. These insights have renewed interest in these circuit classes and spurred substantial research activity on PIT within these settings, as seen in [AM10, KMSV13, SV18, BMS13, OSV16, For15, ASSS16, Guo21, DDS21, PS21, GOS25a, GW25] and references therein.

The work [DS07], which first studied the PIT problem for the class of depth-3 circuits of bounded top fan-in (denoted $\Sigma^k\Pi\Sigma$), defined the *rank* of a depth-3 circuit as the dimension of the vector space of the linear forms appearing in the bottom layer of the circuit. Their high-level strategy for obtaining a deterministic, polynomial-time PIT algorithm for this class was as follows. Suppose there exists a function $R : \mathbb{N} \to \mathbb{N}$ such that any "non-trivial" (simple and minimal) $\Sigma^k\Pi\Sigma$ identity has rank less than $R(k)$. If one can devise a variable reduction procedure that maps the variables into an $R(k)$-dimensional space while preserving the rank of any $R(k)$ linear forms, then composing this map with the circuit yields a nonzero circuit in only $R(k)$ variables. A simple interpolation on this reduced circuit suffices to verify nonzeroness.

While [DS07] was not able to prove the existence of such a function R, they managed to prove a rank bound which depended both on the top fan-in k and on the logarithm of the degree of the circuit. This bound, combined with the variable reduction procedure was enough to yield a quasi-polynomial time PIT algorithm for $\Sigma^k\Pi\Sigma$ circuits. In [DS07] the above variable reduction strategy was achieved in the *white-box* setting, where the circuit is given explicitly as input and one can inspect the linear functions appearing in it. Later, building on [GR08], Karnin and Shpilka obtained a corresponding variable reduction procedure in the *black-box* setting, where the algorithm only has query access to the circuit [KS11].

The work [DS07] also highlighted a connection between rank bounds for depth-3 identities and discrete geometry, specifically to colored versions of the classical Sylvester-Gallai theorem. In particular, [DS07] suggested that insights from discrete geometry could lead to improved rank bounds. This direction was effectively pursued by Kayal and Saraf [KS09], who observed that one of the conjectures posed in [DS07] corresponds to the Edelstein-Kelly theorem [EK66]. By applying higher dimensional incidence theorems, [KS09] obtained the first rank bounds which depended only on the top fan-in k. Subsequent research further improved the upper bound on $R(k)$ culminating in $R(k) = 3k^2$ over fields of characteristic zero [SS13].

Motivated by the success of rank-bounds for depth-3 circuits, Beecken, Mittmann and Saxena [BMS13] conjectured that if one defines an analogous notion of rank for depth-4 circuits, replacing linear dimension with algebraic rank, then "nontrivial" depth-4 identities should have small rank. Motivated by the connection between depth-3 identities and questions in incidence geometry, Gupta suggested far-reaching generalizations of the Sylvester-Gallai theorem and its colored and high-dimensional variants [Gup14]. These conjectures suggest a program of generalizing incidence geometry from studying intersections of lines, hyperplanes or curves, to studying intersections between zero sets of low-degree polynomials (or more generally sparse polynomials).

The first progress on these conjectures was made by Shpilka [Shp20], who established an analogue of the Sylvester-Gallai theorem for quadratic polynomials. Subsequent work gradually confirmed several other conjectures of Gupta [PS20, PS21, PS22, OS22, GOS22, GOPS23, OS24, GOS25a, GOS25b]. The results of [PS21, GOS25a], combined with [BMS13, Theorem 2], immediately yield black-box PIT algorithms for $\Sigma^3\Pi\Sigma\Pi^d$ circuits, for $d = 2$ and more generally for any $d = O(1)$, respectively.

In this work we obtain the first polynomial time black-box PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^2$ circuits, for any constant top fan-in k. We achieve this by combining the conceptual approach of [KS09] with the techniques developed in [OS24]. This allows us to prove an upper bound on the rank of $\Sigma^k\Pi\Sigma\Pi^2$ identities.

Before stating our main results we discuss the relation between PIT and (non-linear) incidence geometry.

1.1 Edelstein-Kelly Theorem and Small Depth Identities

The Sylvester-Gallai theorem, posed independently by Sylvester [Syl93] and Erdös [EBW+43], and solved independently by Melchior [Mel40] and Gallai [Gal44], asserts that if a finite set of points in $\mathbb{R}^n$ has the property that every line passing through any two points in the set also contains a third point from the set, then all the points in the set are collinear. In [Kel86], Kelly observed that a result of Hirzebruch [Hir83] proves that over $\mathbb{C}^n$ the points must lie on a 2-dimensional affine space, thereby resolving a question of Serre [Ser66]. Many variants of this theorem were studied: extensions to higher dimensions, colored versions, robust versions and many more. For a survey on the Sylvester-Gallai theorem and its variants see [BM90]. Of particular relevance to PIT is the Edelstein-Kelly theorem [EK66], a colored variant of the Sylvester-Gallai theorem. We begin by defining the notion of an Edgelstein-Kelly configuration.

Definition 1.1 (Edelstein-Kelly configurations)**.** Let $\mathbb{K}$ be a field, $\mathcal{A} \coloneqq \{u_1, \dots, u_a\}$, $\mathcal{B} \coloneqq \{v_1, \dots, v_b\}$ and $\mathcal{C} \coloneqq \{w_1, \dots, w_c\}$ be pairwise disjoint subsets of $\mathbb{P}(\mathbb{K}^N)$. We say that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ forms a *linear Edelstein-Kelly configuration* if any pair of points taken from distinct sets is collinear with a point from the third set. The *rank* of an Edelstein-Kelly configuration $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is given by $\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}\}$.

The Edelstein-Kelly theorem asserts that, for $\mathbb{K} = \mathbb{R}$, the rank of any linear Edelstein-Kelly configuration is at most 3. Over $\mathbb{C}$, the fractional Sylvester-Gallai theorems of [BDYW11, DSW14, DGOS18] imply that the dimension is bounded by some absolute constant.

Consider now the simplest nontrivial $\Sigma^3\Pi\Sigma$ identity over $S \coloneqq \mathbb{K}[\mathbf{x}]$:

$$\prod_i \ell_{1,i}(\mathbf{x}) + \prod_j \ell_{2,j}(\mathbf{x}) + \prod_k \ell_{3,k}(\mathbf{x}) \equiv 0\,, \tag{1}$$

where the $\ell_{a,b}$ are linear forms. By factoring out the common gcd we may assume that the products in (1) are pairwise prime. Indeed, if a form divides two terms then it must divide the third term as well. The *rank* of the identity is defined as $\dim \operatorname{span}_{\mathbb{K}}\{\ell_{a,b}\}_{a,b}$. Since the three terms sum to zero, if we pick any two linear forms from distinct terms, say $\ell_{1,i}$ and $\ell_{2,j}$, and restrict to the subspace $U = \{\mathbf{a} : \ell_{1,i}(\mathbf{a}) = \ell_{2,j}(\mathbf{a}) = 0\}$, we obtain that also $(\prod_k \ell_{3,k}(\mathbf{x}))|_U \equiv 0$. In other words, if we consider the corresponding zero sets,[1]

$$V(\ell_{1,i}, \ell_{2,j}) \subseteq V\left(\prod_k \ell_{3,k}(\mathbf{x})\right) = \bigcup_k V(\ell_{3,k}(\mathbf{x})). \tag{2}$$

Since $V(\ell_{1,i}, \ell_{2,j})$ is irreducible[2] there exists some $k = k(i,j)$ such that $\ell_{3,k} \in \operatorname{span}_{\mathbb{K}}\{\ell_{1,i}, \ell_{2,j}\}$. Thus, any two linear forms from any two distinct terms span a linear form from the third term.

To fit this into the setting of Definition 1.1, identify each linear form $\ell_{a,b}$ with the point $p_{a,b} \in \mathbb{P}(\mathbb{K}^N)$ spanned by its coefficient vector $\mathbf{a}_{a,b}$. Then, $\ell_{3,k} \in \operatorname{span}\{\ell_{1,i}, \ell_{2,j}\}$ if and only if $p_{3,k}$ is collinear with $p_{1,i}$ and $p_{2,j}$. Hence, the sets $\mathcal{A} = \{p_{1,i}\}$, $\mathcal{B} = \{p_{2,j}\}$ and $\mathcal{C} = \{p_{3,k}\}$ satisfy the conditions of Definition 1.1. The Edelstein-Kelly theorem then implies that, for $\mathbb{K}$ of characteristic zero, all the points $p_{a,b}$ lie on an $O(1)$-dimensional space, and consequently, $\dim_{\mathbb{K}} \operatorname{span}\{\ell_{a,b}\} = O(1)$. In conclusion, this argument proves that any nontrivial (i.e., gcd-free) depth-3 identity depends essentially on only $O(1)$ variables, thereby enabling an efficient PIT algorithm.

Consider now a nontrivial $\Sigma^4\Pi\Sigma$ identity. Henceforth, by nontrivial we mean gcd-free and that any 3 terms are linearly independent (i.e., there is no shorter nontrivial identity):

$$\prod_{i=1}^{d} \ell_{1,i}(\mathbf{x}) + \prod_{i=1}^{d} \ell_{2,i}(\mathbf{x}) + \prod_{i=1}^{d} \ell_{3,i}(\mathbf{x}) + \prod_{i=1}^{d} \ell_{4,i}(\mathbf{x}) \equiv 0\,. \tag{3}$$

The natural attempt to bound the rank is to reduce to the case of three terms as in (1). For this we can consider the restriction to $U = \{\mathbf{a} : \ell_{1,1}(\mathbf{a}) = 0\}$. This restriction cancels out the first term. However, it may now be the case that after restricting to U, the resulting identity

$$\prod_{i=1}^{d} \ell_{2,i}(\mathbf{x})|_U + \prod_{i=1}^{d} \ell_{3,i}(\mathbf{x})|_U + \prod_{i=1}^{d} \ell_{4,i}(\mathbf{x})|_U \equiv 0$$

is no longer nontrivial. This complication becomes more evident as we increase the number of terms, and we will discuss it more in Section 1.3. Thus, moving from 3 terms to 4 already poses nontrivial challenges.

Let us now consider a $\Sigma^3\Pi\Sigma\Pi^d$ identity:

$$\prod_i A_i(\mathbf{x}) + \prod_j B_j(\mathbf{x}) + \prod_k C_k(\mathbf{x}) \equiv 0\,, \tag{4}$$

where each A_i, B_j, C_k is a homogeneous polynomial of degree $\leqslant d$, and, as before, we assume that (4) is gcd-free. In analogy to (2) we obtain[3]

$$V(A_i, B_j) \subseteq V\left(\prod_k C_k(\mathbf{x})\right). \tag{5}$$

[1] By symmetry, a similar containment holds for any two forms from distinct terms and the third term.
[2] This can be shown directly by simple linear algebra, but we use the more general terminology for later purposes.
[3] By symmetry, a similar containment holds for any two forms from distinct terms and the third term.

Trying to repeat the previous argument, we immediately encounter a new difficulty. Since A_i and B_j are forms of degree higher than 1, their variety may not be irreducible. Equivalently, the ideal $I = (A_i, B_j) \subset \mathbb{K}[\mathbf{x}]$ is not necessarily prime. As a result, it may no longer be true that there exists $C_k \in \text{span}\{A_i, B_j\}$. Even worse, we are not guaranteed that there is a C_k such that $V(A_i, B_j) \subseteq V(C_k)$.

When $d = 2$, i.e., the forms are quadratic or linear polynomials, Peleg and Shpilka [PS20, PS21] proved that if the ideal generated by two quadratic polynomials is not prime, then some of its associated primes have very simple form, and with that information [PS21] managed to extend the Edelstein-Kelly theorem to quadratic polynomials. Oliveira and Sengupta [OS22] generalized this approach to the case of cubic forms by characterizing non-radical ideals generated by two cubic forms. However, when $d > 3$ the situation becomes significantly more complex and there is no such "structure theorem" for ideals generated by two higher degree forms. On the other hand, structure theorems such as the primality conditions in [GOS25b], allowed [GOS25b] to control the number and structure of "bad events". To use this result inductively, Garg, Oliveira and Sengupta [OS24, GOS25a] asked more difficult questions, but which are more amenable to induction. Instead of considering dependencies over $S = \mathbb{K}[\mathbf{x}]$, they considered dependencies over quotient rings $R = S/(U)$ where U is a vector space of forms up to degree d, such that R is a unique factorization domain (UFD). In addition, and to generalize the technique of projection introduced in [KS09, Shp20] and generalized in [OS24], [GOS25a] gave the following definition that greatly generalizes Definition 1.1

Definition 1.2 (High-degree EK configurations)**.** Let $U \subseteq S$ be a graded, finitely generated vector space such that $R := S/(U)$ is a UFD, and let $z \in R_1$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset R$ be pairwise disjoint finite sets of irreducible forms of degree at most d. We say that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a (d, z, R)–*EK configuration* if the following hold:

1. $z \notin \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, and the union $\{z\} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ consists of pairwise non-associate forms.

2. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have

$$z \cdot \prod_{C \in \mathcal{C}} C \in \text{rad}(A, B),$$

and similarly for any two forms from any two distinct sets, where $\text{rad}(A, B)$ is the radical of the ideal (A, B) in the quotient ring R.

While this definition is more intricate than Definition 1.1, it has the important advantage of greater flexibility. In particular, it allows one to add forms to U while remaining within the same framework. Their high level approach was to prove an upper bound on the rank using induction. The case $d = 1$ can be handled as in previous work, even though the setting is more general. Now, if all the forms in the identity Equation 4 are of degree at most $d - 1$, then we are done by induction. Otherwise, they consider two cases. The first is when there is a form $A \in \mathcal{A}$ such that for any other form $B \in \mathcal{B}$ (similarly, for $C \in \mathcal{C}$), $\text{span}\{A, B\}$ does not contain any form of small "strength". That is, no nonzero linear combination of A, B can be written as a sum of a few terms of the form $P_i P_i'$, where $\deg(P_i), \deg(P_i') < d$. [GOS25a] prove that in such a case, we can add A to U and by going to $R' = S/(U, A)$, obtain a simpler identity. In the other extreme case, assume that every form in the identity has small strength. Moreover, let us assume that it depends on a few variables. Consider now a forms $A \in \mathcal{A}$. If for many forms $B_j \in \mathcal{B}$, the ideal (A, B_j) is prime, then A spans a form C_k with each of these "good" B_j. If this holds for most $A \in \mathcal{A}$ then we can conclude by a *robust* version of the linear Edelstein-Kelly theorem. If this is not the case, then the following important structural result of [GOS25b, GOS25a] is needed: there is a universal upper bound, depending only on the degree d, on the number of irreducible forms P such that (A, P) is not a prime ideal. Thus, modulo A, many forms in $\mathcal{B}$ become reducible, and we've made progress. Furthermore, since A depends on a few variables, we can pass to a smaller polynomial ring in which these forms factor.

While the description above greatly simplifies the approach of [GOS25a], it highlights some challenges that arise when trying to go from three terms to more terms and from linear forms to higher degree forms.

1.2 Our results

In this work, we overcome the challenges posed by depth-4 identities of constant top fan-in, at the expense of restricting the bottom fan-in to be bounded by 2.

Before we state our main result, we need to establish standard definitions of "nontrivial identities" for $\Sigma^k\Pi\Sigma\Pi^2$ circuits. As we mentioned in the previous subsection, one way in which an identity $\Phi = T_1 + \cdots + T_k$ can be trivial is if each gate is a multiple of a quadratic (or linear) polynomial. To disallow such triviality, we say that Φ is a simple circuit if $\gcd(T_1, \ldots, T_k) = 1$. Another way to obtain a trivial identity is by adding two smaller identities together. To disallow such triviality, we say that Φ is a minimal circuit if no subset of its gates sum to zero.

We also need to define the rank of a circuit. In this work, the rank of a $\Sigma^k\Pi\Sigma\Pi^2$ circuit $\Phi = T_1 + \cdots + T_k$ is the dimension of the $\mathbb{K}$-span of all the forms appearing in any of the gates T_i. While in [BMS13] the authors use an algebraic notion of rank of a depth-4 circuit, we use a stricter (and simpler to state) notion of rank of a circuit, which in particular upper bounds the notion of rank in [BMS13]. With these definitions in mind, we can state our main theorem.

Theorem 1.3. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $S := \mathbb{K}[x_1, \ldots, x_n]$. There is a function $R : \mathbb{N} \to \mathbb{N}$, independent of $\mathbb{K}, n$, such that for any simple and minimal $\Sigma^k\Pi\Sigma\Pi^2$ identity Φ over S, we have $\operatorname{rank}(\Phi) \leqslant R(k)$.*

The above theorem makes progress on [BMS13, Conjecture 28], being the first work to obtain bounds for circuits with top fan-in larger than 3. Combining the above theorem with [BMS13, Theorem 2], we obtain the following corollary.

Corollary 1.4. *There is a deterministic, polynomial-time black-box PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^2$ circuits.*

Theorem 1.3 above is a corollary of our main technical result Theorem 6.1. We do not state it here since it requires definitions about strong vector spaces and lifted strength. Theorem 1.6 allows us to reduce the case of general $\Sigma^k\Pi\Sigma\Pi^2$ identities to the case of identities where every quadratic is close to an algebra. In Section 5 we handle the latter case, and lastly in Section 6 we prove our main technical result and prove Theorem 1.3 as a corollary.

An important ingredient in the proof of Theorem 1.3 (more accurately, of Theorem 6.1) is an extension of Hansen's theorem to quadratic polynomials [Han65], which we prove in Section 4.

Our result both extends Hansen's theorem to the quadratic setting and relaxes its exact linear dependence condition to an approximate one. This can also be viewed as a high-dimensional quadratic analogue of the approximate collinear Sylvester–Gallai theorem of Ai, Dvir, Saraf, and Wigderson [ADSW14].

To state the theorem, we first introduce the notion of an (M, b)-strong configuration. The following simplified version captures the main idea, while the precise definition and theorem appear later.

Definition 1.5 (Simplified version of Definition 4.2)**.** We say that a finite set of quadratic polynomials $\mathcal{F}$ is an (M, b)-strong span configuration if, for any $b-1$ polynomials $Q_1, \dots, Q_{b-1} \in \mathcal{F}$, every other $Q \in \mathcal{F}$ satisfies one of the following:

1. $|\text{span}_{\mathbb{K}}\{Q_1, \dots, Q_{b-1}, Q\} \cap \mathcal{F}| > b$, or
2. Q is M-close to $\text{span}_{\mathbb{K}}\{Q_1, \dots, Q_{b-1}\}$.

Here, M-closeness means that there exists $F \in \text{span}_{\mathbb{K}}\{Q_1, \dots, Q_{b-1}\}$ such that the quadratic form $Q - F$ has rank less than M.

Theorem 1.6 (Simplified version of Theorem 4.6)**.** *For any (M, b)-strong span configuration $\mathcal{F}$, there exists a subspace V of dimension $O_b(1)$ such that every quadratic form in $\mathcal{F}$ is $O_b(M)$-close to V. In particular, the dimension of V is independent of the number of variables, of $|\mathcal{F}|$, and of M.*

Interestingly, for the proof of Theorem 1.6 we actually first need to prove a *robust* version of it, in the spirit of the robust version of Hansen's theorem proved in [BDWY13]. We think that this result is also interesting on it own.

Definition 1.7 (Simplified version of Definition 4.7)**.** Let $0 < \nu \leqslant 1$ We say that a finite set of quadratic polynomials $\mathcal{F}$ is a (ν, M, b)-strong span configuration if, for any $b-1$ polynomials $Q_1, \dots, Q_{b-1} \in \mathcal{F}$, for ν fraction of the remaining $Q \in \mathcal{F}$ the following holds:

1. $|\text{span}_{\mathbb{K}}\{Q_1, \dots, Q_{b-1}, Q\} \cap \mathcal{F}| > b$, or
2. Q is M-close to $\text{span}_{\mathbb{K}}\{Q_1, \dots, Q_{b-1}\}$.

Theorem 1.8 (Simplified version of Theorem 4.10)**.** *For any (ν, M, b)-strong span configuration $\mathcal{F}$, there exists a subspace V of dimension $O_b(1/\nu)$, such that $O_b(\nu)$ fraction of the forms in $\mathcal{F}$ are M-close to V.*

We now proceed to describe an outline of our proof of Theorem 1.3.

1.3 Proof outline

As we previously mentioned, our approach generalizes the framework of [KS09] via the techniques developed in [OS24, GOS25b]. In this subsection, we present our proof strategy and its implementation, emphasizing the conceptual contributions and the technical challenges that we overcome in developing the inductive framework that underlies our rank bound proof for $\Sigma^k\Pi\Sigma\Pi^2$ identities. To better understand our proof, we briefly outline the inductive approach of [KS09] to construct their rank upper bound function $R : \mathbb{N} \to \mathbb{N}$ for $\Sigma^k\Pi\Sigma$ circuits.

Overview of the framework in [KS09]. When $k \leqslant 2$, only the trivial circuit is a simple and minimal identity, by the unique factorization property in polynomial rings. This gives us a trivial starting upper bound on the rank of such identities. In particular, we can define $R(0) = R(1) = R(2) = 1$.

Now, suppose that we have constructed our function R up to $k-1$, where $k \geqslant 3$, and let $\Phi = T_1 + \cdots + T_k$ be our simple and minimal $\Sigma^k\Pi\Sigma$ identity, where each $T_i = \prod_{j=1}^{m} \ell_{ij}$ (note that ℓ_{ij}'s may or may not be scalar multiples of one another). We can still use unique factorization of the polynomial ring to quantify "how different two distinct gates are." This motivated the authors to define the "symmetric difference" of two gates as follows: $T_i \Delta T_j$ equals the set of linear forms (up to scalar multiples) which appear in T_i and T_j with different multiplicities (if $\ell \nmid T_i$ then we say that the multiplicity of ℓ in T_i is zero). Note that the size of the symmetric difference is not too important as a parameter, but a better choice of parameter is the dimension of the span of this set, i.e. $\dim \mathrm{span}_{\mathbb{K}} \{T_i \Delta T_j\}$, as this quantifies the "distinct number of variables" that distinguish the two gates. This leads to the definition of *pairwise rank* of Φ: it is defined as $\min_{1 \leqslant i < j \leqslant k} \dim \mathrm{span}_{\mathbb{K}} \{T_i \Delta T_j\}$.

If two gates T_i and T_j "are almost the same," that is, $\dim \mathrm{span}_{\mathbb{K}} \{T_i \Delta T_j\}$ is "small enough," then it is possible to "merge" these two gates via a *general quotient* by the linear forms in $T_i \Delta T_j$. This essentially preserves simplicity and minimality of the circuit, and since we are quotienting by linear forms the new circuit is still over a polynomial ring, in which case we would get an $\Sigma^{k-1}\Pi\Sigma$ identity, for which we know rank bounds by induction. This would allow us to lift the rank bounds. Note that if one believes that such rank bounds ought to exist, then in particular it must be the case that any two gates should "depend on almost the same variables", in which case the above scenario must always happen. The main strategy of [KS09] is to show that this case always happens, and it takes up most of the work in [KS09].

Assume we are not in the above case, that is, we have a simple and minimal circuit $\Phi = T_1 + \cdots + T_k$ in which *every* pair of gates T_i, T_j are "far enough apart." We would like to show that Φ is not an identity. Since we have an inductive proof, we assume that we know rank bounds for $\Sigma^{k-1}\Pi\Sigma$ identities. One way to show this is via a *fan-in reduction lemma*: to show the existence of a linear form ℓ which appears in one of the gates of the circuit, such that by "zeroing out" ℓ – in other words, work over the quotient ring $\mathbb{K}[\mathbf{x}]/(\ell)$, which is still a polynomial ring – the gates of our new circuit $\overline{\Phi}$ (which will now have at most $k-1$ gates) are still "far enough apart from one another", which implies that $\mathrm{rank}(\overline{\Phi})$ is too large for it to be an identity.

This linear form ℓ, if exists, is rather special, as it could happen that by taking a quotient (which is not "general"), the new circuit is no longer simple and minimal. One terrible thing that could happen because of failure of simplicity is that the simple part of the new circuit "is missing" many linear forms from the original circuit – this could happen if many forms "go into the GCD" after the quotient. For instance, if all gates but two in our circuit were divisible by ℓ and the remaining two gates are $\prod_{i=1}^{m} x_i - \prod_{i=1}^{m}(\ell + x_i)$, when we quotient by ℓ these two gates become an identity with trivial simple part, but our bounds say nothing about the rank of the GCD (which in this case is everything!).

One ingredient in the fan-in reduction lemma of [KS09] is Hansen's incidence theorem [Han65], which in essence states that if a set $\mathcal{F}$ of linear forms is such that $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}\}$ is large, then it must contain a $(k+1)$-subset of forms $\ell_1, \ldots, \ell_{k+1} \in \mathcal{F}$ such that $|\mathrm{span}_{\mathbb{K}} \{\ell_1, \ldots, \ell_{k+1}\} \cap \mathcal{F}| = k+1$.[4] In other words, these forms do not span any other form in $\mathcal{F}$. With this theorem in mind, [KS09] proceed as follows: since the pairwise rank of Φ is large enough (as a function of k), one can construct a large enough "core" $\mathcal{C} := \bigcup_{i,j} \mathcal{C}_{ij}$ of linear forms, where each $\mathcal{C}_{ij} \subset T_i \Delta T_j$ has "enough distinct variables" appearing in the symmetric difference. The objective of having this core is that we want to prove that there is a linear form ℓ in the circuit that "does not affect any subcore $\mathcal{C}_{ij}$" if $\ell \nmid T_i \cdot T_j$. If one achieves this, we will be able to quotient by (ℓ), thereby reducing fan-in of Φ, and preserving "too many variables" in the symmetric differences, which implies that the new circuit $\overline{\Phi}$ will have too large of a rank, in which case it cannot be an identity.

The above summarizes the core ideas behind (a simplification of) the approach of [KS09], but several challenges present themselves in implementing such approach. In general they are not always able to pre-

[4] Hansen's theorem is over $\mathbb{K} = \mathbb{R}$ only, but a generalization of it is proved over any characteristic zero field in [BDYW11, DSW14].

serve the core itself, but a "proxy core," which emerges from the interaction of the core with the remainder of the circuit. This will be the same for us in our generalization of their approach.

Generalizing the above approach. Several new challenges arise in the above approach when trying to generalize it to the setting of $\Sigma^k\Pi\Sigma\Pi^2$ circuits, as we have to deal with quadratic forms, in addition to linear forms. These challenges are both algebro-geometric and combinatorial. We now outline the main conceptual and technical challenges we face, and our approaches to overcome them.

1. In order to reduce the top fan-in, we may have to "zero out" a set of quadratic forms, that is, work over a polynomial ring modulo an ideal with quadratic generators. This forces us to work with rings which are not polynomial rings. If one is not careful, such rings will not even be unique factorization domains (UFDs), which makes the entire approach collapse.

   We overcome this issue by building on the concept of *general quotients* developed in previous works [KS09, Shp20, OS24, GOS25b, GOS25a]. While the general quotients developed in these previous works enable us to reduce the top fan-in in certain situations, they are not able to handle the case where we want to set a *particular* quadratic form to zero. To achieve this, we develop the notion of *targeted quotients* and establish some of its algebro-geometric properties, which will allow us to handle all the necessary quotients that "zero out" a particular quadratic form. We establish the properties that we need from our quotients in Section 3.5.

   Moreover, to be able to always work within "nice enough rings", we invoke the strong algebras and vector spaces developed in [AH20, OS24]. Analogously to previous works, we generalize the notion of $\Sigma^k\Pi\Sigma\Pi^2$ identities to rings of the form $S/(U)$ where S is a polynomial ring and U is a strong enough vector space in S. This generalized notion of identities allows our inductive approach more flexibility to handle the general and targeted quotients that we need to apply in order to reduce the top fan-in. However, unlike in previous works, it is important for us to be able to define the concept of "essential variables" inside of quotient rings, as these variables allow us to exploit the geometric techniques coming from absolute irreducibility. We define strong vector spaces and algebras in Section 3, and define and establish properties of essential variables in quotient rings in Section 3.4.

2. In [KS09], the drop in pairwise rank of the circuit can only happen due to linear relations between linear forms. This allows the authors to both argue about large pairwise rank with Hansen's theorem and to control the drop in pairwise rank by preserving non-colinearity relations. However, the presence of quadratic forms yields more intricate algebro-geometric phenomena, which makes the understanding of the behaviour of pairwise rank more complex.

   We overcome this issue by proving structural results on primality of certain special ideals generated by quadratic forms using absolute irreducibility over algebras, in a similar way as was done in [OS24, GOS25b, GOS25a]. Since we are dealing with quadratic forms only, the primality results we need are simpler and stronger, and thus we do not need to use the primality results in these works. Moreover, we use resultant-based methods to control the drop in pairwise rank under targeted quotients. Once we find a desired set of forms that we want to quotient out, both general quotients and targeted quotients will take care of pairwise rank preservation under quotients. It is important to note that, unlike for general quotients, we do not need to prove any lifting results for the targeted quotients. This phenomenon happens because of the same reasons as its linear analogue in [KS09]: if the pairwise rank is large enough (by assumption), then our circuit must not be an identity. Hence, to obtain the rank bounds one only need to lift the general quotients that one applies when dealing with the cases of small pairwise rank.

3. Unlike the linear case, we do not have at hand an algebro-geometric analog of Hansen's theorem. Note that the main result of [GOPS23] does not seem to be the right analog to what is needed in our case, since they only handle radical dependencies (whereas we would need product dependencies), and they do not handle the same combinatorial conditions as Hansen's theorem.

   We overcome this issue in a 2-step approach: in the first step, we "control" the high strength quadratics (the ones that "depend on too many variables") in any $\Sigma^k\Pi\Sigma\Pi^2$ identity by reducing our identity to a non-trivial $\Sigma^k\Pi\Sigma\Pi^2$ identity where all the quadratics have bounded strength (i.e., "depend on few variables"). Interestingly enough, to control the high strength quadratics, we need to combine the approach of [KS09] to reduce the top fan-in – which is done in Section 4.1 – together with a Hansen-like result, which we establish in Theorem 4.6. Note that Theorem 4.6 is different from the statements of [GOPS23, Section 6.2] (where they consider configurations of strong forms), since in our case we have to handle a Hansen-like condition, whereas [GOPS23] only requires to handle a stronger condition, similar to the SG_k^* conditions from [BDYW11]. We establish our Hansen-like result by reducing our strong span configurations to fractional configurations, which then we can further reduce to several fractional Sylvester-Gallai configurations, with an approach inspired by [BDYW11]. For details, see Section 4 for the entire approach and Section 4.2 for our Hansen-like result.

   Once we complete the strength reduction above, the second step is to prove rank bounds for $\Sigma^k\Pi\Sigma\Pi^2$ identities where all quadratics "depend on few variables." In this case we are able to combine our primality structure theorems similar to the ones from [GOS25b, GOS25a] and the integral sequences developed in [GOPS23] with the approach of [KS09] described above and construct a core, along with a "proxy core," which allows us to upper bound the rank of such $\Sigma^k\Pi\Sigma\Pi^2$ identities.

While the above outlines how we overcome the challenges posed by working with quadratic and linear forms, we now elaborate on our technical contributions and how they compare with previous works.

Elaboration of our techniques. The two main conceptual and technical contributions of this work are the concept of *targeted quotients* and *essential variables in (strong) quotient rings*. Once we have these two concepts and their properties established, we can execute our strategy above with a substantial amount of intricate combinatorial work.

- **Targeted quotients:** Previous works, such as [KS09, Shp20, PS21, OS24, GOS25a], only required working with general quotients – where forms are mapped to general multiples of (appropriate powers of) a special linear form z. These quotients have very useful properties, and allowed these works to reduce the complexity of their main objects of study (polynomial identities or Sylvester-Gallai type configurations). However, in [KS09] the authors need to also quotient by especially chosen linear forms. This type of quotient was not captured (and not needed) in subsequent works [Shp20, PS21, OS24, GOS25a], but it is crucial here. While the definition is clear on what quotient we will need for linear forms and for high strength quadratics, as these essentially behave "as variables in a polynomial ring," it is much less clear what to do in the case where we need to make a quadratic of low strength vanish "in a generic way" that does not substantially affect the other forms in the circuit. The latter case of a low strength quadratic F is addressed by our notion of targeted quotients, whose idea is as follows: if F only depends on the variables $x_1, \dots, x_r$, to "make F vanish", we need to set the variables $x_1, \dots, x_r$ to a zero of F (i.e. a point in the variety defined by F) in "a generic way." The way to obtain such a generic zero of F is to simply pick a general point $(\alpha_1, \dots, \alpha_r)$ in the zero set of F and set $x_i \mapsto \alpha_i z$, where z is a new variable. It turns out that this choice of quotients map F to zero, and at the same time "preserves" the algebraic and geometric properties of other forms that we need. We also show that the only instance where another form G is not preserved under such quotients is when G also depends only on the variables $x_1, \dots, x_r$, and in this case, we also show that G will be sent to zero iff it is a multiple of F. Proposition 3.42 establishes the properties we need from targeted quotients and in Section 5 we show how such quotients interact with the other forms in our circuit.

- **Essential variables in quotient rings:** previous works on Sylvester-Gallai type theorems involving quadratics [Shp20, PS20, PS21, PS22, GOS22] heavily used the fact that they were working over polynomial rings, which in particular was very useful when defining the space of linear forms that defined a quadratic. In this setting, the uniqueness of representation of any polynomial in the polynomial ring was key in their definitions of the relative linear space of forms of a quadratic of low strength over an algebra. However, in our more general setting, where we must work with quotients of a polynomial ring by an ideal generated by a strong vector space, it is much less clear how to define the relative space of a quadratic element. The main difficulty stems from the fact that an element of the quotient ring has many distinct representatives in the corresponding polynomial ring, and in our setting we must make sure that any such representative must be compatible with a unique space of linear forms. To address this issue, in Section 3.4, we establish some necessary results needed in order to define two key concepts: the *pseudo-distance* of a quadratic to an algebra in a quotient ring (Definition 3.28) and then after establishing some results about the pseudo-distance, we can finally define the *relative space of essential variables* (Definition 3.32), which generalize the related notions in previous works. With these notions and their properties at hand, we can analyze the case where all of our quadratics have bounded strength, which we do in Section 5. In this section, we crucially use the relative space of essential variables, combined with the geometric properties established in previous sections, to obtain our rank bounds.

- **Controlling Hansen-like configurations:** in our reduction from general $\Sigma^k\Pi\Sigma\Pi^2$ identities to the case of $\Sigma^k\Pi\Sigma\Pi^2$ identities where all quadratics are close to a small strong algebra, one important step to control the strong quadratics (i.e. the ones depending on many variables) is our Hansen-like result Theorem 4.6. This result bounds the size of a subset needed to bring a set of forms satisfying certain Hansen-like conditions (see Definition 4.2) "close to a given algebra." Although the result of Theorem 4.6 is stated in a manner which seems specific to depth-4 identities, note that Definition 4.2 can be stated independently of the setting of depth-4 identities. We believe that this result is also an interesting geometric-combinatorial result on its own right, as it allows one to control such sets of forms. As we previously mentioned, a weaker statement was proved in [GOPS23, Section 6], where the authors handle a geometric-combinatorial condition similar to the SG_k^* configurations defined in [BDYW11, Definition 5.1]. In our case, our configurations are more related to the fractional configurations generalizing Hansen, given in [BDYW11, Definition 5.2].

Choice of approach, challenges of higher degree and open questions. Given the above overview and the history about rank bounds in the depth-3 case, one may wonder what makes the approach of [KS09] the preferred one in this paper. One of the main reasons that motivated this choice is the fact that the approach of [KS09] proves rank bounds by controlling pairs of gates, whereas the other approaches make simultaneous use of more gates in their analysis. The latter approaches require geometric results involving ideals generated by several forms (of potentially unbounded degree), such as the ideal chinese remaindering results of [KS07, SS13], whereas the approach of [KS09] only requires the use of geometric statements about ideals generated by forms of constant degree. Extending these other approaches is an interesting open problem, and the geometric statements needed to do so may be important in their own right.

Another question that may arise, in light of the recent rank bounds for $\Sigma^3\Pi\Sigma\Pi^d$ circuits and structural results for ideals generated by pairs of degree d forms from [GOS25b, GOS25a], is what makes the approach above not work for bottom fan-in larger than 2. The main issue in extending the above result to bottom fan-in larger than 2 is the fact that we currently lack the correct analogue of "essential variables" when decomposing a low strength form of degree larger than 2, Note that, even for the case of degree 3, a minimum decomposition of a low strength form may have quadratic factors, which may interact non-trivially with the remainder of the forms in the circuit. On the other hand, in our case, once we establish the uniqueness of the space of linear forms that we will quotient by, their infinite strength (as they are truly variables in the polynomial ring) allows us to argue that they do not interact non-trivially with any other linear form lying outside of the given space. Generalizing our notion of essential variables to the higher degree setting is an interesting open question, as it would yield a valuable tool which can have uses not only for PIT but also for other problems in the computational complexity of other computer algebra problems.

Lastly, we conclude by emphasizing that a full resolution of [BMS13, Conjecture 28] is the biggest problem left open by this work: proving rank bounds for $\Sigma^k\Pi\Sigma\Pi^d$ circuits which depend only on k, d.

1.3.1 Flow of argument for rank bound

Starting assumption: rank bounds for simple and minimal $\Sigma^{k-1}\Pi\Sigma\Pi^2$ identities over rings $S/(U)$, where U is a strong enough vector space (the required lower bound on the strength of U is a function of $k-1$).

Suppose we are given a simple minimal $\Sigma^k\Pi\Sigma\Pi^2$ identity $\Phi = T_1 + \cdots + T_k \equiv 0$ in the ring $R = S/(U)$, where U is a strong enough vector space.

- Step 1: Are two gates close to each other? If so then apply a general quotient (see Section 3.5) to merge two gates, and invoke inductive rank bounds for circuits with top fan-in $k-1$. Then lift the rank bound to deduce a rank bound for Φ using Proposition 3.41.

  Here the measure of closeness of two gates is "pairwise rank" or the size of their symmetric difference (see Lemma 4.1, item 1 for quantitative details).

- Step 2: Suppose that no two gates are close to each other, i.e. all pairwise ranks are high.

  Case 2.1. (Fanin-reduction) If there exists a strong form H in some gate T_i such that after applying a quotient by H, the resulting circuit has high pairwise rank, then we obtain a contradiction to the starting assumption.

Case 2.2. If there is no strong form H with the property above, then we have a (M, b)-strong configuration (see Section 4.1 where we prove the contrapositive: a fanin-reduction lemma). This argument follows the strategy of Kayal and Saraf [KS09].

- Step 3: If a set of forms is an (M, b)-configuration, then there is a vector space V such that all forms in the set are close to V (see Section 4.2). This result is itself proved by induction. Refer to Lemma 4.1, item 3, Theorem 4.6, Theorem 4.10 for qualitative versions that prove approximate and robust Hansen-type results.

  Therefore, all the forms in the circuit Φ are close to some vector space V.

- Step 4: Now, all the forms in the circuit Φ are close to some vector space V, i.e. Φ is of bounded pseudo-distance with respect to V (see Section 3.4 for the notions of essential variables and pseudo-distance in quotient rings).

  In Section 5 we handle such circuits and Lemma 5.3 shows that we will land in one of the following cases, each handled separately. These statements are proved using a potential function (see Section 5.1) and an iterative process that decreases this potential (see Section 5.3).

  Case 4.1. There exists a vector space Y such that some symmetric difference lies in the algebra generated by Y. Apply a general quotient with respect to Y and lift the inductive bounds.

  Case 4.2. There exists a vector space Y such that either every quadratic form in Φ is absolutely reducible with respect to Y. Apply a general quotient with respect to Y to reduce the degree of forms in the configuration and use inductive rank bounds on $\Sigma^k\Pi\Sigma$ circuits similar to [KS09].

  Case 4.3. (Targeted fanin-reduction). There exists a form H such that the targeted quotient with respect to H (see Proposition 3.42) results in a circuit with high pairwise rank (see Section 5.4). Then we have a contradiction to the starting assumption.

At the end of these steps, we have obtained a rank bound for $\Sigma^k\Pi\Sigma\Pi^2$ circuits.

1.4 Previous and related work

Polynomial identity testing: Depth-4 circuits with bounded top and bottom fan-in are the simplest known circuit class for which no deterministic polynomial-time algorithm for PIT is currently known. Consequently, this model has been studied using techniques that extend beyond the incidence geometry–based methods discussed earlier.

In [DDS21], the authors devise and prove correctness of a quasipolynomial-time deterministic PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^d$ circuits, leveraging the Jacobian method introduced in [ASSS16]. Their key insight involves applying the logarithmic derivative and its associated power series expansion to transform the top summation gate of the circuit into a powering gate. While this transformation technically violates the bounded top fan-in constraint, circuits with powering gates are well studied, and efficient PIT algorithms are known for such models. This reduction enables the use of existing tools for PIT in powering-gate circuits, ultimately leading to their quasipolynomial-time result.

The breakthrough work of [LST22] on proving lower bounds for bounded-depth arithmetic circuits offers another route toward derandomizing PIT for this model. The *hardness-versus-randomness paradigm* has been a powerful tool for establishing connections between circuit lower bounds and derandomization [Agr05, KI04]. Notably, the results of [CKS19] extended these tradeoffs to the bounded-depth regime, showing that analogous principles continue to hold even in this restricted setting. By leveraging these developments, one obtains a *subexponential-time* PIT algorithm for depth-4 circuits. Building further on the lower bound techniques of [LST22], [AF22] constructed another *hitting set generator* tailored to bounded-depth circuits, yielding an alternative subexponential-time PIT algorithm with improved parameters. This construction exploits the fact that such circuits are inherently unable to detect low-rank matrices, due to the algebraic hardness of computing the determinant in low-depth settings. It is worth emphasizing, however, that neither of these approaches appears capable of yielding a *fully polynomial-time* PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^d$ circuits.

Karnin et al. [KMSV13] and Saraf and Volkovich [SV18] studied *multilinear* $\Sigma^k\Pi\Sigma\Pi$ circuits and obtained quasi-polynomial- and subsequently polynomial-sized hitting sets, respectively. However, as their techniques crucially rely on multilinearity, none of these works can handle even $\Sigma^3\Pi\Sigma$ circuits.

Peleg and Shpilka [PS21] gave the first polynomial sized hitting sets for $\Sigma^3\Pi\Sigma\Pi^2$ circuits. As their technique relied on their structural theorem, which argues about the structure of ideals generated by two quadratic polynomials, they were not able to extend their proof beyond three terms, as more terms would involve ideals generated by three or more quadratics.

Garg et al. [GOPS23] studied high-dimensional Sylvester-Gallai configurations and obtained a generalization of both [Han65, Shp20]. This can be viewed as a problem analogous to ours, but in the *single color class* case, where each and every k-tuple of forms satisfies some nontrivial relation. In contrast, to obtain a PIT algorithm, we need to consider the case where the forms belong to different sets and we can only argue about k-tuples coming from k disjoint sets.

Recently, two independent works [GOS25a, GW25] have made progress in the PIT problem for $\Sigma^3\Pi\Sigma\Pi^d$ circuits. Note that progress in these works is made on the *bottom fan-in* of the circuit, while the current work focuses on the *top fanin* of depth-4 circuits. These two settings present different challenges, which must be overcome in order to fully resolve the PIT problem for $\Sigma^k\Pi\Sigma\Pi^d$ circuits.

In [GW25], the authors use normalization of algebraic varieties to give a deterministic, polynomial time PIT algorithm for $\Sigma^3\Pi\Sigma\Pi^d$ circuits over *any base field* (of any characteristic), but with the *additional assumption* that one summand of the top gate is *square-free* (that is, a multiplication of coprime polynomials). Their approach circumvents the need of incidence theorems and their generalizations, which provides an advantage over finite fields, as it is known that even in the case of depth-3 circuits over fields of positive characteristic, incidence-based approaches can only yield quasi-polynomial time algorithms. On the other hand, the requirement on one summand to be square-free seems to be very restrictive.

In [GOS25a, Theorem 1.4], as discussed above, the authors prove that there is a function $R_{EK} : \mathbb{N} \to \mathbb{N}$ such that any degree d EK configurations over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, where $\operatorname{char} \mathbb{K} = 0$, has rank upper bounded by $R_{EK}(d)$. As any simple and minimal $\Sigma^3\Pi\Sigma\Pi^d$ identity gives rise to a degree d EK configuration over $\mathbb{K}[x_1, \ldots, x_n]$, the EK rank bound implies a rank bound for these identities, and when one combines this result with [BMS13, Theorem 2], one gets a deterministic, polynomial time algorithm for the PIT problem for this class.

High-dimensional and approximate variants of Sylvester-Gallai theorem: let us now elaborate on the relation between Theorem 1.6, Hansen's theorem [Han65], the strong sequences of [GOPS23] and the result of Ai, Dvir, Saraf, and Wigderson [ADSW14].

Both [Han65, ADSW14] consider configurations of points in Euclidean space, so in that context $\mathcal{F}$ should be viewed as a set of points rather than quadratic polynomials. Hansen requires that for every $b-1$ points $p_1, \ldots, p_{b-1}$ and any other point $p \in \mathcal{F}$, either the flat they define contains another point of $\mathcal{F}$, or that p lies exactly in the flat defined by $p_1, \ldots, p_{b-1}$. Thus, Hansen's condition enforces exact containment in the span rather than proximity to it.

Ai et al. consider an approximate version of this condition. Roughly speaking, an ϵ-approximate collinear Sylvester–Gallai configuration is a finite set of points $\mathcal{F} \subset \mathbb{R}^n$ that are neither "too close" nor "too far" from one another, such that for any two points in $\mathcal{F}$, there exists a third point in $\mathcal{F}$ that is ϵ-close to the line spanned by them. In other words, the standard Sylvester–Gallai requirement that the third point lie exactly on the line is replaced by an approximate one. The conclusion in this setting is that there exists a subspace V of dimension $O(1)$ such that all the points in $\mathcal{F}$ are $O(\epsilon)$-close to V. This is similar in spirit to our result, except that it concerns points rather than polynomials and corresponds to the case $b = 3$.

Hence, our Theorem 1.6 (and the formal Theorem 4.6) generalizes both Hansen's theorem and the result of Ai et al., yielding a quadratic, high-dimensional, approximate Sylvester–Gallai theorem.

Barak, Dvir, Wigderson, and Yehudayoff [BDWY13] proved a robust version of Hansen's theorem. They call a $(b-1)$-dimensional flat spanned by b points of $\mathcal{F}$ *elementary* if its intersection with $\mathcal{F}$ does not contain any other point. They showed that if, for every choice of $b-1$ points in $\mathcal{F}$, at least a ν-fraction of the remaining points are such that each of them either lies in the flat spanned by these $b-1$ points or spans with them a flat that is not elementary, then $\mathcal{F}$ is contained in an $O_b(1/\nu^2)$-dimensional flat. This bound was later improved in [DSW14, DGOS18] to $O_b(1/\nu)$.

If we now replace points by quadratic polynomials, and the condition of belonging to a flat by being close to it, we obtain a setting similar to that of Theorem 1.8. The conclusion we derive is also analogous to that of [BDWY13]: to approximate all the quadratics, we must repeat the argument about $1/\nu$ times, resulting in an $O_b(1/\nu^2)$-dimensional subspace. The work of [GOPS23] generalizes the version of *ordinary* spaces defined by [BDYW11, Definition 5.1].

2 Preliminaries

In this section we establish notation that will be used in the remainder of the paper, as well as some basic definitions and claims. We will be working over an algebraically closed field $\mathbb{K}$ with characteristic zero. As customary, a homogeneous polynomial will be called a form. As we need to work over graded $\mathbb{K}$-algebras, we will take R to be a finitely generated and graded $\mathbb{K}$-algebra with $R_0 = \mathbb{K}$ which is a UFD. In this setting, we take R_d to be the set of elements of R of degree d, which we also call forms (of degree d). Additionally, we write $R_{\leqslant d} := \bigoplus_{e \leqslant d} R_e$. Over such rings, we can generalize all the standard main concepts that we need about depths 3 and 4 circuits.

2.1 Circuits over quotient rings

Definition 2.1 (Circuits over a graded UFD)**.** For any $k \in \mathbb{N}$, a $\Sigma^k\Pi\Sigma\Pi_R^d$ circuit is an expression of the form $\Phi = \sum_{i=1}^k T_i$, where $T_i := \prod_{j=1}^{m_i} Q_{ij}$ is the product of irreducible elements $Q_{ij} \in R_{\leqslant d}$. We say that T_i are the gates of the circuit Φ, and for $Q \in R$ we say that Q belongs to gate T_i if $Q \mid T_i$. We say that $Q \in R$ belongs to

Φ if Q belongs to one of the gates of Φ. We say that Φ is a homogeneous circuit if all Q_{ij} are homogeneous elements of R and $\deg T_i = \deg T_j$ for any two gates of Φ.

Since R is a UFD, we know that in the above definition, each gate T_i of Φ can be uniquely factored into irreducible factors, and thus all of the above is well-defined.

Definition 2.2 (GCD, Simplicity and minimality). Given a $\Sigma^k\Pi\Sigma\Pi_R^d$ circuit $\Phi = T_1 + \cdots T_k$, the GCD of Φ, denoted by $\gcd(\Phi)$ is defined as: $\gcd(\Phi) := \gcd(T_1, \ldots, T_k)$. The simple part of Φ, denoted $\text{sim}(\Phi)$, is defined as $\text{sim}(\Phi) := \Phi / \gcd(\Phi)$. We say that Φ is minimal if no strict subset of the gates of Φ form an identity. That is, for any $S \subsetneqq [k]$, we have $\sum_{i \in S} T_i \not\equiv 0$.

We now define the main parameters of $\Sigma^k\Pi\Sigma\Pi_R^d$ circuits.

Definition 2.3. Let $\Phi = \sum_{i=1}^k T_i$ be a $\Sigma^k\Pi\Sigma\Pi_R^d$ circuit.

- For $Q \in R$, the multiplicity of Q in T_i is defined as $\text{mult}(Q, T_i) := \max\{e \in \mathbb{N} \mid T_i \in (Q^e)\}$.
- The symmetric difference of gates T_i and T_j is

$$T_i \Delta T_j := \{(Q) \mid Q \text{ is irreducible and } \text{mult}(Q, T_i) \neq \text{mult}(Q, T_j)\}.$$

  That is, $T_i \Delta T_j$ captures the set of prime principal ideals (Q) in R such that the multiplicity of Q is different in the gates T_i, T_j. Since each prime ideal (Q) is uniquely defined up to multiplication by a unit, we will abuse notation and regard $T_i \Delta T_j$ to be a set of forms.

- It will also be useful to define the set difference as

$$T_i \setminus T_j := \{(Q) \mid Q \text{ is irreducible and } \text{mult}(Q, T_i) > \text{mult}(Q, T_j)\}.$$

  Similarly to the symmetric difference, we will abuse notation and consider it as a set of forms. Note that $T_i \Delta T_j = (T_i \setminus T_j) \cup (T_j \setminus T_i)$.

- The pairwise rank of two gates T_i, T_j is defined as $\dim \text{span}_{\mathbb{K}} \{T_i \Delta T_j\}$, where here we see the symmetric difference as a set of forms. The pairwise rank of Φ is defined as $\min_{i \neq j} \dim \text{span}_{\mathbb{K}} \{T_i \Delta T_j\}$.

2.2 Resultants

In this section we recall some useful facts about resultants that we will use in later sections.

Proposition 2.4. *Let* $A = \mathbb{K}[x_1, \cdots, x_n]$ *and* $S = A[y]$ *be polynomial rings, and* $P, Q \in S$ *be non-associate irreducible polynomials of positive degree in* y*. Suppose* $P = a_d y^d + \cdots + a_0$ *and* $Q = b_e y^e + \cdots + b_0$*, where* $a_i, b_i \in A$*. Let* $I := (P, Q) \cap A$*. If* $\gcd(a_d, b_e, \text{Res}_y(P, Q)) = 1$*, then* $\text{rad}(\text{Res}_y(P, Q)) = \text{rad}(I)$*.*

Proof. Let $f_1, \cdots, f_k$ be the distinct irreducible factors of $\text{Res}_y(P, Q)$. If $\text{rad}(I) \subset (f_i)$ for all i, then $\text{rad}(I) \subset \Pi_{i=1}^k f_i = \text{rad}(\text{Res}_y(P, Q))$. Then $\text{rad}(\text{Res}_y(P, Q)) = \text{rad}(I)$, as $\text{Res}_y(P, Q) \in I$. Therefore it is enough to show that $\text{rad}(I) \subseteq (f)$ for any irreducible factor f of $\text{Res}_y(P, Q)$.

Let $X_f \subset \mathbb{A}^n$ be the hypersurface defined by f. Since $\gcd(a_d, b_e, f) = 1$, we have $X_f \setminus V(a_d, b_e) \neq \emptyset$. Therefore $U = X_f \setminus V(a_d, b_e)$ is a non-empty open subset of X_f. Since X_f is irreducible, we have $\overline{U} = X_f$.

Since f is an irreducible factor of $\mathrm{Res}_y(P,Q)$, we see that $\mathrm{Res}_y(P,Q)(\mathbf{p}) = 0$ for any $\mathbf{p} \in X_f$. Now for $\mathbf{p} \in U$, we have $a_d(\mathbf{p}) \neq 0$ or $b_e(\mathbf{p}) \neq 0$. Therefore, for any $\mathbf{p} \in U$ we have $\deg_y(P(y,\mathbf{p})) = \deg_y(P)$ or $\deg_y(Q(y,\mathbf{p})) = \deg_y(Q)$. Hence $\mathrm{Res}_y(P(y,\mathbf{p}), Q(y,\mathbf{p})) = \mathrm{Res}_y(P,Q)(\mathbf{p}) = 0$ for any $\mathbf{p} \in U$ by [CLO07, Chapter 3, Proposition 6]. Therefore, for any $\mathbf{p} \in U$, the polynomials $P(y,\mathbf{p})$ and $Q(y,\mathbf{p})$ have a common root $y = c$ and we have $(c,\mathbf{p}) \in V(P,Q) \subset \mathbb{A}^{n+1}$. Hence, for any $\mathbf{p} \in U$, there exists a c such that $(c,\mathbf{p}) \in V(P,Q)$ and hence $U \subseteq \pi(V(P,Q)) \subseteq V(I)$. Thus we have $X_f = \overline{U} \subseteq V(I)$ and $\mathrm{rad}(I) \subseteq (f)$. □

Corollary 2.5. *Let* $A = \mathbb{K}[x_1, \cdots, x_n]$ *and* $S = A[y]$ *be polynomial rings, and* $F, Q, H \in S$ *be non-associate irreducible polynomials. Suppose that* $F \in A$, $Q = a_d y^d + \cdots + a_0$ *and* $H = b_e y^e + \cdots + b_0$, *where* $a_i, b_i \in A$ *and* $d, e > 0$. *Suppose that* (F,Q) *is prime and* $\gcd(a_d, b_e, \mathrm{Res}_y(Q,H)) = 1$. *If* F *divides* $\mathrm{Res}_y(Q,H)$, *then* $(Q,H) \subseteq (F,Q)$. *In particular, if* F, Q, H *are quadratic forms, then* $(Q,H) = (F,Q)$.

Proof. Let $I = (Q,H)$ and $J = I \cap A$. By Proposition 2.4, we have $\mathrm{rad}(J) = \mathrm{rad}(\mathrm{Res}_y(Q,H))$. Since F is an irreducible factor of $\mathrm{Res}_y(Q,H)$, we have that $(F) \supseteq \mathrm{rad}(J)$. Now $\mathrm{rad}(J) = \mathrm{rad}(I) \cap A = \cap_{i=1}^k (\mathfrak{p}_i \cap A)$, where $\mathfrak{p}_1, \cdots, \mathfrak{p}_k$ are the minimal primes over I. Therefore $(F) \supseteq \cap_{i=1}^k (\mathfrak{p}_i \cap A)$, and hence $(F) \supseteq (\mathfrak{p}_i \cap A) \neq 0$ for some $i \in [k]$. Since $\mathrm{ht}(F) = 1$, we have $(F) = \mathfrak{p}_i \cap A$. Now (F,Q) and $\mathfrak{p}_i$ are both prime ideals of height 2 and $(F,Q) \subseteq \mathfrak{p}_i$. Therefore, we must have $(F,Q) = \mathfrak{p}_i$. Hence, $(Q,H) \subseteq (F,Q)$.

Now, if in addition F, Q, H are homogeneous of degree 2, then $H \in (F,Q) \Rightarrow H \in \mathrm{span}_{\mathbb{K}}\{F,Q\}$. Since Q, H are not scalar multiples of each other, we have that $F \in \mathrm{span}_{\mathbb{K}}\{Q,H\}$. Therefore $(Q,H) = (F,Q)$. □

2.3 Absolute irreducibility

We now define absolute irreducibility over an algebra and state an effective bound on the number of non-prime ideals involving an absolutely irreducible form and elements of the algebra.

Definition 2.6. If $S := \mathbb{K}[z_1, \cdots, z_n]$ be a polynomial ring and $V \subseteq S_1$ be a vector space of linear forms. Letting $x_1, \ldots, x_r$ be any basis for V and extending it to a basis $x_1, \ldots, x_r, y_1, \ldots, y_s$ of S_1, we say that $F \in S$ is absolutely irreducible over V if F is irreducible over $\overline{\mathbb{K}(V)}[y_1, \ldots, y_s]$.

3 Strong algebras, lifted strength and graded quotients

We start by recalling a few useful notions from commutative algebra.

3.1 Regular, prime and $\mathcal{R}_\eta$-sequences

Definition 3.1 (Regular sequence)**.** Let R be commutative ring. A sequence of elements $F_1, F_2, \cdots F_n \in R$ is called a regular sequence if

(1) $(F_1, F_2, \cdots, F_n)R \neq R$, and

(2) for $i = 1, \cdots, n$, F_i is a non-zerodivisor on $R/(F_1, \cdots, F_{i-1})$.

Remark 3.2. If $F_1, \cdots, F_n$ is a regular sequence of forms in S, then $F_1, \cdots, F_n$ are algebraically independent. Thus the subalgebra generated by $F_1, \cdots, F_n$ is isomorphic to a polynomial ring. In particular, the homomorphism $\mathbb{K}[y_1, \cdots, y_n] \to S$ defined by $y_i \mapsto F_i$ is an isomorphism onto its image.

Given a regular sequence $F_1, \cdots, F_n \in S = \mathbb{K}[x_1, \cdots, x_N]$, we may consider the affine algebraic variety $\mathcal{Z}(F_1, \cdots, F_n) \subseteq \mathbb{K}^N$. In general, this variety can be highly singular and the quotient ring $S/(F_1, \cdots, F_n)$ might not be a UFD. The following notion captures the size of the singular locus of the zero set of $F_1, \cdots, F_n$, and is key to obtaining a quotient UFD (see Corollary 3.13).

Definition 3.3 (Serre's $\mathcal{R}_\eta$-property)**.** Let $\eta \in \mathbb{N}$. We say that a Noetherian ring R satisfies the $\mathcal{R}_\eta$ property if the local ring $R_\mathfrak{p}$ is a regular local ring for all prime ideals $\mathfrak{p} \subset R$ such that $\text{height}(\mathfrak{p}) \leqslant \eta$.

Definition 3.4. [Prime and $\mathcal{R}_\eta$-sequences] Let $\eta \in \mathbb{N}$ and R a Noetherian ring. A sequence of elements $F_1, \dots, F_n \in R$ is called a prime sequence (respectively an $\mathcal{R}_\eta$-sequence) if

1. $F_1, \cdots, F_n$ is a regular sequence, and
2. $R/(F_1, \cdots, F_i)$ is an integral domain (respectively, satisfies the $\mathcal{R}_\eta$ property) for all $i \in [n]$.

Remark 3.5. Since prime sequences and $\mathcal{R}_\eta$-sequences are regular sequences, we know that if $F_1, \cdots, F_n$ is a prime sequence or $\mathcal{R}_\eta$-sequence in the polynomial ring S, then $F_1, \cdots, F_n$ are algebraically independent. In particular, the algebra $\mathbb{K}[F_1, \cdots, F_n]$ is isomorphic to a polynomial ring.

3.2 Strength, lifted strength, and strong vector spaces

In this section, we recall some basic definitions about strength of forms in a polynomial ring as well as strong vector spaces and lifted strength of forms in certain quotient rings. Some results are stated for an arbitrary degree d, but we will only use the results with $d = 2$. We recall once again that we maintain our convention that S is a polynomial ring in finitely many variables.

Let $R = \oplus_{d \geqslant 0} R_d$ be a finitely generated graded $\mathbb{K}$-algebra, generated by R_1. In [AH20] the notions of collapse and strength were defined for a polynomial ring. These definitions were extended to certain special finitely generated graded $\mathbb{K}$-algebras by [OS24]. In this section, we collect the definitions and properties we need from [AH20, OS24], and establish some additional properties that we need.

Definition 3.6 (Collapse)**.** Given a non-zero form $F \in R_d$, we say that F has a k-*collapse* if there exist k forms $G_1, \dots, G_k$ such that $1 \leqslant \deg(G_i) < d$ and $F \in (G_1, \dots, G_k)$.

Definition 3.7 (Strength)**.** Given a non-zero form $F \in R_d$, the *strength* of F, denoted by $s(F)$, is the least positive integer such that F has a $(s(F) + 1)$-collapse but it has no $s(F)$-collapse. We say that $s(F) \geqslant t$ whenever F does not have a t-collapse.

Remark 3.8. By the definitions above, a form $x \in R_1$ does not have a k-collapse for any $k \in \mathbb{N}$. Thus, we say that for any $x \in R_1$, $s(x) = \infty$. In particular, linear forms in the polynomial ring S have infinite strength. We will make the convention that $s(0) = -1$.

Definition 3.9 (Minimum collapse)**.** Given $F \in R_d \setminus 0$ and $s \in \mathbb{N}^*$ such that $s(F) = s - 1$, a *minimum collapse* of F is any identity of the form $F = G_1 H_1 + \cdots + G_s H_s$, where G_i, H_i are forms of degree in $[d-1]$.

It is useful to define the min and max strength of a vector space of forms of the same degree.

Definition 3.10 (Min and max strength)**.** Given any non-zero finite dimensional vector space $V \subset R_d$, define $s_{\min}(V)$ ($s_{\max}(V)$) as the minimum (maximum) strength of any non-zero form in V. If $V = (0)$, then there are no non-zero forms in V. In this case, by convention we define $s_{\min}((0)) = s_{\max}((0)) = \infty$. We will say that a vector space V is k-strong if $s_{\min}(V) \geqslant k$. Note that the zero vector space is infinitely strong.

In particular, given forms $F_1, \dots, F_r \in R_d$, we will denote $s_{\min}(F_1, \dots, F_r) := s_{\min}(\operatorname{span}_{\mathbb{K}}\{F_1, \dots, F_r\})$ and $s_{\max}(F_1, \dots, F_r) := s_{\max}(\operatorname{span}_{\mathbb{K}}\{F_1, \dots, F_r\})$.

Definition 3.11 (Dimension sequence)**.** Given a graded $\mathbb{K}$-vector space $V = \bigoplus_{i=1}^{d} V_i \subset R$, where we denote $\delta_i := \dim V_i$, we denote its dimension sequence by $\delta_V := (\delta_1, \dots, \delta_d)$.

We shall consider the reverse lexicographic ordering, which is a well ordering, on dimension sequences $\delta \in \mathbb{N}^d$. Whenever we say that a function $f : \mathbb{N}^d \to \mathbb{N}^{d'}$ is ascending, we mean that it is ascending with respect to this well ordering.

Definition 3.12 (Strong Ananyan-Hochster vector spaces)**.** Let $R = \oplus_{d \geqslant 0} R_d$ be a finitely generated graded $\mathbb{K}$-algebra, generated by R_1. For any function $B = (B_1, \cdots, B_d) : \mathbb{N}^d \to \mathbb{N}^d$, we say that a non-zero graded vector subspace $V = \oplus_{i=1}^{d} V_i \subset R$, with dimension sequence δ_V, is a B-strong AH vector space if V_i is $B_i(\delta_V)$-strong for all i, i.e. $s_{\min}(V_i) \geqslant B_i(\delta_V)$. The subalgebra $\mathbb{K}[V] \subset R$ generated by a B-strong AH vector space V is called a B-strong AH algebra.

Note that if $V = (0)$, then V is B-strong for any function B, since $s_{\min}((0)) = \infty$. The following result is a corollary of [AH20, Theorem A], and a proof can be found in [OS24, Corollary 5.9]. In the following lemma and the rest of this article, the function $A(\eta, d) : \mathbb{N}^2 \to \mathbb{N}$ is the function defined in [AH20, Theorem A].

Corollary 3.13. *Let $V = \oplus_{i=1}^{d} V_i \subset S$ be a B-strong AH vector space for some function $B : \mathbb{N}^d \to \mathbb{N}^d$. Suppose $B_i(\delta_V) \geqslant A(\eta, i) + 3(\sum_i \delta_i - 1)$ for some $\eta \in \mathbb{N}$. Then any sequence of $\mathbb{K}$-linearly independent forms in V is an $\mathcal{R}_\eta$-sequence. If $\eta \geqslant 3$, then $S/(V)$ is a Cohen-Macaulay, unique factorization domain.*

A number of results we invoke only hold in Cohen-Macaulay (CM) rings, thus this is a useful property for a ring to possess. We recall the notion of lifted strength, originally defined in [OS24, Section 5.3].

Definition 3.14 (Lifted strength)**.** Let $U \subset S$ be a graded vector space and $R = S/(U)$. Let $F \in R_d$ be a non-zero form. We define the lifted strength of F with respect to U as

$$\widetilde{s}_{\min}(U, F) := \min\{s_{\min}(U_d + \operatorname{span}_{\mathbb{K}}\left\{\widetilde{F}\right\})\}$$

where $\widetilde{F}$ varies over all forms in S_d such that the image of $\widetilde{F}$ in R is F. Given a set of forms $F_1, \cdots, F_m \in R_d$, we define

$$\widetilde{s}_{\min}(U, F_1, \cdots, F_m) = \min\{s_{\min}(U_d + \operatorname{span}_{\mathbb{K}}\left\{\widetilde{F}_1, \cdots, \widetilde{F}_m\right\})\},$$

where $\widetilde{F}_i$ varies over all forms in S_d such that the image of $\widetilde{F}_i$ in R is F_i. Given a non-zero vector space $V \subset R_d$, we define

$$\widetilde{s}_{\min}(U, V) = \min\{\widetilde{s}_{\min}(U_d, F_1, \cdots, F_m)\},$$

where $F_1, \cdots, F_m$ vary over all possible bases of V. We say that $V \subset R_d$ is k-lifted strong with respect to U if $\widetilde{s}_{\min}(U, V) \geqslant k$. For simplicity, we omit U from the notation and write $\widetilde{s}_{\min}(V)$ when U is clear from the context.

Suppose that $U \subseteq S_{\leqslant d}$ is of dimension sequence δ_U. Let $V = \oplus_{i=1}^{d} V_i \subset R = S/(U)$ be a graded vector space of dimension sequence δ_V. For any function, $B : \mathbb{N}^d \to \mathbb{N}^d$ we will say that V is B-lifted strong with respect to U, if V_i is $B_i(\delta_U + \delta_V)$-lifted strong, i.e. $\widetilde{s}_{\min}(U, V_i) \geqslant B_i(\delta_U + \delta_V)$ for all $i \in [d]$. In other words, V is B-lifted strong with respect to U, if the vector space $U + \mathrm{span}_{\mathbb{K}}\left\{\widetilde{F}_1, \cdots, \widetilde{F}_m\right\}$ is B-strong in S, for any homogeneous basis $F_1, \cdots, F_m \in R$ of V and any set of homogeneous lifts $\widetilde{F}_1, \cdots, \widetilde{F}_m \in S$.

3.3 Iterative strengthening processes

We recall the main strengthening result that we need, proved in [OS24, Lemma 5.15].

Lemma 3.15 (Strengthening of Algebras)**.** *For any $d \in \mathbb{N}$ and function $B : \mathbb{N}^d \to \mathbb{N}^d$, there exist functions $C_B : \mathbb{N}^d \to \mathbb{N}^d$ and $h_B : \mathbb{N}^d \to \mathbb{N}^d$, depending on B, such that the following holds:*

Given a graded vector space $U = \oplus_{i=1}^d U_i \subset S$ with dimension sequence $\delta_U \in \mathbb{N}^d$, there exists a B-strong AH vector space $V = \oplus_{i=1}^d V_i$ such that

1. *$\mathbb{K}[U] \subset \mathbb{K}[V]$,*
2. *$\forall i \in [d]$, $\dim(V_i) \leqslant C_{B,i}(\delta_U)$, where $C_{B,i}$ denotes the i-th component of $C_B = (C_{B,1}, \cdots, C_{B,d}) : \mathbb{N}^d \to \mathbb{N}^d$.*

Furthermore, suppose $H = \oplus_{i=1}^d H_i \subset U$ is a graded subspace such that $s_{\min}(H_i) \geqslant h_{B,i}(\delta_U)$ for all $i \in [d]$. Then there exists a B-strong AH vector space V satisfying (1) and (2) above such that $H \subset V$.

A corollary that will be useful is the following.

Corollary 3.16 (Strengthening of Algebras)**.** *Let $d \in \mathbb{N}$. Let $B : \mathbb{N}^d \to \mathbb{N}^d$ and $f : \mathbb{N}^d \to \mathbb{N}$ be functions. There exists functions $h_{B,f} : \mathbb{N}^d \to \mathbb{N}^d$ and $C_{B,f} : \mathbb{N}^d \to \mathbb{N}^d$ with the following properties. Given any vector space U that is $h_{B,f}$-strong, and any set of $f(\delta_U)$ many forms $Q_1, \ldots, Q_{f(\delta_U)}$ of degree at most d, we can construct a vector space U' with $U \subset U'$ and $Q_i \in \mathbb{K}[U']$ such that U' is B-strong. Further, the dimension sequence of U' is bounded coordinate wise by $C_{B,f}(\delta_U)$.*

Proof. Given U, we construct the vector space $W := U + \mathrm{span}_{\mathbb{K}}\{Q_1, \ldots, Q_{f(\delta_U)}\}$. The dimension sequence of W depends on the degrees of the forms Q_i and their linear independence with respect to U. Let Δ_U be the set of all obtainable dimension sequences, every element of Δ_U is coordinate wise bounded by $\delta_{U,i} + f(\delta_U)$.

We define $h_{B,f,i}(\delta_U) := \max_{\delta \in \Delta_U} h_{B,i}(\delta)$ and similarly $C_{B,f,i}(\delta_U) := \max_{\delta \in \Delta_U} C_{b,i}(\delta)$. These are well defined since Δ_U is a finite set, and since Δ_U only depends on δ_U and f. Note that the defined functions $h_{B,f}$ and $C_{B,f}$ only depend on B and f. The claim follows by Lemma 3.15 applied to W with these parameters. □

The following corollary corresponds to [OS24, Corollary 5.17]. As in the previous corollary, the number of forms k in the below corollary can be a function of the dimension sequence of U.

Corollary 3.17. *Let $\mu : \mathbb{N}^d \to \mathbb{N}^d$. Let $U \subset S$ be a graded vector space with dimension sequence $\delta_U \in \mathbb{N}^d$ and let $R = S/(U)$. Let $V \subset R$ is a graded vector space with dimension sequence $\delta_V \in \mathbb{N}^d$. Suppose V is $h_{2\mu} \circ t_k$-lifted strong with respect to U. Let $P_1, \cdots, P_k \in R_{\leqslant d}$ be homogeneous elements. Then there exists a graded vector space $V' \subset R_{\leqslant d}$ such that:*

1. *V' is μ-lifted strong with respect to U.*
2. *$P_1, \cdots, P_k \in \mathbb{K}[V']$.*
3. *$V \subset V'$.*

4. *for all* $i \in [d]$, *we have* $\dim(V'_i) \leqslant C_{2\mu,i}(t_k(\delta_U + \delta_V)) - \delta_{U,i}$. *In particular,* $\dim(V') \leqslant C(\mu, \delta_U, \delta_V, k) := \sum(C_{2\mu,i}(t_k(\delta_U + \delta_V)) - \delta_{U,i})$.

Definition 3.18 (Ananyan-Hochster spaces)**.** In the situation of Corollary 3.16, we define $AH(U, W)$ to be any graded vector space V provided by Corollary 3.16. Similarly, in the situation of Corollary 3.17, we define $AH_R(\mu, V, P_1, \cdots, P_k)$ to be any vector space V' provided by Corollary 3.17. When the function μ or the ring R is evident from the context, we will omit it from the notation.

Remark 3.19. Note that, for given μ and $V, P_1, \cdots, P_k$, the vector space $AH_R(\mu, V, P_1, \cdots, P_k)$ is not necessarily unique. As stated in our definition, we use the notation $AH_R(\mu, V, P_1, \cdots, P_k)$ to denote any vector space that satisfies the properties in Corollary 3.17, whose existence is guaranteed. We will only use these properties of these spaces and in all our arguments we work with any fixed choice of such a vector space $AH_R(\mu, V, P_1, \cdots, P_k)$.

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An iterative AH-process. A recurring strategy in our constructions throughout the paper will be the following: we run an iterative process to pick out a set of forms with useful properties and, at each iteration of the process, we would like to have the ability to quotient with respect to these forms to simplify the circuit. As discussed in the introduction, for a quotient ring $R/(V)$ to be a UFD, we need the vector space V to be strong. Hence we will apply the AH-construction above to strengthen the vector space spanned by the chosen forms at each iteration. We make a convenient definition for such iterative processes.

Definition 3.20 ((k, r)-process on $(V, \mathcal{F})$)**.** Let $U \subseteq S$ be a graded finitely generated vector space such that $R := S/(U)$ is a UFD, and $k, r \geqslant 1$ be integers. Let $V \subseteq R$ be a vector space and let $\mathcal{F} \subseteq R$. A (k, r) process on $(V, \mathcal{F})$ is defined to be a process that starts with the vector space V, and performs at most r rounds. In each round, at most k forms $F_1, \ldots, F_k$ of degree at most d are picked from the set $\mathcal{F}$, and V is updated to $AH_R(V, F_1, \ldots, F_k)$. In each iteration, the picked forms are allowed to be chosen based on the new V. Note that we are dropping the dependence on d from the notation for convenience, as d will always be clear from context.

We introduce auxiliary functions that will provide uniform bounds for (k, r)-processes (see Lemma 3.22).

Definition 3.21 (Strength and dimension bound functions)**.** Let $\delta \in \mathbb{N}^d$ be a dimension sequence, $B : \mathbb{N}^d \to \mathbb{N}^d$ be a function and $k, r \in \mathbb{N}$. We have the following definitions:

- *Translation functions.* For any $\boldsymbol{\mu} \in \mathbb{N}^d$, we define the translation function $t_{\boldsymbol{\mu}} : \mathbb{N}^d \to \mathbb{N}^d$ as $t_{\boldsymbol{\mu}} = (t_{\boldsymbol{\mu},1}, \ldots, t_{\boldsymbol{\mu},d})$, where the i-th component is defined by $t_{\boldsymbol{\mu},i}(\boldsymbol{\delta}) = \delta_i + \mu_i$. In other words, for all $i \in [d]$ we add μ_i to the i-th component of $\boldsymbol{\delta}$. For any $n \in \mathbb{N}$, we let $t_n := t_{(n,\ldots,n)}$. More generally, given a function $f : \mathbb{N}^d \to \mathbb{N}^d$, we define the translation function $t_f : \mathbb{N}^d \to \mathbb{N}^d$ as $t_f(\delta) = \delta + f(\delta)$.

- *Strength bound function.* Define a function $H(B, k, r) : \mathbb{N}^d \to \mathbb{N}^d$ recursively as follows. Let $H(B, k, 0) := B$, and

$$H(B, k, r) := h_{2H(B,k,r-1)} \circ t_{k+1},$$

  where h is as defined in Lemma 3.15. This function $H(B, k, r)$ captures the strength needed for a vector space V, so that after applying a (k, r)-process to V, the resulting vector space V' is still B-lifted strong and this (k, r)-process preserves V, i.e. $V \subseteq V'$.

- *Dimension bound function.* We define a function $D(B,k,r,\delta) : \mathbb{N}^d \to \mathbb{N}^d$ recursively as follows. Let $D(B,k,0,\delta)$ be the identity function. To define $D(B,k,r,\delta)(\mu)$, assuming $D(B,k,r-1,\delta)$ is already defined, we first define

$$\beta := \max_{\substack{x \in \mathbb{N}^d \\ \|x\|_1 = \|\mu\|_1}} \|D(B,k,r-1,\delta)(x)\|_1 ,$$

  where $\|x\|_1 = x_1 + \cdots + x_d$ for $x = (x_1, \cdots, x_d) \in \mathbb{N}^d$. We then define the i^{th} coordinate function $D_i(B,k,r,\delta)(\mu)$ as

$$D_i(B,k,r,\delta)(\mu) = \max_{\substack{x \in \mathbb{N}^d, \\ \|x\|_1 = \beta}} C_{2H(B,k,r),i}(t_{k+1}(x+\delta)),$$

  where again C is the function defined in Lemma 3.15. This function $D(B,k,r,\delta)$ captures the dimension upper bound for the result of a (k,r)-process applied on a vector space V.

We note these bounds and basic properties of (k,r) processes below.

Lemma 3.22. *Let $U \subseteq S$ be a graded finitely generated vector space s.t. $R := S/(U)$ is a UFD, and $k, r, k', r' \in \mathbb{N}^*$.*

1. *The composition of a (k,r) process and a (k',r') process is a $(\max(k,k'), r+r')$ process.*
2. *If V is $H(B,k,r)$-lifted strong, and if V' is the result of a (k,r) process on V, then $V \subset V'$ and V' is B-lifted strong. Moreover, each intermediate vector space that appears as part of the process is $h_{2B} \circ t_1$-lifted strong.*
3. *If V' is the result of a (k,r) process on V, then the dimension of V' is bounded by $D(B,k,r,\delta_U)(\delta_V)$, where δ_U, δ_V are the dimension vectors of V, U respectively.*

Proof. The first property follows by definition. Properties (2) and (3) follow by induction on r. □

3.4 Essential variables in quotient rings

In this subsection we define the space of essential variables in strong quotient rings and establish its necessary properties. We will restrict ourselves to having $d = 2$, as it is the case of interest in this work.

Convention. Fix $\eta \geqslant 3$. Throughout this section we will assume that all our ascending functions, such as $\Lambda, \Gamma, \mu : \mathbb{N}^2 \to \mathbb{N}^2$, satisfy the following property: the i-th component is larger than the function

$$B_i(\delta) := A(\eta, i) + 3\left(\sum_i \delta_i\right)$$

for all $\delta \in \mathbb{N}^2$. Here the functions $A(\eta, i)$ are given by [AH20, Theorem A].

In [Shp20], the space of "essential variables" of quadratic forms in polynomial rings was considered. This notion was modified and generalized in [GOS22, Definition 2.3] to include a cut-off where one preserves the quadratic. We generalize this notion to quadratic forms in strong quotient rings. First, let us recall the results of [Shp20, PS20] which hold in the polynomial ring.

Proposition 3.23. *Let $Q \in S = \mathbb{K}[x_1, \cdots, x_N]$ be a quadratic form. Consider the following collection of vector spaces of linear forms*

$$\mathcal{W} := \{W \subseteq S_1 \mid Q \in \mathbb{K}[W]\}.$$

The following statements hold.

1. *There exists a unique* $W \in \mathcal{W}$ *such that* $\dim(W) = \min\{\dim(W') \mid W' \in \mathcal{W}\}$.

2. *Moreover, if* $W \in \mathcal{W}$ *is of minimum dimension, then* $W \subseteq W'$ *for any* $W' \in \mathcal{W}$.

Proof. Item (1) follows from [Shp20, Claim 14], and (2) is given by [PS20, Fact 2.15]. □

Since we want to generalize Proposition 3.23 to quotient rings, we need to assume a fixed bound for the size of algebras considered in the collection $\mathcal{W}$. Let us first define this auxiliary collection of vector spaces corresponding to a quadratic form.

Definition 3.24. Fix $a \in \mathbb{N}$. For $G \in R_2$, we define $\mathcal{W}_a(G)$ to be the collection of all subspaces $W \subseteq R_1$ such that $G \in \mathbb{K}[W]$ and $\dim(W) \leqslant a$.

Proposition 3.25. *Suppose that* $U \subseteq S$ *is* Λ*-strong and* $R = S/(U)$. *Let* $a \in \mathbb{N}$ *be such that* $\Lambda_i(\delta_U) \geqslant B \circ t_{(2a,0)}(\delta_U)$. *Let* $G \in R_2$. *For* $r \in \mathbb{N}$, *if* G *is not* r*-strong, then the following holds.*

1. *We have* $\mathcal{W}_{2r}(G) \neq \emptyset$.

2. *If* $a \geqslant 2r$, *then there is an unique vector space* $W \in \mathcal{W}_a(G)$ *of minimum dimension. Moreover, for any* $W' \in \mathcal{W}_a(G)$, *we have* $W \subseteq W'$.

Proof. (1) Since G is not r-strong, there exist linear forms $\ell_1, \cdots, \ell_{2r} \in R_1$ such that $G \in \mathbb{K}[\ell_1, \cdots, \ell_{2r}] \subseteq R$. We let $W = \operatorname{span}_{\mathbb{K}}\{\ell_1, \cdots, \ell_{2r}\}$.

(2) Let $W_1, W_2 \subseteq R_1$ be two vector spaces of minimum dimension such that $G \in \mathbb{K}[W_i]$ and $\dim(W_i) \leqslant a$. Let $\ell_1, \cdots, \ell_m$ be a basis of $W_1 + W_2$ for some $m \leqslant 2a$. Note that we must have $\dim(W_i) \leqslant a$, by part (1). Since U is $B \circ t_{2a}$-strong, and the number forms $\ell_1, \cdots, \ell_m$ is at most $2a$, we may apply [OS24, Proposition 5.10.5], which shows that $\ell_1, \cdots, \ell_m$ is a prime sequence in R (see Definition 3.4). Let $A = \mathbb{K}[\ell_1, \cdots, \ell_m]$, which is isomorphic to a polynomial ring in m variables (see Remark 3.5). Therefore, we may apply [Shp20, Claim 14] (see Proposition 3.23) in the polynomial ring A, and conclude that $W_1 = W_2$.

Let $W \in \mathcal{W}_a(G)$ be the vector space of minimum dimension. Now let $W' \in \mathcal{W}_a(G)$ be another subspace. We may again find a subalgebra $A \subseteq R$, which is isomorphic to a polynomial ring in at most $2a$ variables that contains basis of $W + W'$. By applying [PS20, Fact 2.15] (see Proposition 3.23) in the polynomial ring A, we obtain $W \subseteq W'$. □

Corollary 3.26. *Let* $\mu, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ *be ascending functions with* $\Lambda \geqslant B \circ t_{(4\mu_2,0)}$. *Suppose that* $U \subseteq S$ *is* $\Lambda \circ t_{(0,1)}$*-strong and* $R = S/(U)$. *Let* $G \in R_2$ *be a form that is not* μ*-lifted strong. Then there exists an unique vector space* $W \subseteq R_1$ *of minimum dimension such that* $G \in \mathbb{K}[W]$. *Moreover, we have* $\dim(W) \leqslant 2\mu_2(\delta_U + (0,1))$.

Proof. By [OS24, Proposition 5.14.1], we know that strength of G in R is upper bounded by its lifted strength, i.e. we have

$$s(G) \leqslant \tilde{s}_{\min}(U, G).$$

Since G is not μ-lifted strong, we know that there is a lift $\widetilde{G}$ such that $U' := \operatorname{span}_{\mathbb{K}}\left\{U + \widetilde{G}\right\}$ is not μ-strong. Now the degree 1 graded piece $U'_1 = \operatorname{span}_{\mathbb{K}}\left\{U + \widetilde{G}\right\} \cap R_1$ is μ-strong, hence we must have that $U'_2 = \operatorname{span}_{\mathbb{K}}\left\{U + \widetilde{G}\right\} \cap R_2$ is not $\mu_2(\delta_U + (0,1))$-strong. In particular, $\tilde{s}_{\min}(U, G) \leqslant \mu_2(\delta_U + (0,1))$, and hence $s(G) \leqslant \mu_2(\delta_U + (1,0))$. Let $r = \mu_2(\delta_U + (0,1))$ and $\Lambda' = \Lambda \circ t_{(0,1)}$. Then we are done by applying Proposition 3.25, as U is Λ'-strong and $\Lambda' \geqslant B \circ t_{(4r,0)}$. □

Proposition 3.27. *Let $\mu, \Gamma, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ be ascending functions such that $\Gamma \geqslant 2\mu$. Suppose that $U \subseteq S$ is Λ-strong and $R = S/(U)$. Let $V \subseteq R_{\leqslant 2}$ be a Γ-lifted strong vector space. Let $F \in R_2$. Up to scalar multiples, there exists at most one quadratic form $G \in \operatorname{span}_{\mathbb{K}}\{V, F\}$ such that $\operatorname{span}_{\mathbb{K}}\{G\}$ is not μ-lifted strong in R.*

Proof. If $V_2 = (0)$, then the only quadratic forms in $\operatorname{span}_{\mathbb{K}}\{V, F\}$ are scalar multiples of F. Hence we may assume that $V_2 \neq (0)$ and $\delta_V \geqslant (0, 1)$. Let $G_1, G_2 \in \operatorname{span}_{\mathbb{K}}\{V, F\}$ be two non-associate forms which are not μ-lifted strong in R. Then we have $\widetilde{s}_{\min}(U, G_i) < \mu_2(\delta_U + (0, 1))$. Let $G_1 = \alpha_1 H_1 + \beta_1 F$ and $G_2 = \alpha_2 H_2 + \beta_2 F$ for $H_1, H_2 \in V$. Since V is Γ-lifted strong and $\Gamma \geqslant \mu$, we must have $\beta_1, \beta_2 \neq 0$. Let $H := \beta_2 G_1 - \beta_1 G_2 \neq 0$. The lifted strength of H satisfies

$$\widetilde{s}_{\min}(U, H) \leqslant \widetilde{s}_{\min}(U, G_1) + \widetilde{s}_{\min}(U, G_1) < 2\mu_2(\delta_U + (0, 1))$$

Now $\Gamma \geqslant 2\mu$, hence $2\mu_2(\delta_U + (0, 1)) \leqslant \Gamma_2(\delta_U + \delta_V)$. Therefore, we have a contradiction as V is Γ-lifted strong and $H \in V$ is a non-zero form. $\square$

Definition 3.28. Let $R = S/(U)$. For a vector space $V \subseteq R_{\leqslant 2}$ and a form $F \in R_2$, we define the pseudo-distance of F from the algebra $\mathbb{K}[V]$ as

$$\text{ps-dist}(F, V) = \begin{cases} 1 + \min_Q\{s(F - Q) \mid Q \in \mathbb{K}[V] \cap R_2\} & \text{if } F \notin \mathbb{K}[V] \\ 0 & \text{if } F \in \mathbb{K}[V] \end{cases}$$

We say that F is *s-close* to the algebra $\mathbb{K}[V]$ if $\text{ps-dist}(F, V) < s$.

We note a few properties of the pseudo-distance.

Proposition 3.29. *Let $R = S/(U)$. Let $V \subseteq R_{\leqslant 2}$. We have the following.*

1. *Let $F \in R_2$. If $\text{ps-dist}(F, V) \leqslant s$, then there exist linear forms $\ell_1, \cdots, \ell_{2s} \in R_1$ such that $F \in \mathbb{K}[V, \ell_1, \cdots, \ell_{2s}]$.*
2. *For any $F_1, F_2 \in R_2$, we have*

$$\text{ps-dist}(F_1 + F_2, V) \leqslant \text{ps-dist}(F_1, V) + \text{ps-dist}(F_2, V).$$

3. *If $V' \subseteq R_{\leqslant 2}$ is such that $V \subseteq \mathbb{K}[V']$, then*

$$\text{ps-dist}(F, V') \leqslant \text{ps-dist}(F, V)$$

for all $F \in R_2$.

Proof. (1) There exists $Q \in \mathbb{K}[V]$ such that $s(F - Q) < s$. Hence there exist $\ell_1, \cdots, \ell_s \in R_1$ such that $F - Q \in (\ell_1, \cdots, \ell_s)$. Since $F - Q \in R_2$, there exist $\ell_{s+1}, \cdots, \ell_{2s}$ such that $F - Q \in \mathbb{K}[\ell_1, \cdots, \ell_{2s}]$. Hence $F = Q + (F - Q) \in \mathbb{K}[V, \ell_1, \cdots, \ell_{2s}]$.

(2) If $F_1 - Q_1 \in (\ell_1, \cdots, \ell_{s_1})$ and $F_2 - Q_2 \in (\ell'_1, \cdots, \ell'_{s_2})$ for some $Q_1, Q_2 \in \mathbb{K}[V]$, then $F_1 + F_2 - (Q_1 + Q_2) \in (\ell_1, \cdots, \ell_{s_1}, \ell'_1, \cdots, \ell'_{s_2})$. Hence $s(F_1 + F_2 - (Q_1 + Q_2)) \leqslant s_1 + s_2 - 1$. We are done by taking $s_1 = \text{ps-dist}(F_1, V)$ and $s_2 = \text{ps-dist}(F_2, V)$.

(3) Let $Q \in \mathbb{K}[V]$ such that $s(F - Q) = \text{ps-dist}(F, V) - 1$. Then $Q \in \mathbb{K}[V']$, as $V \subseteq \mathbb{K}[V']$. Hence $\text{ps-dist}(F, V') \leqslant s(F - Q) + 1 = \text{ps-dist}(F, V)$. $\square$

Proposition 3.30. *Let $\mu, \Gamma, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ be ascending functions. Let $R = S/(U)$, where $U \subseteq S_{\leqslant 2}$ is $\Lambda \circ t_{(0,1)}$-strong. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong. Let $F \in R_2$.*

1. *Suppose $\Gamma \geqslant \mu$. If $\mathrm{span}_{\mathbb{K}}\{V, F\}$ is not μ-lifted strong, then*

$$\text{ps-dist}(F, V) \leqslant \mu_2(\delta_U + \delta_V + (0,1)).$$

2. *If we have*

$$\text{ps-dist}(F, V) + \dim(V_1) + \dim(U_1) \leqslant \mu_2(\delta_U + \delta_V + (0,1)),$$

then $\mathrm{span}_{\mathbb{K}}\{V, F\}$ is not μ-lifted strong.

Proof. (1) Note that $F \notin V$ as $\Gamma \geqslant \mu$. Since V is $\Gamma \circ t_{(0,1)}$-lifted strong, we know that $\widetilde{s}_{\min}(U, G) \geqslant \mu_2(\delta_U + \delta_V + (0,1))$ for all $G \in V_2$. Since $\mathrm{span}_{\mathbb{K}}\{V, F\}$ is not μ-lifted strong, there exist lifts of a basis of V and F, given by $\widetilde{F_1}, \cdots, \widetilde{F_m}, \widetilde{F}$ and $\widetilde{Q} \in \mathrm{span}_{\mathbb{K}}\left\{\widetilde{F_1}, \cdots, \widetilde{F_m}\right\}$ such that $s_{\min}(U, \widetilde{F} - \widetilde{Q}) < \mu_2(\delta_U + \delta_V + (0,1))$. Let Q be the image of $\widetilde{Q}$ in the quotient ring $R = S/(U)$. Then we have $Q \in \mathbb{K}[V]$ and $s(F - Q) < \mu_2(\delta_U + \delta_V + (1,0))$.

(2) Let $Q \in \mathbb{K}[V]$ such that $s(F - Q) = \text{ps-dist}(F, V) - 1$. Let $P = F - Q$ and $r = \text{ps-dist}(F, V)$. Let us first choose appropriate lifts of all the forms involved as follows. Note that there exist $\ell_1, \cdots, \ell_{2r} \in R_1$ such that $P \in \mathbb{K}[\ell_1, \cdots, \ell_{2r}]$. Let $\widetilde{\ell_1}, \cdots, \widetilde{\ell_{2r}}$ be lifts in S. Then there exists $\widetilde{P} \in \mathbb{K}[\widetilde{\ell_1}, \cdots, \widetilde{\ell_{2r}}]$ such that $\widetilde{P}$ is a lift of P. Let $Q = G + H$, where $G \in \mathbb{K}[V_1]$ and $H \in V_2$. Now there exists a lift $\widetilde{G} \in S$ such that $s(\widetilde{G}) < \dim(V_1)$. Let $\widetilde{F}, \widetilde{H}$ be lifts of F, H respectively. Now we have $(\widetilde{F} - \widetilde{G} - \widetilde{H}) - \widetilde{P} \in (U)$. Therefore, there exists a form $H' \in U_2$ such that $(\widetilde{F} - \widetilde{G} - \widetilde{H}) - \widetilde{P} - H' \in (U_1)$. Hence, we have

$$s(\widetilde{F} - \widetilde{H} - H') < r + \dim(V_1) + \dim(U_1),$$

as $s(\widetilde{P}) < r$ and $s(\widetilde{G}) < \dim(V_1)$. Recall that $\widetilde{H}, \widetilde{F}$ are lifts of forms in $\mathrm{span}_{\mathbb{K}}\{V, F\}$. Since $\mu_2(\delta_U + \delta_V + (0,1)) \geqslant r + \dim(V_1) + \dim(U_1)$, we have that $\mathrm{span}_{\mathbb{K}}\{V, F\}$ is not μ-lifted strong. □

In Proposition 3.25, we saw that in a strong enough quotient ring, given a weak form, there is a unique vector space of linear forms of minimal dimension that contains it in it's algebra. Given a form F that is not weak, but is close (of low pseudo-distance) to some $V \subseteq R_{\leqslant 2}$, we can apply that proposition to the weak form $F - Q$ which shows the low pseudo-distance, resulting in a low-dimension addition to V_1 whose algebra contains F. The following proposition shows that and proves a similar uniqueness guarantee.

Proposition 3.31. *Let $\Gamma, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ be ascending functions. Let $R = S/(U)$, where $U \subseteq S_{\leqslant 2}$ is $\Lambda \circ t_{(0,1)}$-strong. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong. Let $F \in R_2$. Let $a, r \in \mathbb{N}$ such that $\text{ps-dist}(F, V) \leqslant r$ and $a \geqslant 2r + \dim(V_1)$. Suppose that $\Lambda_i(\delta_U) \geqslant B_i \circ t_{(2a,0)}(\delta_U)$ and $\Gamma_2(\delta_U + \delta_V) \geqslant a + \dim(U_1)$. Then there exists a vector space $W \subseteq R_1$ such that the following holds.*

1. *There exists $Q \in \mathbb{K}[V]$ such that $s(F - Q) = \text{ps-dist}(V, F) - 1$ and $F - Q \in \mathbb{K}[W]$.*

2. *We have $V_1 \subseteq W$.*

3. *We have $\dim(W) \leqslant 2\,\text{ps-dist}(F, V) + \dim(V_1)$.*

4. *Let $W' \subseteq R_1$ be a vector space such that $V_1 \subseteq W'$ and $\dim(W') \leqslant a$. Suppose that $F - Q' \in \mathbb{K}[W']$ for some $Q' \in \mathbb{K}[V]$ with $s(F - Q) < r$. Then we have $W \subseteq W'$.*

5. *A vector space W of minimum dimension that satisfies properties (1) and (2) above is unique. Moreover, if W is the unique vector space of minimum dimension satisfying (1) and (2), then for all $Q' \in \mathbb{K}[V]$ with $s(F - Q') = \text{ps-dist}(F, V) - 1$, we have $F - Q' \in \mathbb{K}[W]$.*

Proof. Let $s = \text{ps-dist}(F, V) \leqslant r$. So we have $a \geqslant 2s + \dim(V_1)$. Let $Q_1 \in \mathbb{K}[V]$ such that $s(F - Q_1) = \text{ps-dist}(V, F) - 1$. Since $\Lambda_i(\delta_U) \geqslant B \circ t_{(2a,0)}(\delta_U)$, we may apply Proposition 3.25 to the form $F - Q_1 \in R$ which is not s-strong. Let $W_1 \in \mathcal{W}_a(F - Q_1)$ be the unique vector space of minimal dimension provided by Proposition 3.25. We define $W := W_1 + V_1$. We will show that W satisfies the properties stated above.

Property (1) is satisfied by construction. We also have $V_1 \subseteq W$ by construction, so (2) is satisfied. Moreover, by Proposition 3.25, we have $\dim(W_1) \leqslant 2s$. Hence we have (3).

Now we will show property (4) for W as defined above. Let $W' \subseteq R_1$ be a vector space of $\dim(W') \leqslant a$ such that $V_1 \subseteq W'$. Let $Q_2 \in \mathbb{K}[V] \cap R_2$ be a form such that $s(F - Q_2) < r$ and $F - Q_2 \in \mathbb{K}[W']$. Let $W_2 \in \mathcal{W}_a(F - Q_2)$ be the unique vector space of minimal dimension given by applying Proposition 3.25 to $F - Q_2$. Then $W_2 \subseteq W'$ by Proposition 3.25. It is enough to show that $W_2 + V_1 = W$.

First, we will show that $Q_1 - Q_2 \in \mathbb{K}[V_1]$. Let $P := Q_1 - Q_2 \in \mathbb{K}[V]$. Note that $s(P) < s + r \leqslant 2r$. We may choose a lift $\widetilde{P} \in S$ such that $s(\widetilde{P}) \leqslant s(P) < 2r$. Let $P = G + H$ where $G \in \mathbb{K}[V_1]$ and $H \in V_2$. Now, let us show that $H = 0$. Suppose that $H \neq 0$. Then for any lift $\widetilde{H}$ in S, there exists $H' \in U_2$ such that $\widetilde{H} + \widetilde{G} - \widetilde{P} - H' \in (U_1)$. Hence we have $s(\widetilde{H} - H') < 2r + \dim(V_1) + \dim(U_1)$. This is a contradiction, since V is Γ-lifted strong and $\Gamma_2(\delta_U + \delta_V) \geqslant 2r + \dim(V_1) + \dim(U_1)$. Therefore we must have $Q_1 - Q_2 \in \mathbb{K}[V_1]$.

Now we have $F - Q_2 = (F - Q_1) + (Q_1 - Q_2) \in \mathbb{K}[W_1 + V_1] = \mathbb{K}[W]$. Since $\dim(W) \leqslant a$, we have that $W \in \mathcal{W}_q(F - Q_2)$. Therefore by Proposition 3.25, we have $W_2 \subseteq W$. Similarly, we have $F - Q_1 = (F - Q_2) + (Q_2 - Q_1) \in \mathbb{K}[W_2 + V_1]$. Now $\dim(W_2 + V_1) \leqslant a$ and hence $W_2 + V_1 \in \mathcal{W}_a(F - Q_1)$. Again, by minimality of W_1, we have that $W = W_1 + V_1 \subseteq W_2 + V_1$. Hence $W = W_2 + V_1 \subseteq W'$ as desired.

If W' is of minimal dimension satisfying property (1) and (2), then $\dim(W') \leqslant \dim(W)$. Hence we must have $W' = W$, as $W \subseteq W'$ by (4) above. Moreover, let $Q_2 \in \mathbb{K}[V] \cap R_2$ be a form such that $s(F - Q_2) = \text{ps-dist}(F, V) - 1 = s - 1$ and $F - Q_2 \in \mathbb{K}[W']$. Let $W_2 \in \mathcal{W}_a(F - Q_2)$ be the unique vector space of minimal dimension given by applying Proposition 3.25 to $F - Q_2$. Then the argument above shows that $W_2 + V_1 = W$. Hence $F - Q_2 \in \mathbb{K}[W]$. ☐

We now define the notion of "relative space of essential variables" in quotient rings, generalizing the notions from [Shp20, PS20, PS21, PS22, GOS22, GOPS23].

Definition 3.32 (Relative space of essential variables)**.** Let $\Gamma, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ be ascending functions. Let $R = S/(U)$, where $U \subseteq S_{\leqslant 2}$ is $\Lambda \circ t_{(0,1)}$-strong. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong. Suppose that $\Lambda_i(\delta_U) \geqslant B_i \circ t_{(2a,0)}(\delta_U)$ and $\Gamma_2(\delta_U + \delta_V) \geqslant a + \dim(U_1)$ for some $a \in \mathbb{N}$. We define the space of essential variables separately for quadratic forms and linear forms as follows.

Quadratics. For a quadratic form $F \in R_2$ such that ps-dist(V, F) satisfies

$$2 \cdot \text{ps-dist}(F, V) + \dim(V_1) \leqslant a,$$

we will define the space of essential variables of F relative to V in R and we denote it by $\mathbb{L}_{R,V}(a, F)$.

Let $W \subseteq R_1$ be the unique vector space of minimal dimension provided by applying Proposition 3.31. We will denote

$$\mathbb{L}_R(a, V, F) := W.$$

Recall that $V_1 \subseteq W$. We define the relative space of essential variables $\mathbb{L}_{R,V}(a, F)$ as the vector space

$$\mathbb{L}_{R,V}(a, F) = \mathbb{L}_R(a, V, F)/V_1.$$

Linear forms. For a linear form $F \in R_1$, we define the relative spaces as follows:

$$\mathbb{L}_R(a, V, F) = V_1 + \text{span}_{\mathbb{K}}\{F\},$$

and

$$\mathbb{L}_{R,V}(a, F) = \mathbb{L}_R(a, V, F)/V_1.$$

We will omit R from the notation and write $\mathbb{L}(a, V, F)$ and $\mathbb{L}_V(a, F)$, when the ambient ring is evident from the context.

Remark 3.33. A key difference between the definition above and its polynomial ring version $\text{Lin}(F)$ in [Shp20] is the use of a cut-off on pseudo-distance. This cut-off is more in line with [GOS22, Definition 2.3]. We only define the relative space of essential variables $\mathbb{L}_{R,V}(a, F)$ for forms F which have pseudo-distance in terms of V and the cut-off a. When $U = (0)$, i.e. $R = S$ is a polynomial ring, and $V = (0)$, we may conceptually think of the space $\text{Lin}(F)$ (defined in [Shp20]), as the relative space $\text{Lin}_{S,(0)}(a, F)$, where a is ∞. In the latter setting, the relative space is exactly the generalization of [GOS22, Definition 2.3] when replacing 5 by a.

We note the following properties of the relative space of essential variables.

Proposition 3.34. *Let $\Gamma, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ be ascending functions. Let $R = S/(U)$, where $U \subseteq S_{\leqslant 2}$ is $\Lambda \circ t_{(0,1)}$-strong. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong. Let $F \in R_2$. Let $a \in \mathbb{N}$ such that $a \geqslant 2 \cdot \text{ps-dist}(F, V) + \dim(V_1)$. Suppose that $\Lambda_i(\delta_U) \geqslant B_i \circ t_{(2a,0)}(\delta_U)$ and $\Gamma_2(\delta_U + \delta_V) \geqslant a + \dim(U_1)$. The following holds.*

1. *We have $\text{span}_{\mathbb{K}}\{V, F\} \subseteq \mathbb{K}[V_2, \mathbb{L}_R(a, V, F)]$ and*

$$\dim(\mathbb{L}_{R,V}(a, F)) \leqslant \dim(\mathbb{L}_R(a, V, F)) \leqslant a.$$

2. *Let V' be a $\Gamma \circ t_{(0,1)}$-lifted strong vector space such that $V \subseteq \mathbb{K}[V']$. Suppose that $a \geqslant 2 \cdot \text{ps-dist}(F, V') + \dim(V'_1)$. Then we have*

$$\dim(\mathbb{L}_{R,V'}(a, F)) \leqslant \dim(\mathbb{L}_{R,V}(a, F)).$$

Proof. (1) We have $V_1 \subseteq \mathbb{L}_R(a, V, F)$ by Proposition 3.31. Hence $\mathbb{K}[V, \mathbb{L}_R(a, V, F)] = \mathbb{K}[V_2, \mathbb{L}_R(a, V, F)]$. Moreover, there exists $Q \in \mathbb{K}[V]$ such that $F - Q \in \mathbb{K}[\mathbb{L}_r(a, V, F)]$. Hence $F \in \mathbb{K}[V_2, \mathbb{L}_R(a, V, F)]$. Moreover, we have the dimension upper bound by part (3) of Proposition 3.31.

(2) First let us note that $\mathbb{L}_R(a, V', F)$ is well-defined. Since Γ is an ascending function, we have $\Gamma_2(\delta_U + \delta_{V'}) \geqslant \Gamma_2(\delta_U + \delta_V) \geqslant a + \dim(U_1)$. Hence the conditions in Definition 3.32 are satisfied. Let $v_1, \cdots, v_m$ be a basis of V_1 and $v_1, \cdots, v_m, w_1, \cdots, w_n$ be an extension to a basis of $\mathbb{L}_R(a, V, F)$. We have $n = \dim(\mathbb{L}_{R,V}(a, F))$. Now there exists $Q \in \mathbb{K}[V]$ such that $F - Q \in \mathbb{K}[\mathbb{L}_R(a, V, F)]$ and $s(F - Q) = \text{ps-dist}(F, V) - 1$. Consider $W' := V'_1 + \text{span}_{\mathbb{K}}\{w_1, \cdots, w_n\}$. We have $\dim(W') \leqslant a$ and $V'_1 \subseteq W'$. Moreover, we have $F - Q \in \mathbb{K}[W']$ and $s(F - Q) = \text{ps-dist}(F, V) - 1$. Set $r := \text{ps-dist}(F, V)$. Then we have $\text{ps-dist}(F, V') \leqslant r$ by Proposition 3.29.

Hence we can apply Proposition 3.31 to the vector space V' and the form F to obtain that $\mathbb{L}_R(a, V', F) \subseteq W'$. Hence $\mathbb{L}_{R,V'}(a, F) \subseteq W'/V'_1$ and

$$\dim(\mathbb{L}_{R,V'}(a, F)) \leqslant \dim(W'/V'_1) \leqslant n = \dim(\mathbb{L}_{R,V}(a, F)). \qquad \square$$

Definition 3.35 (Lin-separated forms)**.** Let $\Gamma, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ be ascending functions. Let $R = S/(U)$, where $U \subseteq S_{\leqslant 2}$ is $\Lambda \circ t_{(0,1)}$-strong. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong. Let $F \in R_2$. Let $a \in \mathbb{N}$. Suppose that $\Lambda_i(\delta_U) \geqslant B \circ t_{(2a,0)}(\delta_U)$ and $\Gamma_2(\delta_U + \delta_V) \geqslant a + \dim(U_1)$.

Let $\mathcal{F} \subseteq R_{\leqslant 2}$ be a finite set of forms such that for any $F \in \mathcal{F} \cap R_2$ we have $2 \cdot \text{ps-dist}(F, V) + \dim(V_1) \leqslant a$. We will say that $\mathcal{F}$ is *lin-separated* modulo V if $\mathcal{F} \cap \mathbb{K}[V] = \emptyset$ and

$$\dim(\sum_{F \in \mathcal{F}} \mathbb{L}_{R,V}(a, F)) = \sum_{F \in \mathcal{F}} \dim(\mathbb{L}_{R,V}(a, F)).$$

Proposition 3.36. *Let $\Gamma, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ be ascending functions. Let $R = S/(U)$, where $U \subseteq S_{\leqslant 2}$ is $\Lambda \circ t_{(0,1)}$-strong. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong. Let $F \in R_2$. Let $a \in \mathbb{N}$. Suppose that $\Lambda_i(\delta_U) \geqslant B_i \circ t_{(2a,0)}(\delta_U)$ and $\Gamma_2(\delta_U + \delta_V) \geqslant a + \dim(U_1)$. The following holds.*

1. *Let $\mathcal{F} \subseteq R_{\leqslant 2}$ be a finite set of forms that is lin-separated modulo V. Then any subset $\mathcal{G} \subseteq \mathcal{F}$ is also lin-separated modulo V.*
2. *Let $F, G \in R_{\leqslant 2} \setminus \mathbb{K}[V]$ be such $\mathbb{L}_R(a, V, F)$ and $\mathbb{L}_R(a, V, G)$ are well-defined. Then F, G are lin-separated modulo V iff $\mathbb{L}_R(a, V, F) \cap \mathbb{L}_R(a, V, G) = V_1$.*

Proof. (1) Let $\mathcal{F} = \{F_1, \cdots, F_m\}$. Without loss of generality, we may assume that $\mathcal{G} = \{F_1, \cdots, F_n\}$ for some $n \leqslant m$. Suppose that $\mathcal{G}$ is not lin-separated modulo V. Then we have

$$\dim(\sum_{i=1}^{n} \mathbb{L}_{R,V}(a, F_i)) < \sum_{i=1}^{n} \dim(\mathbb{L}_{R,V}(a, F_i)).$$

Therefore, we have

$$\dim(\sum_{i=1}^{m} \mathbb{L}_{R,V}(a, F_i)) \leqslant \dim(\sum_{i=1}^{n} \mathbb{L}_{R,V}(a, F_i)) + \dim(\sum_{i=n+1}^{m} \mathbb{L}_{R,V}(a, F_i)) < \sum_{i=1}^{m} \dim(\mathbb{L}_{R,V}(a, F_i)).$$

This is a contradiction as $\mathcal{F}$ is lin-separated modulo V.

(2) Let $W_1 = \mathbb{L}_{R,V}(a, F)$ and $W_2 = \mathbb{L}_{R,V}(a, G)$. We have $\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2)$ iff $W_1 \cap W_2 = (0)$ in R_1/V_1. By lifting back to R_1, we obtain that F, G are lin-separated modulo V iff $\mathbb{L}_R(a, V, F) \cap \mathbb{L}_R(a, V, G) = V_1$.

$\square$

We note the following property of lin-separated forms under quotients.

Lemma 3.37. *Let $\Gamma, \Lambda : \mathbb{N}^2 \to \mathbb{N}^2$ be ascending functions. Let $R = S/(U)$, where $U \subseteq S_{\leqslant 2}$ is $\Lambda \circ t_{(0,1)}$-strong. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong. Let $a \in \mathbb{N}$. Suppose that $\Lambda_i(\delta_U) \geqslant B_i \circ t_{(2a,0)}(\delta_U)$ and $\Gamma_i(\delta_U + \delta_V) \geqslant B_i \circ t_{(2a,0)}(\delta_U + \delta_V)$.*

Let $F_1, F_2 \in R_2$ *be such that* $F_1, F_2 \notin (V)$*. Suppose that* $2\,\text{ps-dist}(F_i, V) + \dim(V_1) \leqslant a$*, i.e.* $\mathbb{L}_{R,V}(a, F_i)$ *is well-defined. Consider the quotient ring* $R' = R/(V)$ *and let* F_1', F_2' *be the images of* F_1, F_2 *in* R'*. Then the following holds.*

1. *The relative spaces of essential variables* $\mathbb{L}_{R',(0)}(a, F_i')$ *are well-defined.*
2. *In the vector space* $R_1' \simeq R_1/V_1$*, we have* $\mathbb{L}_{R',(0)}(a, F_i') \subseteq \mathbb{L}_{R,V}(a, F_i')$*.*
3. *Suppose that* F_1, F_2 *are lin-separated modulo* V *in* R*, then* F_1', F_2' *are lin-separated modulo* (0) *in* R'*.*
4. *Suppose that* F_1, F_2 *are lin-separated modulo* V*. Let* $P = F_1 + F_2 + H \in R$ *where* $H \in \mathbb{K}[V]$ *such that* $\mathbb{L}_{R,V}(a, P)$ *is well-defined. Let* $P' \in R'$ *be the image of* P*. Then we have* $\mathbb{L}_{R',(0)}(a, P') = \mathbb{L}_{R',(0)}(a, F_1') + \mathbb{L}_{R',(0)}(a, F_2')$*. Moreover, we have*

$$\dim(\mathbb{L}_{R',(0)}(a, P')) = \dim(\mathbb{L}_{R',(0)}(a, F_1')) + \dim(\mathbb{L}_{R,V}(a, F_2')).$$

Proof. (1) Let us check that the forms F_i', the vector space $V' = (0)$ and the ring R' satisfy the conditions in Definition 3.32. Let $\widetilde{V} \subseteq S$ be the vector space generated by lifts of a basis of V. Then $R' = S/(U + \widetilde{V})$. Now note that $U + \widetilde{V}$ is $\Gamma \circ t_{(0,1)}$-strong, as V is $\Gamma \circ t_{(0,1)}$-lifted strong. By assumption, we have $\Gamma_i(\delta_U + \delta_V) \geqslant B \circ t_{(2a,0)}(\delta_U + \delta_V)$. Now, $B \circ t_{(2a,0)}(\delta_U + \delta_V) \geqslant a + \dim(U_1) + \dim(V_1)$. Moreover, we have $\text{ps-dist}(F_i', (0)) \leqslant \text{ps-dist}(F, V)$. Hence in the quotient ring R' we have $\text{ps-dist}(F_i', (0)) \leqslant a$. Hence the spaces $\mathbb{L}_{R',(0)}(a, F_i')$ are well-defined.

(2) Suppose $Q_i \in \mathbb{K}[V]$ is such that $s(F_i - Q_i) = \text{ps-dist}(F_i, V) - 1$. Let $W_i \in \mathcal{W}_a(F_i - Q_i)$ be the unique vector space of minimal dimension such that $F_i - Q_i \in \mathbb{K}[W_i]$. Recall that $\mathbb{L}_R(a, V, F_i) = W_i + V_1$ and $\mathbb{L}_{R,V}(a, F_i) = (W_i + V_1)/V_1$. Let $\phi : R_1 \to R_1' = R_1/V_1$ be the quotient map. Let $G_i = F_i - Q_i$ and $G_i' \in R'$ be the image. Then $G_i' = F_i'$ in R'. Moreover, $\phi(\mathbb{L}_R(a, V, F_i)) = \mathbb{L}_{R,V}(a, F_i) \subseteq R_1/V_1 = R_1'$ is a vector space of linear forms with dimension at most a and such that $F_i' \in \mathbb{K}[\phi(\mathbb{L}_{R,V}(a, F_i))]$. Therefore, by minimality, we have $\mathbb{L}_{R',(0)}(a, F_i') \subseteq \mathbb{L}_{R,V}(a, F_i)$.

(3) Follows from part (2). Indeed, we have $\mathbb{L}_{R',(0)}(a, F_1) \cap \mathbb{L}_{R',(0)}(F_2) \subseteq \mathbb{L}_{R,V}(a, F_1) \cap \mathbb{L}_{R,V}(F_2) = (0)$.

(4) Note that $F_1', F_2' \neq 0$ as $F_1, F_2 \notin (V)$. Let P' be the image of P in R'. Note that $P' = F_1' + F_2'$. Recall that $\dim(\mathbb{L}_{R'}(a, F_i')) \leqslant a$ and $\mathbb{L}_{R'}(a, F_1') \cap \mathbb{L}_{R'}(a, F_2') = (0)$. Let $x_1, \cdots, x_m \in R_1'$ be a basis of $\mathbb{L}_{R'}(a, F_1')$. Similarly, let $y_1, \cdots, y_n \in R_1'$ be a basis of $\mathbb{L}_{R'}(a, F_2')$. Since $\Gamma_i(\delta_U + \delta_V) \geqslant B_i \circ t_{(2a,0)}(\delta_U + \delta_V)$, we have that $x_1, \cdots, x_m, y_1, \cdots, y_n \in R'$ is a prime sequence. Therefore the subalgebra $A = \mathbb{K}[x_1, \cdots, x_m, y_1, \cdots, y_n] \subseteq R'$ is isomorphic to a polynomial ring in $m + n$ variables. Now $P' = F - 1' + F_2'$ and $F_1' \in \mathbb{K}[x_1, \cdots, x_m]$ and $F_2' \in \mathbb{K}[y_1, \cdots, y_n]$. Therefore, by applying [PS20, Fact 2.17] repeatedly, we obtain that $\mathbb{L}_{R'}(a, P') = \mathbb{L}_{R'}(a, F_1') + \mathbb{L}_{R'}(a, F_2')$. Moreover, the dimensions add up as $\mathbb{L}_{R'}(a, F_1') \cap \mathbb{L}_{R'}(a, F_2') = (0)$ □

The following result shows that, under suitable conditions, the relative space of essential variables $\mathbb{L}_{R,V}(a, V, F)$ is well-defined if $\text{span}_{\mathbb{K}}\{V, F\}$ is not μ-lifted strong.

Corollary 3.38. *Let* $\mu, \Gamma, \Lambda, f : \mathbb{N}^2 \to \mathbb{N}^2$ *be ascending functions. Suppose that the functions* μ, Γ, Λ, f *satisfy the following conditions:*

1. $\Gamma_i(\delta) \geqslant 2\mu_i \circ t_{(0,1)}(\delta) + \delta_1$ *for all* $\delta \in \mathbb{N}$*, and* $i = 1, 2$*.*
2. $\Lambda \geqslant B \circ t_{(2\mu_2 + f_1, 0)} \circ t_f \circ t_{(0,1)}$*.*

Let $R = S/(U)$, *where* $U \subseteq S_{\leqslant 2}$ *is* $\Lambda \circ t_{(0,1)}$*-strong. Let* $V \subseteq R_{\leqslant 2}$ *be* $\Gamma \circ t_{(0,1)}$*-lifted strong such that* $\dim(V_i) < f_i(\delta_U)$. *Let* $F \in R_2$ *be such that* $\operatorname{span}_{\mathbb{K}}\{V, F\}$ *is not* μ*-lifted strong. Then the spaces* $\mathbb{L}_R(a, V, F)$ *and* $\mathbb{L}_{R,V}(a, F)$ *are well-defined for* $a = 2\mu_2(\delta_U + \delta_V + (0, 1)) + f_1(\delta_U)$.

Proof. We have $\Gamma \geqslant \mu$. Hence by Proposition 3.30, we have

$$\text{ps-dist}(V, F) \leqslant 2\mu_2(\delta_U + \delta_V + (0, 1)).$$

Therefore $a = 2\mu_2(\delta_U + \delta_V + (0, 1)) + f_1(\delta_U) \geqslant 2\,\text{ps-dist}(V, F) + \dim(V_1)$. Now we have

$$\Lambda_i(\delta_U) \geqslant B_i \circ t_{(2\mu_2+f_1,0)} \circ t_f \circ t_{(0,1)}(\delta_U) \geqslant B \circ t_{(2a,0)}(\delta_U).$$

Moreover, we have $\Gamma_2(\delta_U + \delta_V) \geqslant 2\mu_2(\delta_U + \delta_V + (0, 1)) + \dim(U_1) + \dim(V_1) \geqslant a + \dim(U_1)$. Hence Proposition 3.31 and Definition 3.32 apply and the spaces $\mathbb{L}_R(a, V, F)$ and $\mathbb{L}_{R,V}(a, F)$ are well-defined. ☐

3.5 Graded, general and targeted quotients

Throughout this section, we fix positive integers $d, \eta \in \mathbb{N}$ with $\eta \geqslant 3$. Additionally, $B : \mathbb{N}^d \to \mathbb{N}^d$ denotes an ascending function such that $B_i(\delta) \geqslant A(\eta, i) + 3(\sum_i \delta_i - 1)$ for all $i \in [d]$. Here $A(\eta, i)$ is the function defined in Corollary 3.13. Before we define our quotients, we remark what we mean by general points.

General points. Let $X \subseteq \mathbb{K}^m$ be an algebraic variety. We say that a property $\mathcal{P}$ holds for a *general* $\alpha \in X$, if there exists a non-empty open subset $\mathcal{U} \subseteq X$ such that the property $\mathcal{P}$ holds for all $\alpha \in \mathcal{U}$. Here $\mathcal{U} \subseteq X$ is open with respect to the Zariski topology. Hence $\mathcal{U}$ is the complement of the zero set of finitely many polynomial functions on $\mathbb{K}^m$. Note that, equivalently a property $\mathcal{P}$ holds for a general $\alpha \in X$, if there is a closed subset $\mathcal{Z} \subseteq X$ such that the $\mathcal{P}$ holds for all $\alpha \notin \mathcal{Z}$. When $X = \mathbb{K}^m$, we simply say that α is general (without mentioning X).

We recall the definitions of graded and general quotients from [OS24].

Definition 3.39 (Graded and General Quotients)**.** Let $U = \oplus_{i=1}^d U_i \subset S$ be a graded vector space of dimension sequence δ_U and $R := S/(U)$ be the quotient ring. Let $V = \oplus_{i=1}^d V_i \subset R$ be a graded subspace of dimension sequence μ.

Let $F_1, \dots, F_n$ be a homogeneous basis for V and z be a new variable. For $\alpha \in \mathbb{K}^n$, let $I_\alpha \subseteq R[z]$ be the homogeneous ideal generated by the forms $\{F_1 - \alpha_1 z^{\deg(F_1)}, \dots, F_n - \alpha_n z^{\deg(F_n)}\}$. We define the *graded quotient* map $\varphi_{V,\alpha}$ as the quotient homomorphism of finitely generated graded $\mathbb{K}$-algebras given by

$$\varphi_{V,\alpha} : R[z] \to R[z]/I_\alpha.$$

We say that $\varphi_{V,\alpha}$ is a *general quotient* if α is chosen to be general.

The next proposition and lemma correspond to [OS24, Proposition 6.3] and [OS24, Lemma 6.4].

Proposition 3.40. *Suppose* $V \subset R$ *is* B*-lifted strong with respect to* U*. Then* $R[z]$ *and* $R[z]/I_\alpha$ *are quotients of* $S[z]$ *by* $\mathcal{R}_\eta$*-sequences, for any choice of* $\alpha \in \mathbb{K}^n$*. In particular, they are Cohen-Macaulay UFDs.*

The next proposition, corresponding to [OS24, Proposition 6.9], tells us that if a finite set of forms resulting from a general quotient has small vector space dimension, then it must be the case that the original set of forms has small vector space dimension. We refer to this result as "lifting from general quotients," as we are lifting our upper bounds for the quotiented configurations to the original configurations.

Proposition 3.41 (Lifting from general quotients)**.** *Let $d, e \in \mathbb{N}$ such that $1 \leqslant d \leqslant e$. Let $U \subset S_{\leqslant e}$ be a graded vector space generated by forms $H_1, \cdots, H_t$. Let $R = S/(U)$. Let $V \subset R_{\leqslant e}$ be a B-lifted strong vector space with basis $F_1, \cdots, F_n \in R$. Let $\varphi_\alpha : R[z] \to R[z]/I_\alpha$ be a graded quotient as defined in Definition 3.39. Let $\mathcal{F} \subset R_{\leqslant d}$ be a finite set of homogeneous elements. Suppose that there exists $D \in \mathbb{N}$ and a dense set $\mathcal{U} \subset \mathbb{K}^n$ such that $\dim \operatorname{span}_{\mathbb{K}} \{\varphi_\alpha(\mathcal{F})\} \leqslant D$ for every $\alpha \in \mathcal{U}$. Then*

$$\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant d^2(1+d)^{2n+2} D \cdot \Pi_{i=1}^{t} \deg(H_i) \cdot \Pi_{j=1}^{n} \deg(F_j).$$

While the statement of [OS24, Proposition 6.9] requires that the set of α for which the rank bound holds is an open set, the proof only requires the weaker condition that the set of such α is dense. Indeed the proof just requires that no closed set contains the set of all α for which the bound holds. The fact that this weaker condition suffices will be crucial for us.

We will also consider graded quotients where α is general zero of a polynomial F. The next proposition shows that such projections still preserve various properties of forms in S.

Proposition 3.42 (Targeted Quotients)**.** *Let $S = \mathbb{K}[x_1, \cdots, x_n, y_1, \cdots, y_m]$ and z be a new variable. Fix positive integers $d_1, \cdots, d_n \in \mathbb{N}$. Let $F \in S$ be an irreducible homogeneous polynomial such that $F \in \mathbb{K}[x_1, \cdots, x_n]$. For $\alpha \in \mathbb{K}^n$, let $I_\alpha = (x_1 - \alpha_1 z^{d_1}, \cdots, x_n - \alpha_n z^{d_n})$. Let $\varphi_\alpha : S[z] \to S[z]/I_\alpha$ be the quotient ring homomorphism. Then we have the following.*

1. *Let $Q \in S$ such that $Q \notin (F)$. Then $\varphi_\alpha(Q) \neq 0$ for general $\alpha \in \mathcal{Z}(F)$, where $\mathcal{Z}(F) \subseteq \mathbb{K}^n$ denotes the set of zeroes of F.*
2. *Let $Q, H \in S$ be forms of positive degree in at least one of the variables y_i. Suppose that for a dense set of $\alpha \in \mathcal{Z}(F)$, the forms $\varphi_\alpha(Q), \varphi_\alpha(H)$ have a common factor which is not in $\mathbb{K}[z]$. Then F divides the resultant $\operatorname{Res}_{y_i}(Q, H)$ for some y_i.*

Proof. (1) Let $Q = \sum Q_{\mathbf{e}} y^{\mathbf{e}}$, where $\mathbf{e} = (e_1, \cdots, e_m) \in \mathbb{N}^m$, $y^{\mathbf{e}} = y_1^{e_1} \cdots y_m^{e_m}$ and $Q_{\mathbf{e}} \in \mathbb{K}[x_1, \cdots, x_n]$. Now, $\varphi_\alpha(Q) = 0$ iff $\varphi_\alpha(Q_{\mathbf{e}}) = 0$ for all $\mathbf{e}$. Therefore we may assume that $Q \in \mathbb{K}[x_1, \cdots, x_n]$. Now $\varphi_\alpha(Q) = Q(\alpha_1 z^{d_1}, \cdots, \alpha_n z^{d_n})$ is a polynomial in $\mathbb{K}[z] \subseteq S[z]/I_\alpha$. If $\varphi_\alpha(Q) = 0$, then we may substitute $z = 1$, and obtain $Q(\alpha_1, \cdots, \alpha_n) = 0$. If $Q \notin (F)$, then we have $\mathcal{Z}(F) \setminus \mathcal{Z}(Q)$ is a non-empty open subset of $\mathcal{Z}(F)$. Moreover, $\varphi_\alpha(Q) \neq 0$ for all $\alpha \in \mathcal{Z}(F) \setminus \mathcal{Z}(Q)$, i.e. for a general $\alpha \in \mathcal{Z}(F)$.

(2) If F divides Q or H, then F divides $\operatorname{Res}_{y_i}(Q, H)$ for any y_i such that at least one of Q, H are of positive degree in y_i. So we may assume that $Q, H \notin (F)$. In particular, $\varphi_\alpha(Q), \varphi_\alpha(H) \neq 0$ for a general $\alpha \in \mathcal{Z}(F)$.

For a fixed α, any common factor of $\varphi_\alpha(Q), \varphi_\alpha(H)$, which is not in $\mathbb{K}[z]$, must be of positive degree in at least one of the y_i. For $i \in [m]$, let $\mathcal{T}_i$ be the set of $\alpha \in \mathcal{Z}(F)$ such that $\varphi_\alpha(Q), \varphi_\alpha(H)$ have a common factor of positive degree in the variable y_i. Now $\cup_i \mathcal{T}_i$ contains a dense subset of $\mathcal{Z}(F)$. Since there are finitely many sets $\mathcal{T}_i$, at least one of the sets $\mathcal{T}_i$ must be dense in $\mathcal{Z}(F)$. In particular, we may assume that $\varphi_\alpha(Q), \varphi_\alpha(H)$ have a common factor of positive degree in the same variable y_i, for all α in a dense subset $\mathcal{T} \subseteq \mathcal{Z}(F)$. Therefore, we have $\operatorname{Res}_{y_i}(\varphi_\alpha(Q), \varphi_\alpha(H)) = 0$ for $\alpha \in \mathcal{T}$.

Now let $Q_{\mathbf{d}}$ and $H_{\mathbf{e}}$ be the leading coefficients of Q, H when expressed as a polynomial in y_i. If $\varphi_\alpha(Q_{\mathbf{d}}) = \varphi_\alpha(H_{\mathbf{e}}) = 0$ for a dense set of $\alpha \in \mathcal{Z}(F)$, then F divides both $Q_{\mathbf{d}}$ and $H_{\mathbf{e}}$ by part (1). In particular, F divides

$\mathrm{Res}_{y_i}(Q, H)$. Therefore, we may assume that $\varphi_\alpha(Q_{\mathbf{d}}), \varphi_\alpha(H_{\mathbf{e}}) \neq 0$ for all α in a dense subset $\mathcal{T} \subseteq \mathcal{Z}(F)$. Since the leading coefficients remain non-zero under the homomorphism φ_α, we have $\varphi_\alpha(\mathrm{Res}_{y_i}(Q, H)) = \mathrm{Res}_{y_i}(\varphi_\alpha(Q), \varphi_\alpha(H)) = 0$ for all $\alpha \in \mathcal{T}$. By part (1), we must have that F divides $\mathrm{Res}_{y_i}(Q, H)$. □

3.6 Absolute irreducibility in strong quotient rings

The following definition, from [GOS25b] will be useful in later sections. This generalizes the definition of absolute irreducibility in polynomial rings given by Definition 2.6. Here we will generalize this notion to strong quotient rings $R = S/(U)$.

Definition 3.43. Let $B : \mathbb{N}^d \to \mathbb{N}^d$. Suppose $B_i(\delta) \geqslant A(\eta, i) + 3(\sum_i \delta_i - 1)$ for some $\eta \in \mathbb{N}$. Suppose $R = S/(U)$. Suppose $V \subset R$ is a graded vector space that is $h_{2B} \circ t_1$-lifted strong with respect to U. Suppose $P \in R$ is a form. Let V' be the vector space obtained by applying [GOS25a, Corollary 4.11] to V and P. Let $y_1, \ldots, y_a$ be a basis of homogeneous forms of V, and $y_{a+1}, \ldots, y_b$ extend this to a basis of V'. We say P is absolutely reducible over V if P is absolutely reducible as a polynomial in $\mathbb{K}(y_1, \ldots, y_a)[y_{a+1}, \ldots, y_b]$.

4 Reduction to low strength case

In this section, we prove the following result.

Lemma 4.1. *Suppose $\mu_{k-1} : \mathbb{N}^2 \to \mathbb{N}^2$ and $\xi_{k-1} : \mathbb{N}^2 \to \mathbb{N}$ are fixed functions. Define the function $\xi_k^1 : \mathbb{N}^2 \to \mathbb{N}$ as $\xi_k^1(\delta) := \xi_{k-1}\left(\left(C_{\mu_{k-1}} \circ t_1\right)(\delta)\right) + 1$. Let $\mu_k^{far} := h_{\mu_{k-1}} \circ t_1$. Here, $C_{\mu_{k-1}}, h_{\mu_{k-1}}$ are the functions defined in Lemma 3.15. Define the function $M(\delta) : \mathbb{N}^2 \to \mathbb{N}$ as*

$$M(\delta) := \max_{\delta' \in \mathbb{N}^2, \|\delta'\| = \binom{k}{2}\xi_k^1(\delta)} \mu_{k-1}(\delta + \delta' + (0, k^2 \xi_k^1(\delta))).$$

For every vector space U that is μ_k^{far}-strong, and every $\Sigma^k\Pi\Sigma\Pi^2$ simple minimal identity $\mathcal{C} = \sum_{i=1}^k T_i$ in $R := S/(U)$, one of the following holds:

1. *for some i, j, the symmetric difference $T_i \Delta T_j$ satisfies $\dim \mathrm{span}_{\mathbb{K}}\{T_i \Delta T_j\} \leqslant k^2 \xi_k^1(\delta_U)$.*
2. *there is a form $H \in \mathcal{C}$ that is μ_{k-1}-lifted strong, such that the image of $\mathcal{C}$ in $R/(H)$ has pairwise rank at least $\xi_{k-1}(\delta_U + (0, 1))$.*
3. *let $b := k^2\left(\xi_k^1(\delta_U) + 1\right)$, there exists a graded vector space V generated by forms in $\mathcal{C}$ with $\dim V \leqslant 4^{b+1} + b^3 + b + 48$ such that every quadratic form $Q \in \mathcal{C}$ is $3^b M(\delta)$-close to V.*

Let us explain the intuitive meaning of the different cases. It is good to recall the outline of the Kayal-Saraf proof as explained in Section 1.3. Case 1 corresponds to the situation when two terms in the circuit are “close” to each other and can be merged. In the depth-3 case, this is done by taking a general projection of the linear functions in $T_i \Delta T_j$. Case 2 is when there is a function that by “zeroing it” we maintain a circuit with smaller top fan-in whose rank violates the inductive hypothesis. Finally, Case 3 tells us that there is a small (i.e., constant dimension) space of forms, such quotient the circuit according to it, then all the forms become of low rank. That is, we reduce to the case where all the forms are of low rank.

The rest of this section is devoted to the proof of this lemma. Fix a dimension sequence $\delta \in \mathbb{N}^2$. Let U be a vector space with dimension sequence $\delta_U := \delta$ that is μ_k^{far}-strong. Let $\mathcal{C}$ be any simple and minimal $\Sigma^k\Pi\Sigma\Pi^2$ identity in $R := S/(U)$

We consider all vector spaces U' obtained by adding a single form to U, and making the result vector space μ_{k-1}-strong using Lemma 3.15 while preserving U. Such spaces exist since $\mu_k^{\text{far}} \geqslant h_{\mu_{k-1}} \circ t_1$. Given δ_U, we know by Lemma 3.15 that the dimension sequence of any U' obtained above is bounded coordinate wise by $(C_{\mu_{k-1}} \circ t_1)(\delta_U)$. By definition, $\xi_k^1(\delta_U)$ is greater than $\xi_{k-1}(\delta_{U'})$ for any such U'.

Constructing a core

Start by picking a core set of forms as follows. Go over the $\binom{k}{2}$ many symmetric differences $T_i \Delta T_j$, and pick $\xi_k^1(\delta_U)$ many forms from each symmetric difference, such that the set of all forms picked is linearly independent. For each symmetric difference, we first attempt to pick quadratic forms, and only pick linear forms if we already have a set that spans the quadratics in $T_i \Delta T_j$. If we cannot pick such a core, then some symmetric difference must satisfy $\dim \operatorname{span}_{\mathbb{K}}\{T_i \Delta T_j\} \leqslant k^2 \xi_k^1(\delta_U)$, which gives us Case 1 of Lemma 4.1. Hence, we can assume that the above process succeeds, and we have a core consisting of $\binom{k}{2}\xi_k^1(\delta_U)$ linearly independent forms. Let V^{core} be the span of the forms that are in the core. We have $\dim V^{\text{core}} = \binom{k}{2}\xi_k^1(\delta_U)$. Let $M := M(\delta_U)$ for brevity, where $M(\cdot)$ is the function from the statement of Lemma 4.1.

Strong forms outside V^{core}

Let $\mathcal{B} \subset \mathcal{C}$ be the set of forms $Q \in \mathcal{C}$ such that $\operatorname{span}_{\mathbb{K}}\{Q\}$ is μ_{k-1}-lifted strong with respect to U, and that do not belong to V^{core}. Suppose $P \in \mathcal{C} \setminus \mathcal{B}$. Then either $P \in V^{\text{core}}$, or $\operatorname{span}_{\mathbb{K}}\{P\}$ is not μ_{k-1}-lifted strong. The latter implies that $s(P) \leqslant \mu_{k-1}(\delta_U + (0,1))$, by [OS24, Proposition 5.14.1]. In particular, $s(P) \leqslant M$, and therefore P is M-close to the zero vector space, therefore also V^{core}. The forms in $\mathcal{B}$ are exactly those that we have to now control.

Define $\mathcal{B}' \subset W := \mathbb{P}(R_2/V_2^{\text{core}})$ to be the image of $\mathcal{B} \subset R_2$ under the vector space quotient $R_2 \to R_2/V_2^{\text{core}}$ followed by projectivisation. By definition, no element of $\mathcal{B}$ is mapped to 0 under the quotient map, however it is possible for different elements of $\mathcal{B}$ to get mapped to linearly dependent elements of the vector space W. We will control the set $\mathcal{B}'$, which will imply control over the set $\mathcal{B}$.

Definition 4.2 (Strong span configurations)**.** A set $\mathcal{G}' \subset W := \mathbb{P}(R_2/V_2^{\text{core}})$ is called a (M', b')-strong span configuration if the following holds. For every $h_1, \ldots, h_{b'} \in \mathcal{G}'$ which are linearly independent, either $|\operatorname{span}_{\mathbb{K}}\{h_1, \ldots, h_{b'}\} \cap \mathcal{G}'| > b'$, or there exist preimages $H_1, \ldots, H_{b'} \in R_2$ of $h_1, \ldots, h_{b'}$, and a form $H \in \operatorname{span}_{\mathbb{K}}\{H_1, \ldots, H_{b'}\}$ such that H is M'-close to V^{core}.

Note that in the above definition, the choice of preimages does not matter, since the pseudo-distance to V^{core} is unchanged if different preimages are picked. Returning to the set $\mathcal{B}'$, which is the image of $\mathcal{B}$ in W, we consider two cases. Recall that in Case 3 we defined

$$b := k^2 \left(\xi_k^1(\delta_U) + 1\right).$$

Either $\mathcal{B}'$ is an (M, b)-strong span configuration, or $\mathcal{B}'$ fails to be an (M, b)-strong configuration. If $\mathcal{B}'$ fails to be an (M, b)-strong configuration, then we can deduce that there exist $H_1, \ldots, H_b \in \mathcal{B}$ with the following properties: The forms $H_1, \ldots, H_b$ are linearly independent, and no form in their span is M-close to V_2^{core}. In particular this implies that no non zero form in the span lies in V_2^{core}. Finally, every form in $\mathcal{B} \cap \text{span}_{\mathbb{K}}\{H_1, \ldots, H_b, V_2^{\text{core}}\}$, actually lies in $\text{span}_{\mathbb{K}}\{H_i, V_2^{\text{core}}\}$ for some i, since the images $h_1, \ldots, h_b$ satisfy $\left|\text{span}_{\mathbb{K}}\{h_1, \ldots, h_b\} \cap \mathcal{B}'\right| = b$.

In the rest of this section, we handle these two cases. We will show that given any (M', b')-strong span configuration, we can find a small set of forms $\mathcal{H}$ such that every form in the configuration is close to $\text{span}_{\mathbb{K}}\{\mathcal{H}\} + V^{\text{core}}$, as in Case 3 of Lemma 4.1. We also show that if $\mathcal{B}'$ fails to be an (M, b)-strong configuration, that is, $H_1, \ldots, H_b$ with the above properties exist, then one of the forms H_i satisfies Case 2 of Lemma 4.1. This will use a Kayal–Saraf type argument. The rest of this section is split into two subsections, to handle these two cases. We start with the second case.

4.1 A fan-in reduction lemma when $\mathcal{B}'$ fails to be an (M, b)-strong configuration

Let $\mathcal{H} := \{H_1, \ldots, H_b\}$ be the witness to the fact that $\mathcal{B}'$ is not an (M, b)-strong span configuration. In this section, we will show the existence of a form $H \in \mathcal{H}$, such that the image of $\mathcal{C}$ in the ring $R/(H)$ has rank at least $\xi_k^1(\delta_U)$. Each of the elements in $\mathcal{H}$ has lifted strength at least μ_{k-1}. For any such element H, and any lift $\widetilde{H}$ of H in S, the vector space $U + \widetilde{H}$ is μ_{k-1}-strong. This ring is the same as $R/(H)$, and is a Cohen–Macaulay UFD by Corollary 3.13. The image of $\mathcal{C}$ in these rings is well defined, as is the notion of pairwise rank for these images. The dimension sequence of the vector space $U + \widetilde{H}$ is $\delta_U + (0, 1)$, and by construction we have $\xi_k^1(\delta_U) > \xi_{k-1}(\delta_U + (0, 1))$. Therefore, the existence of such a H shows that Case 2 of Lemma 4.1 is satisfied, as we also show that pairwise rank remains high (Lemmas 4.4 and 4.5).

Recall that V^{core} was constructed by picking $\xi_k^1(\delta_U)$ many linearly independent elements from each symmetric difference $T_i \Delta T_j$. Suppose $G_1, \ldots, G_a$ are the forms we picked from $T_1 \Delta T_2$, with $a = \xi_k^1(\delta_U)$. For a form $H \in \mathcal{H}$, we define the notion of H being *core preserving* and *core collapsing* for $T_1 \Delta T_2$ as follows.

Definition 4.3 (Core preserving/collapsing)**.** Let $H \in \mathcal{H} \setminus (T_1 \cup T_2)$. We say that H is core preserving for $T_1 \Delta T_2$ if $\left|\text{span}_{\mathbb{K}}\{G_i, H\} \cap T_1 \Delta T_2\right| = 1$ for all $i \in [a]$. The form G_i is always in this span, therefore the condition states that no other form is in the span. We say H is core collapsing for $T_1 \Delta T_2$ if there is $i \in [a]$ such that $\left|\text{span}_{\mathbb{K}}\{G_i, H\} \cap T_1 \Delta T_2\right| \geqslant 2$.

The notion extends naturally to every symmetric difference. The following lemma justifies the name core preserving.

Lemma 4.4. *Suppose* $H \in \mathcal{H} \setminus (T_1 \cup T_2)$ *is core preserving for* $T_1 \Delta T_2$*. The image of* T_1, T_2 *in* $R' := R/(H)$ *has pairwise rank at least* $\xi_k^1(\delta_U)$*.*

Proof. We claim that $G_1, \ldots, G_a$ are witnesses to the pairwise strength of T_1, T_2. Condition $H \notin T_1 \cup T_2$ ensures these gates survive in R'. The forms $G_1, \ldots, G_a$ remain linearly independent in R', otherwise $H \in V^{\text{core}}$.

Fix G_1, and suppose without loss of generality that $G_1 \in T_1 \setminus T_2$. The form G_1 is either a linear form, or a quadratic form. If G_1 is a quadratic form, then the image of G_1 in R' remains irreducible: indeed if it was reducible, say $G_1 = \ell_1 \ell_2$ in R', then $H = \alpha G_1 + \ell_1 \ell_2$ in R, contradicting the fact that H is not M-close to V^{core}.

Suppose that G_1 is no longer in the symmetric difference in R'. It must be that for some form $F \in T_2 \setminus T_1$, the image of F in R' has a non-trivial gcd with the image of G_1. We consider two cases. First suppose G_1 is a quadratic form (i.e. $G_1 \in R_2$). Since the image of G_1 is irreducible, it must be that $F \in (G_1, H)$ in the ring R. Since F, G_1, H are quadratics, this implies $F \in \text{span}_{\mathbb{K}}\{G_1, H\}$, contradicting the fact that H is core preserving.

Suppose now that $G_1 \in R_1$. If F were also linear, then G_1, F are associate in R, contradicting the choice of G_1, F (namely that they lie in the symmetric differences). Therefore F must be a quadratic, and we must have $F = H + G_1\ell$ in R. It must also hold that $F \notin V_2^{\mathrm{core}}$, otherwise H will be 1-close to V^{core}. However this contradicts the construction of the core: we always first pick as many quadratics as possible before picking linear forms, but here we show that we picked G_1 as part of the core but did not pick F. $\square$

We now consider the core collapsing case.

Lemma 4.5. *Suppose $H \in \mathcal{H} \setminus (T_1 \cup T_2)$. Assume that there are c other elements in $\mathcal{H} \setminus (T_1 \cup T_2)$ that are core collapsing for $T_1 \Delta T_2$. Then the image of T_1, T_2 in $R' := R/(H)$ has pairwise rank at least c.*

Proof. Without loss of generality, we assume that the c elements that are core collapsing other than H are $H_1, \dots, H_c$. The form H_1 is core collapsing for $T_1 \Delta T_2$. By definition, this means that there is a form G_{i_1} among $G_1, \dots, G_a$ such that $\left|\mathrm{span}_{\mathbb{K}}\{G_{i_1}, H_1\} \cap T_1 \Delta T_2\right| > 1$. If G_{i_1} is a linear form, then the only homogeneous forms in $\mathrm{span}_{\mathbb{K}}\{G_{i_1}, H_1\}$ are multiples of G_{i_1} and multiples of H_1, neither of which are in $T_1 \Delta T_2$. Therefore, in the above setting, it must be that G_{i_1} is a quadratic form. Let F_1 be any form in the above intersection other than G_{i_1}, so $F_1 \in T_1 \Delta T_2$ and $F_1 \in \mathrm{span}_{\mathbb{K}}\{G_{i_1}, H_1\}$. Similarly we define G_{i_j} and F_j for all $j = 2, \dots, c$.

We claim that $F_1, \dots, F_c$ are witnesses to the pairwise rank of T_1, T_2 in R'. First observe that F_1 is irreducible in R': if not, and if $F_1 = \ell_1\ell_2$ in R', then $F_1 = \ell_1\ell_2 + \alpha H$ in R with $\alpha \neq 0$. Further we have $F_1 \in \mathrm{span}_{\mathbb{K}}\{G_{i_1}, H_1\}$, which would imply that an element in the span of $H, H_1, V^{\mathrm{core}}$ has strength 1, contradicting the fact that no form in the span of $H_1, \dots, H_b$ is M-close to V^{core}.

Since F_1 is a quadratic form in R', the only way it is no longer in the symmetric difference of $T_1 \Delta T_2$ is if there is another form $F' \in T_1 \Delta T_2$ such that F', F_1 are associate in R'. This implies $F' = F_1 + \alpha H$ in R, with $\alpha \neq 0$. Further we have $F_1 \in \mathrm{span}_{\mathbb{K}}\{G_{i_1}, H_1\}$, say $F_1 = \alpha' G_{i_1} + \alpha'' H_1$ with $\alpha', \alpha'' \neq 0$. We have $F' = \alpha' G_{i_1} + \alpha'' H_1 + \alpha H$. Now if $F' \notin \mathcal{B}$, then $\alpha'' H_1 + \alpha H$ is M-close to V^{core}, since $s(F') < \mu_{k-1}(\delta_U + (0,1)) \leqslant M$. This cannot hold, because of our choice of $H_1, \dots, H_b$ (as witnesses to the fact that $\mathcal{B}'$ is not an (M, b)-strong span configuration). On the other hand, the only forms in $\mathrm{span}_{\mathbb{K}}\{H_1, \dots, H_b, V^{\mathrm{core}}\} \cap \mathcal{B}$ are those that lie in $\mathrm{span}_{\mathbb{K}}\{H_i, V^{\mathrm{core}}\}$ for some i (again, as they witness to the fact that $\mathcal{B}'$ is not an (M, b)-strong span configuration). However if F' had this structure, it would contradict the fact that no form in $\mathrm{span}_{\mathbb{K}}\{H_1, \dots, H_b\}$ lies in V^{core}. In either case we have a contradiction, therefore such an F' does not exist.

Finally, by the same token, we observe that $F_1, \dots, F_c$ are linearly independent in R'. Any linear dependence between $F_1, \dots, F_c$ would lift to a linear dependence between $H_1, \dots, H_b$ and an element of V^{core}, which contradicts the choice of $H_1, \dots, H_b$. Therefore the pairwise rank of T_1, T_2 in R' is at least c. $\square$

While Lemma 4.4 and Lemma 4.5 are stated for $T_1 \Delta T_2$, the same results hold for other symmetric differences $T_i \Delta T_j$. Recall that our objective was to find a form H such that the pairwise rank of $\mathcal{C}$ in $R/(H)$ is at least ξ_k^1. Given the above two lemmas, a sufficient condition for H to satisfy this property is that for every $i \neq j \in [k]$, either H is core preserving for $T_i \Delta T_j$, or there are $\xi_k^1(\delta_U)$ many forms other than H that are core collapsing for $T_i \Delta T_j$, or H belongs to one of T_i, T_j.

The following counting argument shows that such a $H \in \mathcal{H}$ exists. For each $T_i \Delta T_j$ let $\mathcal{H}^{i,j} \subset \mathcal{H}$ be the set of elements H such that H is core collapsing for $T_i \Delta T_j$. If $\left|\mathcal{H}^{i,j}\right| \geqslant \xi_k^1(\delta_U) + 1$, then for any $H \in \mathcal{H}$, either one of T_i, T_j vanishes in $R/(H)$ (which happens if H belongs to one of T_i, T_j), or their pairwise rank is at least $\xi_k^1(\delta_U)$, by either Lemma 4.4 or Lemma 4.5.

Let $\mathcal{H}'$ be the union of $\mathcal{H}^{i,j}$ over those i, j such that $\left|\mathcal{H}^{i,j}\right| \leqslant \xi_k^1(\delta_U)$. The size of $\mathcal{H}'$ is at most $\binom{k}{2}\xi_k^1(\delta_U)$, and by the choice of b, we can pick $H \in \mathcal{H} \setminus \mathcal{H}'$. We claim that this H has the required property. The pairs i, j with $\mathcal{H}^{i,j} \geqslant \xi_k^1(\delta_U)+1$ have already been handled. If i, j is such that $\mathcal{H}^{i,j} \leqslant \xi_k^1(\delta_U)$, then by construction either H belongs to one of T_i, T_j or H is core preserving for $T_i \Delta T_j$.

4.2 Controlling $\mathcal{B}'$ when it is an (M, b)-strong configuration

In this subsection, we show that strong span configurations, defined in Definition 4.2, can be controlled. This is the last step in completing the proof of Lemma 4.1. Specifically, we show the following result.

Theorem 4.6. *Suppose $\mathcal{G}' \subset W := \mathbb{P}(R_2/V_2^{core})$ is an (M', b')-strong span configuration. There exists a subset $\mathcal{H}' \subset \mathcal{G}'$ of size at most $4^{b'+1} + b'^3 + 48$ such that every form in $\mathcal{G}'$ is $3^{b'}M'$-close to $V^{core} + \mathrm{span}_{\mathbb{K}}\{\mathcal{H}'\}$.*

In our setting, we can apply Theorem 4.6 to $\mathcal{B}'$ with $b' = b$ and $M' = M$. If we then add the elements of $\mathcal{H}'$ obtained by the above application to V^{core}, we deduce Case 3 of Lemma 4.1. To prove Theorem 4.6, we define a generalization of (M', b')-strong span configurations. To make the definition and the arguments convenient we establish some notation. First, we use small letters to denote elements of W, and corresponding capital letters to denote preimages in R_2. For example, if $h, h_i \in W$ then H, H_i will denote a choice of preimage of h, h_i. Our arguments and results will not depend on which preimage is chosen. Given a set of elements $h_1, \ldots, h_c \in W$, we say that $h_1, \ldots, h_c$ are M'-far from V^{core} if $h_1, \ldots, h_c$ are linearly independent, and there is no nonzero form $H \in \mathrm{span}_{\mathbb{K}}\{H_1, \ldots, H_c\}$ such that H is M'-close to V^{core}. Therefore, if we say $h_1, \ldots, h_c$ are not M'-far from V^{core}, we mean either that they are linearly dependent, or that some nonzero element in the span of $H_1, \ldots, H_c$ is M'-close to V^{core}. [5] Given a set $\mathcal{H} \subset W$, with $\mathcal{H} := \{h_1, \ldots, h_c\}$, we say that g is M'-close to $\mathcal{H}, V^{core}$ (or g is M'-close to $h_1, \ldots, h_c, V^{core}$) if G is M'-close to $\mathrm{span}_{\mathbb{K}}\{H_1, \ldots, H_c\} + V^{core}$.

Definition 4.7 (Fractional strong span configuration)**.** Let $0 < \nu \leqslant 1$. A set $\mathcal{G}' \subset W := \mathbb{P}(R_2/V_2^{core})$ is called a (ν, M', b')-strong span configuration if the following holds. For every $h_1, \ldots, h_{b'-1} \in \mathcal{G}'$ which are M'-far from V^{core}, for ν fraction of the remaining elements $h \in \mathcal{G}'$, either $\left|\mathrm{span}_{\mathbb{K}}\{h_1, \ldots, h_{b'-1}, h\} \cap \mathcal{G}'\right| > b'$, or h is M-close to $h_1, \ldots, h_{b'-1}, V^{core}$.

An (M', b')-strong span configuration is the same as a $(1, M', b')$-strong span configuration. As a warm up, we show how $(\nu, M', 2)$-span configurations can be controlled. This will also act as the base case in our induction arguments.

Lemma 4.8. *Suppose $\mathcal{G}'$ is a $(\nu, 2, M')$-strong span configuration such that no $0 \neq g \in \mathcal{G}$ is M'-close to V^{core}. Then there is a set $\mathcal{H} \subset W$ of size at most $8/\nu$, and a set $\mathcal{G}_c$ with $|\mathcal{G}_c| \geqslant (\nu/2)|\mathcal{G}'|$ such that for all $g \in \mathcal{G}_c$, the element g is M'-close to $\mathcal{H}, V^{core}$.*

Proof. For $g \in \mathcal{G}'$, let $\mathcal{F}_{weak}(g)$ be the set of forms $g' \in \mathcal{G}'$ that are M'-close to g, V^{core}. If for some $g \in \mathcal{G}'$ we have $|\mathcal{F}_{weak}(g)| \geqslant \nu/2\,|\mathcal{G}'|$ then we pick $\mathcal{H} = \{g\}$ and we are done with $\mathcal{G}_c = \mathcal{F}_{weak}(g)$. Therefore we assume that for all $g \in \mathcal{G}'$, we have $|\mathcal{F}_{weak}(g)| < \nu/2\,|\mathcal{G}'|$

For $g \in \mathcal{G}'$, let $\mathcal{F}_{span}(g)$ be the set of forms $h \in \mathcal{G}' \setminus \mathcal{F}_{weak}(g)$ such that $\left|\mathrm{span}_{\mathbb{K}}\{g, h\} \cap \mathcal{G}'\right| \geqslant 3$. We have $\left|\mathcal{F}_{span}(g)\right| \geqslant \nu/2\,|\mathcal{G}'|$ for all $g \in \mathcal{G}'$, since $|\mathcal{F}_{weak}(g)| \leqslant \nu/2\,|\mathcal{G}'|$. This shows that the set $\mathcal{G}'$ is a $\nu/2$-linear SG

[5] We could have chosen to say $h_1, \ldots, h_c$ are M'-close to V^{core} in this situation, as opposed to saying not M'-far. However the former makes it seem as though all the forms are close to V^{core}, while what we want to capture is that some form in the span is close to V^{core}.

configuration. By [DGOS18, Theorem 1.6] $\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{G}'\} \leqslant 8/\nu$ and thus it has a basis consisting of at most $8/\nu$ forms. Picking $\mathcal{H}$ to be such a basis, we are done with $\mathcal{G}_c = \mathcal{G}'$. □

Note that if a form g is M'-close to V^{core} in a $(\nu, 2, M')$-strong span configuration, then every other form h automatically satisfies a relation with g, since the second condition in the definition of $(\nu, 2, M')$-strong span configurations holds. This is why when discussing fractional configurations, we focus on the forms that are not M'-close.

Continuing our warm up, we now show how $(1, 2, M')$-strong span configurations can be controlled. This will also act as a base case in our main proof.

Lemma 4.9. *Suppose $\mathcal{G}'$ is a $(1, 2, M')$-strong span configuration. There is a set $\mathcal{H} \subset \mathcal{G}'$ of size at most 8 such that every $g \in \mathcal{G}'$ is $2M'$-close to $\mathcal{H}, V^{\text{core}}$.*

Proof. Let $\mathcal{G}_w$ be the set of forms in $\mathcal{G}'$ that are M'-close to V^{core}. Let $\mathcal{G}_s := \mathcal{G}' \setminus \mathcal{G}_w$. For brevity let $m := |\mathcal{G}_s|$. Suppose $\mathcal{G}_s$ is a 1/2-linear SG configuration. In this case, by [DGOS18, Theorem 1.6] $\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{G}_s\} \leqslant 8$. Hence, picking $\mathcal{H}$ to be a basis of $\mathcal{G}_s$ of size 8, we are done.

Therefore we can assume that $\mathcal{G}_s$ is not a 1/2-linear SG configuration. Let $h_1 \in \mathcal{G}_s$ be a witness to this fact and let $\mathcal{H} := \{h_1\}$. Let $\mathcal{G}_1$ be those elements that are M'-close to $\mathcal{H}, V^{\text{core}}$, we have $\mathcal{G}_w \subset \mathcal{G}_s$. Since $\mathcal{G}'$ is a $(1, 2, M')$-strong span configuration, this means that for at least $m/2$ elements $g \in \mathcal{G}_s$, namely those with which h_1 does not span anything in $\mathcal{G}_s$, either g is M'-close to h_1, V^{core}, or $\left|\operatorname{span}_{\mathbb{K}}\{h_1, g\} \cap \mathcal{G}_w\right| \geqslant 1$. Observe that if $\left|\operatorname{span}_{\mathbb{K}}\{h_1, g\} \cap \mathcal{G}_w\right| \geqslant 1$ then this also implies that g is M'-close to h_1, V^{core}. Therefore $|\mathcal{G}_1 \cap \mathcal{G}_s| \geqslant m/2$.

We now have to show that every element $g \in \mathcal{G}' \setminus \mathcal{G}_1$ is $2M'$-close to $\mathcal{H}, V^{\text{core}}$. It suffices to show that every element $g \in \mathcal{G}' \setminus \mathcal{G}_1$ is either in the span of two elements of $\mathcal{G}_1$ or is M'-close to g', V^{core} for some $g' \in \mathcal{G}_1$.

Let g be any element that does not satisfy the second property. Then for every $g' \in \mathcal{G}_1 \cap \mathcal{G}_s$ we have $\left|\operatorname{span}_{\mathbb{K}}\{g, g'\} \cap \mathcal{G}'\right| \geqslant 3$. If for some g', one of the elements in the span (other than g') lies in $\mathcal{G}_1$, then g is in the span of two elements of $\mathcal{G}_1$. If for $g', g'' \in \mathcal{G}_1 \cap \mathcal{G}_s$, the intersection $\operatorname{span}_{\mathbb{K}}\{g, g'\} \cap \operatorname{span}_{\mathbb{K}}\{g, g''\}$ contains an element other than g, then it must be that $g \in \operatorname{span}_{\mathbb{K}}\{g', g''\}$. By the pigeonhole principle, one of the conditions is forced to hold since $|\mathcal{G}_1 \cap \mathcal{G}_s| > m/2$. □

We now turn to the case $b' > 2$. Our argument is inspired by the arguments in [BDYW11], where rank bounds for higher dimensional linear SG configurations are given. In particular, we induct on b'. If the given (ν, b', M')-strong span configuration is a $(\nu', b' - 1, M')$-strong configuration for some smaller ν' that we choose, then have bounds by induction. If not then we use a witness to the failure to identify a fractional linear SG configuration, where we can apply the bounds from [DGOS18, Theorem 1.6].

Theorem 4.10. *Suppose $\mathcal{G}'$ is a (ν, b', M')-strong span configuration such that no $g \in \mathcal{G}'$ is M'-close to V^{core}. Then there is a set $\mathcal{H}$ of at most $b' + 4^{b'}/\nu$ elements, and a set $\mathcal{G}_c$ with $|\mathcal{G}_c| \geqslant \left(\nu/4^{b'-1}\right)|\mathcal{G}'|$ such that for all $g \in \mathcal{G}_c$, the element g is M'-close to $\mathcal{H}, V^{\text{core}}$.*

Proof. Define $\nu_i := \nu/4^{b'-i}$ for all $i \leqslant b'$. Let r be the smallest positive integer such that $\mathcal{G}'$ is a not a (ν_r, r, M')-strong span configuration, but $\mathcal{G}'$ is a $(\nu_{r+1}, r+1, M')$-strong span configuration. Note that $\nu_{b'} = \nu$, therefore $1 \leqslant r \leqslant b' - 1$. Suppose $r = 1$, so $\mathcal{G}'$ is a $(\nu_2, 2, M')$-strong span configuration. Applying Lemma 4.8 then gives us the required sets $\mathcal{H}, \mathcal{G}_c$.

Now suppose $r > 1$. In particular, $\mathcal{G}'$ is not a (ν_r, r, M')-strong span configuration. Let $h_1, \ldots, h_{r-1}$ be a witness to this fact. This implies that $h_1, \ldots, h_{r-1}$ are M'-far from V^{core}. This also implies that for at least $1 - \nu_r$ fraction of $g \in \mathcal{G}'$, the element g is M'-far from $h_1, \ldots, h_r, V^{\text{core}}$, and further that $\left|\text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}, g\} \cap \mathcal{G}'\right| = r$. In particular, we must have $\left|\text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}\} \cap \mathcal{G}'\right| = r - 1$ (if not, then the condition on the previous line cannot hold for many g). Let $\mathcal{G}_1$ be this set.

For any $g \in \mathcal{G}_1$, define the set $\mathcal{F}_{\text{weak}}(g) \subset \mathcal{G}_1$ to be the set of elements g' such that g' is M'-close to $h_1, \ldots, h_{r-1}, g, V^{\text{core}}$. If for any $g \in \mathcal{G}_1$, we have $|\mathcal{F}_{\text{weak}}(g)| \geqslant \nu_r |\mathcal{G}'|$, then the required statement holds with $\mathcal{H} := \{h_1, \ldots, h_{r-1}, g\}$ and $\mathcal{G}_c := \mathcal{F}_{\text{weak}}(g)$. Therefore we can assume that $|\mathcal{F}_{\text{weak}}(g)| \leqslant \nu_r |\mathcal{G}'|$ for all $g \in \mathcal{G}_1$.

For any $g \in \mathcal{G}_1$ define $\mathcal{F}_{\text{span}}(g) \subset \mathcal{G}'$ to be the set of elements g' such that $\text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}, g, g'\} > r + 1$. Since $\mathcal{G}'$ is a $(\nu_{r+1}, r+1, M')$-strong span configuration, for every $g \in \mathcal{G}_1$ we have

$$\left|\mathcal{F}_{\text{span}}(g) \cup \mathcal{F}_{\text{weak}}(g) \cup (\mathcal{G}' \setminus \mathcal{G}_1)\right| \geqslant 4\nu_r |\mathcal{G}|.$$

In particular, under the above assumptions, for every $g \in \mathcal{G}_1$ we have $\left|\mathcal{F}_{\text{span}}(g)\right| \geqslant 2\nu_r |\mathcal{G}'|$.

Let $Y := \text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}\}$. We now show that under the assumptions above, the image of $\mathcal{G}_1$ in $\mathbb{P}(W/Y)$ is a ν_r-linear SG configuration. Since $\mathcal{G}_1$ consists of more than half the elements of $\mathcal{G}$, setting $\mathcal{H}$ to be a basis of $\mathcal{G}_1$ will complete the proof, with $\mathcal{G}_c = \mathcal{G}_1$. Observe first that the above map is injective: indeed if g, g' get mapped to the same element, then $g' \in \text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}, g\}$, contradicting the definition of $\mathcal{G}_1$. Now for each $g \in \mathcal{G}_1$ we have $\left|\mathcal{F}_{\text{span}}(g)\right| \geqslant 2\nu_r |\mathcal{G}'|$. If we can ensure that for each g, for at least ν_r elements $g' \in \mathcal{F}_{\text{span}}(g)$ we have $\left|\text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}, g, g'\} \cap \mathcal{G}_1\right| > r + 1$ then we will be done. For this, we simply observe that if $\text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}, g, g'\} \cap \text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}, g, g''\}$ has an element in $\mathcal{G}' \setminus \mathcal{G}_1$ then $g'' \in \text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}, g, g'\}$. Here we crucially use the fact that $\text{span}_{\mathbb{K}}\{h_1, \ldots, h_{r-1}\} \cap \mathcal{G}'$ only has $r - 1$ elements. Therefore, there can be at most $\nu_r |\mathcal{G}'|$ elements in $\mathcal{F}_{\text{span}}(g)$ that only span elements of $\mathcal{G}' \setminus \mathcal{G}_1$ with $h_1, \ldots, h_{r-1}, g$. The remaining elements, of which there are at least $\nu_r |\mathcal{G}'|$ many, are witness to necessary SG condition. □

With these lemmas, we can complete the proof of Theorem 4.6, and control $(1, b', M')$-strong span configurations.

Proof of Theorem 4.6. Let $\mathcal{G}'_w$ be the elements in $\mathcal{G}'$ that are M'-close to V^{core}. Let $\mathcal{G}'_s := \mathcal{G}' \setminus \mathcal{G}'_w$. Let $m := |\mathcal{G}'_s|$. The set $\mathcal{G}'_s$ is itself a $(1, b', M')$-strong span configuration. Indeed for some choice of $h_1, \ldots, h_{b'-1} \in \mathcal{G}'_s$ that is M'-far from V^{core}, and some $h \in \mathcal{G}'_s$, if $\text{span}_{\mathbb{K}}\{h_1, \ldots, h_{b'-1}, h\}$ contains an element of $\mathcal{G}'_w$, then this implies that h is M'-close to $h_1, \ldots, h_{b'-1}, V^{\text{core}}$. Therefore we focus on the set $\mathcal{G}'_s$.

We start by applying Theorem 4.10 to the set $\mathcal{G}'_s$ with $\nu = 1$. Let $\mathcal{H}'_1$ be the set of elements guaranteed by Theorem 4.10, we have $|\mathcal{H}'_1| \leqslant b' + 12 \cdot 4^{b'-1}$. Let $\mathcal{G}'_1 \subset \mathcal{G}'_s$ be the set of elements that are M'-close to $\mathcal{H}_1, V^{\text{core}}$. We have $|\mathcal{G}_1| \geqslant m/4^{b'-1}$.

Define $\mathcal{J}'_1 := \mathcal{G}'_s \setminus \mathcal{G}'_1$, these are the images of the forms that we still have to control. We pick a set of forms $f_1, \ldots, f_{b'-1} \in \mathcal{J}'_1$ with the following properties: The set $f_1, \ldots, f_{b'-1}$ is $3M'$ far from $\mathcal{H}'_1, V^{\text{core}}$, and $\left|\text{span}_{\mathbb{K}}\{f_1, \ldots, f_{b'-1}\} \cap \mathcal{J}'_1\right| = b' - 1$. The case where such a set of forms does not exist will be handled later in the proof. For each $g \in \mathcal{G}_1$, the element g is not M'-close to $f_1, \ldots, f_{b'-1}$, since it if was, then $f_1, \ldots, f_{b'-1}$ would be not be $3M'$-far from $\mathcal{H}'_1, V^{\text{core}}$. Here we use the fact that g is itself not M'-close to V^{core}. Therefore, we must have $\left|\text{span}_{\mathbb{K}}\{f_1, \ldots, f_{b'-1}, g\} \cap \mathcal{J}'_1\right| > b'$. If an element in this span other than g lies in either $\mathcal{G}'_w$ or $\mathcal{G}'_1$, then this would imply that $f_1, \ldots, f_{b'-1}$ is $3M'$-close to $\mathcal{H}'_1, V^{\text{core}}$, contradicting assumption. Therefore, for each g, there is at least one element in this span other than $f_1, \dots, f_{b'-1}$ that lies in $\mathcal{J}'_1$. Note that since $\left|\mathrm{span}_{\mathbb{K}}\{f_1, \dots, f_{b'-1}\} \cap \mathcal{J}'_1\right| = b' - 1$, the element in the $\mathrm{span}_{\mathbb{K}}\{f_1, \dots, f_{b'-1}, g\}$ non-trivially depends on g. Further note that any element in this span is M'-close to $\mathcal{H}'_1, \{f_1, \dots, f_{b'-1}, V^{\mathrm{core}}\}$.

Suppose now for $g, g' \in \mathcal{G}'_1$, there is an element in $\mathrm{span}_{\mathbb{K}}\{f_1, \dots, f_{b'-1}, g'\} \cap \mathrm{span}_{\mathbb{K}}\{f_1, \dots, f_{b'-1}, g''\}$ that is not one of the f_i. Since elements in these two spans non-trivially depend on g, g', this would imply that $\mathrm{span}_{\mathbb{K}}\{g, g'\} \cap \mathrm{span}_{\mathbb{K}}\{f_1, \dots, f_{b'-1}\} \neq \{0\}$. This again shows that $f_1, \dots, f_{b'-1}$ is not $3M'$-far from $\mathcal{H}'_1, V^{\mathrm{core}}$, contradicting assumption. Define $\mathcal{H}'_2 := \{f_1, \dots, f_{b'-1}\}$, and define $\mathcal{G}'_2 \subset \mathcal{G}'_s$ to be the set of elements that are M'-close to $\mathcal{H}'_1, \mathcal{H}'_2, V^{\mathrm{core}}$. The above argument shows $|\mathcal{H}'_2| \geqslant 2m/4^{b'-1}$. Define $\mathcal{J}'_2 := \mathcal{G}'_s \setminus \mathcal{B}'_2$.

We now repeat this, and look for a set $\mathcal{H}'_3 \subset \mathcal{J}_2$ of size $b' - 1$ such that $\mathcal{H}'_3$ is $3M'$-far from $\mathcal{H}'_1, \mathcal{H}'_2, V^{\mathrm{core}}$ and such that $\left|\mathrm{span}_{\mathbb{K}}\{\mathcal{H}'_3\} \cap \mathcal{J}'_2\right| = b' - 1$. If we can find such a set, then we can repeat this argument, and define $\mathcal{G}'_3$ to be the set of forms that are M'-close to $\mathcal{H}'_1, \mathcal{H}'_2, \mathcal{H}'_3, V^{\mathrm{core}}$ and we will have $|\mathcal{H}'_3| \geqslant 4m/4^{b'-1}$.

We do this process as many times as possible. Suppose we managed to iterate this argument t times, and construct $\mathcal{H}'_1, \dots, \mathcal{H}'_t$. Note that the case where we could not pick $f_1, \dots, f_{b'-1}$ is simply the case when $t = 0$. Since the size of the set that is M'-close doubles each time, we must have $t \leqslant 2(b' - 2)$. At the end, one of two cases holds. Either every element of $\mathcal{G}'_s$ is M'-close to $\mathcal{H}'_1, \cdots, \mathcal{H}'_t, V^{\mathrm{core}}$, in which case the lemma is proved. If not, then every set of $b' - 1$ forms that is $3M'$-far from $\mathcal{H}'_1, \cdots, \mathcal{H}'_t, V^{\mathrm{core}}$ is such that $\left|\mathrm{span}_{\mathbb{K}}\{f_1, \dots, f_{b'-1}\} \cap \mathcal{J}'_t\right| > b' - 1$.

In this latter case, the set $\mathcal{J}'_t$ is essentially a $(1, b' - 1, 3M')$-strong span sequence. The only difference is that we replace the condition $g_1, \dots, g_{b'}$ is M'-close to V^{core} with the condition that $g_1, \dots, g_{b'-1}$ is $3M'$-close to $\mathcal{H}'_1, \cdots, \mathcal{H}'_t, V^{\mathrm{core}}$. Note that the span relations still depend only on V^{core}. All our arguments apply verbatim to this slightly more general setting. Therefore, we can apply induction, with Lemma 4.9 acting as a base case. Letting $\mathcal{H}'$ denote the union of all the sets constructed (namely the $\mathcal{H}'_i$ and the sets obtained by induction) we deduce that every element of $\mathcal{G}'_s$ is $3^{b'}M'$-close to $\mathcal{H}', V^{\mathrm{core}}$. The size bound on $\mathcal{H}'$ also follows by induction. This completes the proof. □

5 Special case: all quadratics close to an algebra

In this section we consider identities $\mathcal{C}$ where all the quadratic forms are close to an algebra $\mathbb{K}[V]$ for some vector space $V \subseteq R_{\leqslant 2}$. We start by defining useful potential functions the capture the proximity to the algebra $\mathbb{K}[V]$ and absolute irreducibility over V.

5.1 Potential functions and absolutely irreducible forms

Let $\Gamma, \Lambda, f : \mathbb{N}^2 \to \mathbb{N}^2$ and $g : \mathbb{N}^2 \to \mathbb{N}$ be ascending functions. Let $R = S/(U)$, where $U \subseteq S_{\leqslant 2}$ is $\Lambda \circ t_{(0,1)}$-strong. Set $a := 2g(\delta_U) + f_1(\delta_U)$. Suppose that U satisfies the following conditions:

1. $\Lambda_i(\delta_U) \geqslant B \circ t_{(2a,0)}(\delta_U)$.
2. $\Gamma \geqslant h_{2B} \circ t_1$.
3. $\Gamma_2(\delta_U + f_1(\delta_U) + f_2(\delta_U)) \geqslant a + \dim(U_1)$.

Relative linear spaces. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong such that $\dim(V_i) \leqslant f_i(\delta_U)$. Note that for $F \in R_2$, such that $\text{ps-dist}(F, V) < g(\delta_U)$, the relative spaces $\mathbb{L}_R(a, V, F) \subseteq R_1$ and $\mathbb{L}_{R,V}(a, F) \subseteq R_1/V_1$ are well-defined by Definition 3.32.

Individual potential. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong and $\dim(V_i) \leqslant f_i(\delta_U)$. Let $F \in R_{\leqslant 2}$ be such that ps-dist$(F, V) < g(\delta_U)$ if $\deg(F) = 2$. We define the individual potential of F with respect to V (and the bounding functions f, g) as follows

$$\Phi(f, g, V, F) = \begin{cases} 0 & F \in V_1 \\ 1 & \text{if } F \in R_1 \setminus V_1 \\ \dim(\mathbb{L}_{R,V}(a, F)) & \text{if } F \in R_2 \end{cases}$$

Absolutely irreducible forms. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong and $\dim(V_i) \leqslant f_i(\delta_U)$. Let $\mathcal{C} = \sum_{i=1}^{k} T_i$ be a $\Sigma^k\Pi\Sigma\Pi^2$ simple and minimal identity in R. For $i, j \in [k]$ we denote the set of absolutely irreducible forms in $T_i \Delta T_j$ as

$$\mathcal{B}_{ij}(V, \mathcal{C}) = \{F \in T_i \Delta T_j \mid F \text{ is absolutely irreducible over } V \text{ and } F \notin (V)\}.$$

We denote the set of absolutely irreducible forms in $\mathcal{C}$ as

$$\mathcal{B}(V, \mathcal{C}) = \{F \in \mathcal{C} \mid F \text{ is absolutely irreducible over } V \text{ and } F \notin (V)\}.$$

Proposition 5.1. *For all irreducible forms $F \in \mathcal{C}$, there exists $i \neq j$ such that $F \in T_i \Delta T_j$. Moreover, we have $\mathcal{B}(V, \mathcal{C}) = \cup_{i \neq j} \mathcal{B}_{ij}(V, \mathcal{C})$.*

Proof. Since $\mathcal{C}$ is a simple and minimal identity, we know that for every irreducible form $F \in \mathcal{C}$, there exists T_i, T_j such that $T_i \in (F)$ and $T_j \notin (F)$. In particular, $F \in T_i \Delta T_j$. Therefore, every form $F \in \mathcal{B}(V, \mathcal{C})$ belongs to some $\mathcal{B}_{ij}(V, \mathcal{C})$. Hence $\mathcal{B}(V, \mathcal{C}) = \cup_{i \neq j} \mathcal{B}_{ij}(V, \mathcal{C})$. □

Potential of a circuit. Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong and $\dim(V_i) \leqslant f_i(\delta_U)$. Let $\mathcal{C} = \sum_{i=1}^{k} T_i$ be a $\Sigma^k\Pi\Sigma\Pi^2$ simple and minimal identity in R such that ps-dist$(F, V) < g(\delta_U)$ for all $F \in \mathcal{C} \cap R_2$. We define the potential of the circuit $\mathcal{C}$ as follows:

$$\Phi(f, g, V, \mathcal{C}) = \sum_{i \neq j} \max_{F \in T_i \Delta T_j} \Phi(f, g, V, F).$$

We define the *irreducibility potential* of the circuit $\mathcal{C}$ as follows:

$$\Phi_{\text{irr}}(f, g, V, \mathcal{C}) = \sum_{i \neq j} \max_{F \in \mathcal{B}_{ij}(V, \mathcal{C})} \Phi(f, g, V, F).$$

Proposition 5.2. *Let $V \subseteq R_{\leqslant 2}$ be $\Gamma \circ t_{(0,1)}$-lifted strong and $\dim(V_i) \leqslant f_i(\delta_U)$. Let $\mathcal{C} = \sum_{i=1}^{k} T_i$ be a $\Sigma^k\Pi\Sigma\Pi^2$ simple and minimal identity in R such that ps-dist$(F, V) < g(\delta_U)$ for all $F \in \mathcal{C} \cap R_2$. Then we have $\Phi(f, g, V, F) \leqslant a$ for all $F \in T_i \Delta T_j$, and*

$$\Phi_{\text{irr}}(f, g, V, \mathcal{C}) \leqslant \Phi(f, g, V, \mathcal{C}) \leqslant a\binom{k}{2}.$$

Proof. If $F \in R_1$, then $\Phi(f, g, V, F) \leqslant 1 \leqslant a$. If $F \in R_2$, then we have $\Phi(f, g, V, F) = \dim(\mathbb{L}_{R,V}(a, F)) \leqslant a$ by Proposition 3.34. By summing over all the $\binom{k}{2}$ symmetric differences, we obtain $\Phi(f, g, V, \mathcal{C}) \leqslant a\binom{k}{2}$. We have $\Phi_{\text{irr}}(f, g, V, \mathcal{C}) \leqslant \Phi(f, g, V, \mathcal{C})$ as $\mathcal{B}_{ij}(V, \mathcal{C}) \subseteq T_i \Delta T_j$. □

5.2 Identities with bounded pseudo-distance

In this subsection, we state our main result that controls identities where all forms $F \in \mathcal{C}$ have bounded pseudo-distance with respect to a vector space V.

Notation. Suppose $\mu_{k-1} : \mathbb{N}^2 \to \mathbb{N}^2$ and $\xi_{k-1} : \mathbb{N}^2 \to \mathbb{N}$ are fixed functions. We define the function $\xi_k^2 : \mathbb{N}^2 \to \mathbb{N}$ as

$$\xi_k^2(\delta) := \xi_{k-1}\left(\left(C_{2\mu_{k-1}} \circ t_1\right)(\delta)\right) + 1.$$

Let $g^{\dim} : \mathbb{N}^2 \to \mathbb{N}^2$ and $g^{ps} : \mathbb{N}^2 \to \mathbb{N}$ be given functions. These functions will upper bound the dimension of the initial vector space V and the pseudo-distances $\text{ps-dist}(F, V)$ for forms $F \in \mathcal{C}$.

Auxiliary functions. Define $a_k : \mathbb{N}^2 \to \mathbb{N}$ as $a_k(\delta) := 2g^{ps}(\delta) + \delta_1$ for $\delta \in \mathbb{N}^2$. We define $\lambda_k(\delta) := 2a_k(\delta) + \xi_k^2(\delta) + 1$ and $\beta_k(\delta) := k^2 \cdot \xi_k^2(\delta)$. Moreover, define the function $c_k : \mathbb{N}^2 \to \mathbb{N}$ as

$$c_k := a_k\binom{k}{2}\left(a_k\left(\binom{k}{2} + \beta_k\right) + 2a_k^2\lambda_k\binom{k}{2} + a_k\lambda_k\binom{k}{2}\right).$$

Strength and dimension bounds under strengthening process. We define two new functions $\Gamma, f : \mathbb{N}^2 \to \mathbb{N}^2$ (depending on the functions above). Let $\Gamma = h_{2B} \circ t_{(2c_k,1)}$ and $f = C_\Gamma \circ t_{g^{\dim}}$.

In the situation of Lemma 5.3, we will have a vector space V of dimension sequence at most $g^{\dim}(\delta_U)$. The function Γ will be the strength lower bound that we will enforce when we apply a strengthening process on V. Moreover, the function f will upper bound the dimension sequence of the resulting vector space after strengthening V.

Strength lower bound function for U. Let $\mu_k^{low} := h_{2\Gamma} \circ t_{g^{\dim}}$. This function will provide the strength lower bound required on U in Lemma 5.3.

Constants. Given $U \subseteq S_{\leqslant 2}$ as in Lemma 5.3 below, let us set $a := 2g^{ps}(\delta_U) + f_1(\delta_U)$. We define $\lambda(k) := 2a + \xi_k^2(\delta_U) + 1$ and we have $\beta_k(\delta_U) = k^2 \cdot \xi_k^2(\delta_U)$.

Dimension upper bound. Let $\rho_k(\delta_U) := g_1^{\dim}(\delta_U) + g_2^{\dim}(\delta_U) + c_k(\delta_U + f_1(\delta_U) + f_2(\delta_U)) + C_\Gamma(\delta_U + g^{\dim}(\delta_U))$. This quantity will provide the dimension upper bound on vector spaces Y in outcomes (1), (2) of Lemma 5.3.

Lemma 5.3. *Suppose $\mu_{k-1} : \mathbb{N}^2 \to \mathbb{N}^2$ and $\xi_{k-1} : \mathbb{N}^2 \to \mathbb{N}$ are fixed functions. Let $g^{\dim} : \mathbb{N}^2 \to \mathbb{N}^2$ and $g^{ps} : \mathbb{N}^2 \to \mathbb{N}$ be functions. Let $\xi_k^2, \mu_k^{low}, \rho_k$ be the functions defined above.*

Let $U \subseteq S_{\leqslant 2}$ be a μ_k^{low}-strong vector space and $R := S/(U)$. Let $\mathcal{C} = \sum_{i=1}^k T_i$ be a $\Sigma^k\Pi\Sigma\Pi^2$ simple and minimal identity in R. Suppose there exists a vector space $V \subseteq R_{\leqslant 2}$ such that

- $\dim(V_i) \leqslant g_i^{\dim}(\delta_U)$, *and*
- *for all quadratic forms $Q \in \mathcal{C} \cap R_2$, we have $\text{ps-dist}(Q, V) < g^{ps}(\delta_U)$, i.e. all quadratic forms of $\mathcal{C}$ are $g^{ps}(\delta_U)$-close to $\mathbb{K}[V]$.*

Then one of the following holds.

1. *There exists a vector space $Y \subseteq R_{\leqslant 2}$ with $\dim(Y) \leqslant \rho_k(\delta_U)$, and a pair $i, j \in [k]$, such that*

$$T_i \Delta T_j \subseteq \mathbb{K}[Y].$$

2. *There exists a vector space $Y \subseteq R_{\leqslant 2}$ with $\dim(Y) \leqslant \rho_k(\delta_U)$, such that for all $Q \in \mathcal{C} \cap R_2$ we have that Q is not absolutely irreducible over Y.*

3. *There exists* $H \in \mathcal{C}$ *with the following property. Let* $W := AH_R(\mu_{k-1}, H)$ *be the* μ_{k-1}*-lifted strong space obtained by applying* Corollary 3.17 *to* $\text{span}_{\mathbb{K}}\{H\}$ *in* R. *Let* $\varphi_{W,\alpha} : R[y] \to R' := R[y]/I_{W,\alpha}$ *be the graded quotient corresponding to* $\alpha \in \mathbb{K}^{\dim(W)}$. *For a general* $\alpha \in \mathcal{Z}(H) \subseteq \mathbb{K}^{\dim(W)}$, *we have that the circuit* $\varphi_{W,\alpha}(\mathcal{C}) \subseteq R'$ *has pairwise rank at least* $\xi_{k-1}(\delta_W)$.

The rest of this section will be devoted to the proof of Lemma 5.3. We will construct a vector space Y via an iterative process, and show that if Y does not satisfy condition 1, then there exists a form $H \in \mathcal{C}$ that enables us to carry out fan-in reduction.

5.3 Constructing cores and disjoint sequences

We will now iteratively construct a set of forms that will make up the core. We will then use another iterative process to pick a sequence of forms that we call a good disjoint sequence. This is analogous to what we did in Section 4. Throughout these iterative processes, we will use our potential functions to measure progress.

Construction of a core.

Set-up. Suppose we are in the setting of Lemma 5.3. Let Γ, f be the functions defined in the notation before Lemma 5.3. In the following, we will write $g := g^{ps}$, where g^{ps} is given in the statement of Lemma 5.3. Recall that $a := 2g(\delta_U) + f_1(\delta_U)$. We also have $\lambda(k) := 2a + \xi_k^2(\delta_U) + 1$. For simplicity, we will write β_k in place of $\beta_k(\delta_U) = k^2 \cdot \xi_k^2(\delta_U)$.

Strengthening. First, let us replace V by the vector space $AH_R(\Gamma, V)$ which is $\Gamma \circ t_{(0,1)}$-lifted strong. Note that, after updating V, we have $\dim(V_i) \leqslant f_i(\delta_U)$ and for all quadratic forms $Q \in \mathcal{C}$, we have $\text{ps-dist}(Q, V) \leqslant g(\delta_U)$ by Proposition 3.29.

Construction of core. We will pick a set of core forms using an iterative process and we will denote the set of these forms as $\mathcal{A}_{ij}^{\text{core}}(V, \mathcal{C}) \subseteq T_i \Delta T_j$. We define

$$\mathcal{A}^{\text{core}}(V, \mathcal{C}) := \cup_{i \neq j} \mathcal{A}_{ij}^{\text{core}}(V, \mathcal{C})$$

and

$$W^{\text{core}}(V, \mathcal{C}) := \sum_{G \in \mathcal{A}^{\text{core}}(V, \mathcal{C})} \mathbb{L}_R(a, V, G) \subseteq R.$$

Iterative process for core. We describe the iterative process below.
Start by setting $Y := V$, let $\mathcal{A}^{\text{core}}(Y, \mathcal{C}) = \emptyset$ and $\mathcal{A}_{ij}^{\text{core}}(Y, \mathcal{C}) = \emptyset$ for $i, j \in [k]$.
If $T_i \Delta T_j \subseteq \mathbb{K}[Y]$ for some $i \neq j \in [k]$, then we stop.
Otherwise, for $i \in [k]$ and $j > i$, we do the following:

- If $\mathcal{B}_{ij}(Y, \mathcal{C}) \neq \emptyset$, i.e. there are quadratics in $T_i \Delta T_j$ that are absolutely irreducible over Y, do the following:
  - If $\mathcal{B}_{ij}(Y, \mathcal{C})$ contains a sequence of quadratic forms $F_1, \cdots, F_{\lambda(k)}$ such that the set

    $$\mathcal{A}^{\text{core}}(Y, \mathcal{C}) \cup \{F_1, \cdots, F_{\lambda(k)}\}$$

    is lin-separated modulo Y (recall Definition 3.35), then we let $\mathcal{A}_{ij}^{\text{core}}(Y, \mathcal{C}) := \{F_1, \cdots, F_{\lambda(k)}\}$.

– Otherwise, we have that the maximum length of a lin-separated sequence in $\mathcal{B}_{ij}(Y, \mathcal{C})$ is less than $\lambda(k)$. Let $F_1, \cdots, F_m$ be such a maximal sequence where $m < \lambda(k)$. We replace Y with $\bigcup_{n=1}^{m} \mathbb{L}_R(a, Y, F_n)$, and restart the iterative process with the new Y and $i = 1$.[6]

- If $\mathcal{B}_{ij}(Y, \mathcal{C}) = \emptyset$, we do the following:
  – If there exist (not necessarily quadratic) forms $F_1, \cdots, F_{\lambda(k)}$ in $T_i \Delta T_j$ such that the set

$$\mathcal{A}^{\text{core}}(Y, \mathcal{C}) \cup \{F_1, \cdots, F_{\lambda(k)}\}$$

  is lin-separated modulo Y, then we let $\mathcal{A}^{\text{core}}_{ij}(Y, \mathcal{C}) := \{F_1, \cdots, F_{\lambda(k)}\}$.

  – Otherwise, we have that the maximum length of a lin-separated sequence in $T_i \Delta T_j$ is less than $\lambda(k)$. Let $F_1, \cdots, F_m$ be such a maximal sequence where $m < \lambda(k)$. We replace Y with $\bigcup_{n=1}^{m} \mathbb{L}_R(a, Y, F_n)$, and restart the iterative process with the new Y and $i = 1$.

Now we prove termination and properties of the outcome of this iterative process.

Proposition 5.4. *We have the following.*

1. *The iterative process restarts at most $2a\binom{k}{2}$ times. In particular, the iterative process terminates.*
2. *At the end of the iterative process, we have a vector space Y, containing V, that is $\Gamma \circ t_{(0,1)}$-lifted strong and $\dim(Y) \leqslant \dim(V) + 2a^2\lambda(k)\binom{k}{2}$. Furthermore, we have $\dim(W^{core}) \leqslant a\lambda(k)\binom{k}{2}$ and the vector space $Y + W^{core}$ is $\Gamma \circ t_{(0,1)}$-lifted strong.*
3. *Moreover, one of the following holds.*
   3.1 *We have $T_i \Delta T_j \subseteq \mathbb{K}[Y]$.*
   3.2 *We have $\mathcal{B}_{ij}(Y, \mathcal{C}) = \emptyset$ for all $i \neq j$. In particular, $\mathcal{B}(Y, \mathcal{C}) = \emptyset$.*
   3.3 *There exists $i \neq j$ such that $\mathcal{B}_{ij}(Y, \mathcal{C}) \neq \emptyset$. In this case, for all i, j, we have one of the following.*
      3.3.1 *If $\mathcal{B}_{ij}(Y, \mathcal{C}) \neq \emptyset$, then $\mathcal{A}^{core}_{ij}(Y, \mathcal{C}) \subseteq \mathcal{B}_{ij}(Y, \mathcal{C})$ and the core $\mathcal{A}^{core}_{ij}(Y, \mathcal{C})$ consists of $\lambda(k)$ absolutely irreducible quadratic forms that are lin-separated modulo Y.*
      3.3.2 *If $\mathcal{B}_{ij}(Y, \mathcal{C}) = \emptyset$, then the core $\mathcal{A}^{core}_{ij}(Y, \mathcal{C})$ consists of $\lambda(k)$ forms that are lin-separated modulo Y. In particular, for each quadratic form F in $\mathcal{A}^{core}_{ij}(Y, \mathcal{C})$, we have $F \in (Y)$ or F is absolutely reducible.*

Proof. (1) It is enough to show that the iterative process restarts at most $2a\binom{k}{2}$ times. Suppose that for certain i, j, we have $\mathcal{B}_{ij}(Y, \mathcal{C}) \neq \emptyset$ and the iterative process restarts because there is no $\lambda(k)$-length sequence which is lin-separated modulo Y. Let $Y' = \bigcup_{n=1}^{m} \mathbb{L}_R(a, Y, F_n)$, which is the new Y when we restart the iterative process. Note that $\mathcal{B}_{ij}(Y', \mathcal{C}) \subseteq \mathcal{B}_{ij}(Y, \mathcal{C})$ as $Y \subseteq Y'$. Moreover, for all $F \in \mathcal{B}_{ij}(Y, \mathcal{C}) \setminus \{F_1, \cdots, F_m\}$, we know that the dimension of the relative linear space drops, i.e. $\dim(\mathbb{L}_{R,Y'}(a, F)) < \dim(\mathbb{L}_{R,Y}(a, F))$, as $F_1, \cdots, F_m, F$ is not lin-separated modulo Y. Also, $\mathbb{L}_{R,Y'}(a, F_i) = (0)$ as $F_i \in \mathbb{K}[Y']$. Whereas we have $\mathbb{L}_{R,Y}(a, F_i) \neq (0)$ for $i \in [m]$, as $F_i \in \mathcal{B}_{ij}(Y, \mathcal{C})$, i.e. it is absolutely irreducible over Y and $F_i \notin (Y)$. Hence $\dim(\mathbb{L}_{R,Y'}(a, F_i)) < \dim(\mathbb{L}_{R,Y}(a, F_i))$ for all $i \in [m]$. Therefore we have

$$\Phi(f, g, Y', F) < \Phi(f, g, Y, F),$$

[6] Proposition 5.4 shows that Y remains $\Gamma \circ t_{(0,1)}$-lifted strong throughout the process, and therefore it satisfies the conditions of Definition 3.35.

for all $F \in \mathcal{B}_{ij}(Y, \mathcal{C})$. Hence the irreducibility potential of $\mathcal{C}$ must decrease, i.e. we have

$$\Phi_{\text{irr}}(f, g, Y', \mathcal{C}) < \Phi_{\text{irr}}(f, g, Y, \mathcal{C}).$$

By Proposition 5.2, we have an upper bound $\Phi_{\text{irr}}(f, g, Y, \mathcal{C}) \leqslant a\binom{k}{2}$. Therefore, the first kind of restart occurs at most $a\binom{k}{2}$ times. Similarly, we see that every time the second kind of restart works, the circuit potential $\Phi(f, g, Y, \mathcal{C})$ must decrease. Hence the second kind of restart also occurs at most $a\binom{k}{2}$ times.

(2) At the beginning of the iterative process we have $Y = V$. Now the vector space Y is updated only when a restart step occurs in the process. Recall that $\dim(\mathbb{L}_R(a, Y, F) \leqslant a$ for all F_i. Hence, whenever a restart step occurs, the dimension of Y increases by at most $a\lambda(k)$. Since there are at most $2a\binom{k}{2}$ restart steps, we have that the final Y satisfies $\dim(Y) \leqslant \dim(V) + 2a^2\lambda(k)\binom{k}{2}$. Let $c := 2a^2\lambda(k)\binom{k}{2}$. Since V was $\Gamma \circ t_{(c,1)}$-lifted strong, we have that Y is $\Gamma \circ t_{(0,1)}$-lifted strong. Note that $\mathcal{A}^{\text{core}}_{ij}(Y, \mathcal{C})$ has $\lambda(k)$ forms. For each such form $F \in \mathcal{A}^{\text{core}}_{ij}(Y, \mathcal{C})$ we have $\dim(\mathbb{L}_{R,Y}(a, F)) \leqslant a$. Therefore, $\dim(W^{\text{core}}) \leqslant a\lambda(k)\binom{k}{2}$. Again, let $c' = a\lambda(k)\binom{k}{2} + 2a^2\lambda(k)\binom{k}{2}$. Since V is $\Gamma \circ t_{(c',1)}$-lifted strong, we have that $Y + W^{\text{core}}$ is $\Gamma \circ t_{(0,1)}$-lifted strong.

(3) If $\mathcal{B}_{ij}(V, \mathcal{C}) = \emptyset$ for all $i \neq j$, then $\mathcal{B}(V, \mathcal{C}) = \emptyset$ by Proposition 5.1. At the last iteration of the process (when no more restart conditions occur), we end up with $\mathcal{A}^{\text{core}}(Y, \mathcal{C})$ with the stated properties due to the construction. □

Definition 5.5. After the iterative process ends, we will say that $T_i \Delta T_j$ is an *irreducible symmetric difference* if $\mathcal{B}_{ij}(Y, \mathcal{C}) \neq \emptyset$ (corresponding to case 3.2.1 in Proposition 5.4). Otherwise, we say that $T_i \Delta T_j$ is a *non-irreducible symmetric difference* (corresponding to case 3.2.2 in Proposition 5.4).

Disjoint sequences.

We continue with the set up above and assume that we have constructed a core using Proposition 5.4. We will now define the notion of *disjoint* sequences, which will be the analogue of the witness set $\mathcal{H}$ in Section 4.1.

Notation. Recall that we defined $\beta_k := k^2 \cdot \xi_k^2(\delta_U)$. Let Y and $W^{\text{core}}(Y, \mathcal{C})$ be the vector spaces given by the core construction above. Let $R' = R/(Y + W^{\text{core}}(Y, \mathcal{C}))$. For $F \in R$ let F' denote the image of F in R'.

Definition 5.6. We say that $H_1, \cdots, H_{\beta_k} \in \mathcal{C} \cap R_2$ is a disjoint sequence with respect to (Y, W^{core}) if the following holds.

1. We have $H_i \in \mathcal{B}(Y + W^{\text{core}}, \mathcal{C})$, i.e. H_i is absolutely irreducible over $Y + W^{\text{core}}$ and $H_i \notin (Y + W^{\text{core}})$.

2. The forms $H_1, \cdots, H_{\beta_k}$ are lin-separated modulo $Y + W^{\text{core}}(Y, \mathcal{C})$.

3. Then for any other $G_1, \cdots, G_{\beta_k}$ with properties (1) and (2) above, we have

$$\sum_{i=1}^{\beta_k} \dim(\mathbb{L}_{R'}(a, G_i')) \leqslant \sum_{i=1}^{\beta_k} \dim(\mathbb{L}_{R'}(a, H_i')).$$

   In other words, among sequences of size β_k with properties (1) and (2) above, the sequence $H_1, \cdots, H_{\beta_k}$ have the maximum sum of dimensions of essential variables in R' .

Definition 5.7 (Good disjoint sequences.)**.** Let $H_1, \cdots, H_{\beta_k}$ be a disjoint sequence with respect to (Y, W^{core}). We say that $H_1, \cdots, H_{\beta_k}$ is a *good* disjoint sequence if there is no form $P \in \mathcal{B}(Y + W^{\text{core}})$ such that $P = \alpha H_i + \beta H_j + Q$ for some $Q \in \mathbb{K}[Y + W^{\text{core}}]$ and $\alpha, \beta \neq 0$.

Lemma 5.8. *Let $H_1, \cdots, H_{\beta_k}$ be a disjoint sequence with respect to (Y, W^{core}) which is not a good disjoint sequence. Let $P \in \mathcal{B}(Y + W^{core}, \mathcal{C})$ such that $P = \alpha H_i + \beta H_j + Q$ for some $Q \in \mathbb{K}[Y + W^{core}]$ and $\alpha, \beta \neq 0$. Then the following holds.*

1. *The sequence*

$$H_1 \cdots, H_{i-1}, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_{\beta_k}, P$$

is lin-separated modulo $Y + W^{core}$.

2. *For any $F \in \mathcal{B}(Y + W^{core}, \mathcal{C})$, the sequence*

$$H_1 \cdots, H_{i-1}, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_{\beta_k}, P, F$$

is not lin-separated modulo $Y + W^{core}$.

Proof. (1) Let $Y' = Y + W^{\text{core}}$. We have that $\mathbb{L}_{R,Y'}(a, P) \subseteq \mathbb{L}_{R,Y'}(a, H_i) + \mathbb{L}_{R,Y'}(a, H_j)$. Since $H_1, \cdots, H_{\beta_k}$ are lin-separated, we have

$$(\mathbb{L}_{R,Y'}(a, H_i) + \mathbb{L}_{R,Y'}(a, H_j)) \cap \sum_{\ell \neq i,j} \mathbb{L}_{R,Y'}(a, H_\ell) = (0).$$

Hence we are done.

(2) By Lemma 3.37, we have that $\dim(\mathbb{L}_{R'}(a, P')) = \dim(\mathbb{L}_{R'}(a, H_i') + \dim(\mathbb{L}_{R'}(a, H_i'))$. Therefore, the sequence obtain by removing H_i, H_j and adding P has the same sum of dimensions of relative spaces in R'. However, it is of length $\beta_k - 1$. If we have a lin-separated sequence after adding F, then this would contradict condition (3) in the definition of disjoint sequences. □

Note that it is not necessary that a disjoint sequence exists, even without the additional good property above. However, if a good disjoint sequence does not exist then we can control the forms in $\mathcal{C}$. In particular, after increasing the vector space Y we will be able to obtain $\mathcal{B}(Y, \mathcal{C}) = \emptyset$. We will use the following iterative process for constructing a good disjoint sequence.

Iterative process for constructing good disjoint sequences. Start with Y and W^{core} as above.

1. If $\mathcal{B}(Y + W^{\text{core}}, \mathcal{C}) \neq \emptyset$, we do the following.

   1.1 If there exists a lin-separated sequence of $H_1, \cdots, H_{\beta_k} \in \mathcal{B}(Y + W^{\text{core}}, \mathcal{C})$: we choose one with maximum dimensional sum and obtain a disjoint sequence. Now we do the following:

   1.1.1 If the disjoint sequence is good, we stop.

   1.1.2 If the disjoint sequence is not good, i.e. there exists $\alpha H_i + \beta H_j + Q \in \mathcal{B}(Y + W^{\text{core}}, \mathcal{C})$. Then we consider

$$H_1 \cdots, H_{i-1}, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_{\beta_k}, P.$$

   We replace Y with $Y + W^{\text{core}} + \sum_{\ell \neq i,j}^{\beta_k} \mathbb{L}_{R,Y+W^{\text{core}}}(a, H_\ell) + \mathbb{L}_{R,Y+W^{\text{core}}}(a, P)$. We run the core constructing iterative process with this new Y.

1.1.2.1 If the core construction ends in $\mathcal{B}(Y, \mathcal{C}) = \emptyset$, we stop.

1.1.2.2 Otherwise, we restart our iterative process with the output (Y, W^{core}) of the core construction process.

1.2 Otherwise, we pick a maximal length lin-separated sequence $H_1, \cdots, H_m \in \mathcal{B}(Y + W^{\text{core}}, \mathcal{C})$ and replace Y with $Y + W^{\text{core}} + \sum_{i=1}^{m} \mathbb{L}_{R,Y+W^{\text{core}}}(a, H_i)$. We run the core constructing iterative process with this new Y.

1.2.1 If the core construction ends in $\mathcal{B}(Y, \mathcal{C}) = \emptyset$, we stop.

1.2.2 Otherwise, we restart our iterative process with the output (Y, W^{core}) of the core construction process.

2. If $\mathcal{B}(Y + W^{\text{core}}, \mathcal{C}) = \emptyset$, we stop.

Lemma 5.9. *The iterative process for good disjoint sequences terminates. At the end of the process, we have one of the following:*

1. *We have an output of the core construction (Y', W^{core}) where Y' is $\Gamma \circ t_{(0,1)}$-lifted strong. Moreover we have a good disjoint sequence $H_1, \cdots, H_{\beta_k}$ with respect to (Y', W^{core}).*

2. *We have a $\Gamma \circ t_{(0,1)}$-lifted strong vector space Y' such that $\mathcal{B}(Y', \mathcal{C}) = \emptyset$.*

Moreover, in both the cases above we have

$$\dim(Y') \leqslant \dim(Y) + a\binom{k}{2}\left(a(\binom{k}{2} + \beta_k) + 2a^2\lambda(k)\binom{k}{2} + a\lambda(k)\binom{k}{2}\right).$$

Proof. Termination. Suppose that for a certain (Y, W^{core}) the iterative process restarts. In each of the two restart steps, i.e. 1.1.2.2 and 1.2.2, we have a maximal lin-separated sequence $F_1, \cdots, F_m \in \mathcal{B}(Y + W^{\text{core}}, \mathcal{C})$ with $m < \beta_k$. Let $Y' = Y + W^{\text{core}} + \sum_{i=1}^{m} \mathbb{L}_{R,Y+W^{\text{core}}}(a, F_i)$. Now any form $F \in \mathcal{B}(Y + W^{\text{core}}, \mathcal{C})$, we have that $\Phi(f, g, Y', F) < \Phi(f, g, Y, F)$, as $F_1, \cdots, F_m, F$ is not lin-separated modulo Y. Therefore, every time a restart occurs the irreducibility potential $\Phi_{\text{irr}}(f, g, Y, \mathcal{C})$ decreases. Since we have an upper bound $\Phi_{\text{irr}}(f, g, Y, \mathcal{C}) \leqslant a\binom{k}{2}$, the iterative process terminates after at most $a\binom{k}{2}$ restart steps.

Dimension bound and strength. Every time we encounter step 1.1.2 or 1.2 above, we first increase the dimension of Y by at most $a(\binom{k}{2}+\beta_k)$. Then if we run the core construction with this Y, the output (Y', W^{core}) satisfies

$$\dim(Y' + W^{\text{core}}) \leqslant \dim(Y) + a(\binom{k}{2} + \beta_k) + 2a^2\lambda(k)\binom{k}{2} + a\lambda(k)\binom{k}{2}.$$

Since there are at most $a\binom{k}{2}$ restart steps, we have an upper bound on dimension of the output Y' (in all cases):

$$\dim(Y') \leqslant \dim(Y) + a\binom{k}{2}\left(a(\binom{k}{2} + \beta_k) + 2a^2\lambda(k)\binom{k}{2} + a\lambda(k)\binom{k}{2}\right).$$

Since if we let c be the summand above such that $\dim(Y') = \dim(Y) + c$, then Y' is $\Gamma \circ t_{(0,1)}$-lifted strong, since Y is $\Gamma \circ t_{(c,1)}$-lifted strong. □

5.4 A fan-in reduction lemma

We now use the core construction (Y, W^{core}) and a good disjoint sequence from Section 5.3 to show a fan-in reduction lemma. As in Section 4, we again define the notions of core preserving and core collapsing forms.

The definitions will be more intricate, because of the changed setting. Let $\mathcal{H}$ denote the good disjoint sequence constructed in the previous subsection.

Definition 5.10. Suppose $G_1, \dots, G_s$ are the forms from $T_1 \Delta T_2$ that were picked when constructing the core, with $s = \lambda(k)$. Suppose H is one of the forms in the good disjoint sequence that does not belong to $T_1 \cup T_2$.

We say H is core preserving for $T_1 \Delta T_2$ if the following happens: There exists at least $\xi_k^2(\delta_U)$ many forms among the G_i, say $G_1, \dots, G_t$ such that

- The forms H and G_i are lin-separated modulo Y for $i \leqslant t$, and
- $\left|\text{span}_{\mathbb{K}}\{G_i, H\} \cap T_i \Delta T_j\right| = 1$ for $i \leqslant t$.

We say H is core collapsing for $T_1 \Delta T_2$ if it is not core preserving.

Recall that we want to make claims about the image of $\mathcal{C}$ under a targeted quotient of a form H from our disjoint sequence. Depending on the lifted strength of H, the targeted quotient takes on one of two different forms. If H is already μ_{k-1}-lifted strong in R, then the targeted quotient is isomorphic to the map $R/(H)$. On the other hand, if H is not already μ_{k-1}-lifted strong, then the vector space $W = AH_R(\mu_{k-1}, H)$ is a vector space of linear forms. The relative linear space with respect to Y of H is exactly the image of this vector space. It will be convenient to analyse these two cases separately.

If H is already μ_{k-1}-lifted strong in R, then we will directly analyse the image of the circuit under the quotient map. If not, then we will analyse the image of the circuit under the composition of two maps: the first is a general projection of Y, and the second is a targeted quotient of the image of H. We will show that under the composition of these maps, the image of $\mathcal{C}$ has large pairwise rank. We can alternatively write this composition as a targeted quotient of H, followed by a general projection of the remaining elements of Y. The fact that the pairwise rank is large under the composition will imply that it will be large under the first of these maps.

Lemma 5.11. *Suppose H is core preserving for $T_1 \Delta T_2$. Let $W := AH_R(\mu_{k-1}, H)$ be the μ_{k-1}-lifted strong space obtained applying Corollary 3.17 to $\text{span}_{\mathbb{K}}\{H\}$ in R. Let $\varphi_{W,\alpha} : R[y] \to R' := R[y]/I_{W,\alpha}$ be the graded quotient corresponding to $\alpha \in \mathbb{K}^{\dim(W)}$. For a general $\alpha \in \mathcal{Z}(H) \subseteq \mathbb{K}^{\dim(W)}$, the image of T_1, T_2 under the map $\varphi_{W,\alpha}$ has pairwise rank at least $\xi_{k-1}(\delta_W)$.*

Proof. As in the definition Definition 5.10, we assume that $G_1, \dots, G_t$ are the forms from the core that are lin-separated modulo Y with H, and such that $\left|\text{span}_{\mathbb{K}}\{G_i, H\} \cap T_i \Delta T_j\right| = 1$. Here we have $t \geqslant \xi_k^2(\delta_U)$ by definition. We will show that under a targeted quotient $\varphi_{W,\alpha}$, the images of these G_i witness the pairwise rank of T_1, T_2. Fix G_1, and assume without loss of generality that $G_1 \in T_1 \setminus T_2$.

We perform a case analysis, based on the lifted strength of H. First, let us assume that H is already μ_{k-1}-lifted strong. In this case, the map $\varphi_{W,\alpha}$ is the same as the map $R \to R/(H)$. Suppose G_1 is a quadratic form. If G_1 becomes reducible in $R/(H)$, then $G_1 = H + \ell_1\ell_2$ in R. However, this contradicts the fact that G_1 and H are lin-separated modulo Y with H absolutely irreducible modulo Y. Therefore, if G_1 is a quadratic, and there is a $F \in T_2 \setminus T_1$ such that G_1 and F have non trivial gcd in $R/(H)$, it must be that $F \in \text{span}_{\mathbb{K}}\{H, G_1\}$, in contradiction to the definition of H. If G_1 is a linear form, then any such F must be a quadratic form, and it can be written as $F = H + G_1\ell$ for a linear form ℓ. Now G_1 is in the core, therefore every element of $T_1 \Delta T_2$ is absolutely reducible over Y, and this includes F. This is in contradiction to H and G_1 being lin-separated modulo Y. This shows our result in the case when H is μ_{k-1}-lifted strong.

We now analyse the case when H is not μ_{k-1}-lifted strong, and therefore W is a vector space of linear forms. We start with a basis of $W \cap Y_1$ (call this basis M_1, and extend this to a basis of Y (call the new elements M_2) and also extend this to a basis of W (call the new elements M_3). Under the map $\varphi_{W,\alpha}$, the elements of M_1 and M_3 are mapped to multiples of a new variable. On M_2, the map acts as identity. We can now compose this map with a general projection of the image of Y, call this map ψ_Y. On the other hand, we can also write the composed map $\psi_Y \circ \varphi_{W,\alpha}$ as the composition of a general quotient of Y (call this map ψ'_Y), followed by a targeted quotient of the image of M_3, denoted φ'_W.

We will show that under the composition of the above maps, T_1, T_2 have pairwise rank at least $\xi_{k-1}(\delta_W)$. This implies the same is true under just the map $\varphi_{W,\alpha}$.

Suppose $F \in T_2 \setminus T_1$ is such that $\gcd(\psi_Y \circ \varphi_{W,\alpha}(F), \psi_Y \circ \varphi_{W,\alpha}(G_1)) \neq 1$. We now do a case analysis, based on whether $T_1 \Delta T_2$ is an irreducible symmetric difference, or a reducible one.

If $T_1 \Delta T_2$ is an irreducible symmetric difference, then $\psi'_Y(G_1)$ is irreducible. Since G_1 and H are lin-separated, this implies that $\psi_Y \circ \varphi_{W,\alpha}(G_1)$ is also irreducible. Therefore, if F exists, then it must be that $\psi_Y \circ \varphi_{W,\alpha}(F)$and $\psi_Y \circ \varphi_{W,\alpha}(G_1)$ are associate. This implies that some linear combination of F and G_1 lies in the kernel of $\psi_Y \circ \varphi_{W,\alpha} = \varphi'_W \circ \psi'_Y$, whence $H \in \mathrm{span}_{\mathbb{K}}\{F, G_1\}$. This contradicts the definition of H.

On the other hand, suppose $T_1 \Delta T_2$ is a reducible symmetric difference. In this case, either G_1 is a linear form and so is $\psi'_Y(G_1)$, or G_1 is a quadratic form and $\psi'_Y(G_1)$ is a product of linear forms. The same is true of F and $\psi_Y(F)$. The forms G_1 and F are relatively prime in R, therefore, the only common factor of $\psi'_Y(F)$ and $\psi'_Y(G_1)$ is z. Suppose $\psi'_Y(G_1) = \ell_1\ell_2$ and $\psi'_Y(F) = \ell_3\ell_4$. For $\gcd(\varphi'_W \circ \psi'_Y(F), \varphi'_W \circ \psi'_Y(G_1))$ to be different from 1, either one of ℓ_1, ℓ_2 has to become equal to ℓ_3, ℓ_4 under the second part of the map (which cannot happen, since H is a quadratic and these forms are linear), or $\ell_1\ell_2$ and $\ell_3\ell_4$ must be associate under the second part of the map, which would imply that their difference is a multiple of $\psi'_Y(H)$. This in turn again implies that $H \in \mathrm{span}_{\mathbb{K}}\{F, G_1\}$, a contradiction.

Therefore, such an F cannot exist in either case, and the pairwise rank of T_1, T_2 is at least $\xi_k^2(\delta)$. By construction, this value is at least $\xi_{k-1}(\delta_W)$. □

Lemma 5.12. *Suppose $H \in \mathcal{H} \setminus (T_1 \cup T_2)$. Assume that there are c other elements in $\mathcal{H} \setminus (T_1 \cup T_2)$ that are core collapsing for $T_1 \Delta T_2$. Let $W := AH_R(\mu_{k-1}, H)$ be the μ_{k-1}-lifted strong space obtained applying Corollary 3.17 to $\mathrm{span}_{\mathbb{K}}\{H\}$ in R. Let $\varphi_{W,\alpha} : R[y] \to R' := R[y]/I_{W,\alpha}$ be the graded quotient corresponding to $\alpha \in \mathbb{K}^{\dim(W)}$. For a general $\alpha \in \mathcal{Z}(H) \subseteq \mathbb{K}^{\dim(W)}$, the image of T_1, T_2 under the map $\varphi_{W,\alpha}$ has pairwise rank at least c.*

Proof. Without loss of generality, we assume that the c elements are $H_1, \ldots, H_c$. We focus on H_1. Recall that the core consists of $G_1, \ldots, G_s$ from $T_1 \Delta T_2$. Among these s forms, there are at most $2a$ forms such that G_i and H_1 are not lin-separated modulo Y among the remaining forms, there are at most $\xi_k^2(\delta)$ many forms such that $\mathrm{span}_{\mathbb{K}}\{G_i, H_1\} \cap T_1 \Delta T_2$ has only one element. In particular, there exists an index i_1 such that G_{i_1} and H_1 are lin-separated modulo Y, and such that $F_1 \in \mathrm{span}_{\mathbb{K}}\{H_1, G_{i_1}\} \cap T_1 \Delta T_2$. Similarly, we construct $F_2, \ldots, F_c$. Note that in this case, G_{i_j} cannot be linear: if it were, then F_i would also be linear, and this would contradict the fact that G_{i_j} and F_i are relatively prime.

We now claim that $F_1, \ldots, F_c$ are witnesses to the pairwise rank of T_1, T_2 under the map $\varphi_{W,\alpha}$. Suppose towards a contradiction that F_1 is no longer in the symmetric difference. Suppose we write $F_1 = \gamma H_1 + \gamma' G_{i_1}$. If G_{i_1} is an absolutely irreducible quadratic over Y, then so is F_1. Further, H and F_1 are lin-separated modulo Y. We are now in the same situation as in the proof of Lemma 5.11, and we can deduce that if F' is such that F_1 and H have non trivial gcd under the map $\varphi_{W,\alpha}$, then $H \in \mathrm{span}_{\mathbb{K}}\{F', F_1\}$. Therefore,

$H \in \mathrm{span}_{\mathbb{K}}\{F', H_1, G_{i_1}\}$, which implies $F' \in \mathrm{span}_{\mathbb{K}}\{H, H_1, G_{i_1}\}$. For the same reason as above, F' must be absolutely irreducible over Y, W^{core}. This is a contradiction to the good disjoint sequence property. □

As in Section 4, while the above lemmas are stated for $T_1 \Delta T_2$, they also apply to every other symmetric difference. The rest of the argument is completed the same way. For each $T_i \Delta T_j$, we let $\mathcal{H}^{i,j} \subset \mathcal{H}$ be the set of forms which are core collapsing for $T_i \Delta T_j$. If $\left|\mathcal{H}^{i,j}\right| \geqslant \xi_k^2(\delta_U) + 1$, then for any $H \in \mathcal{H}$, either one of T_i, T_j vanishes under a targeted quotient of H (which happens if H belongs to one of T_i, T_j), or their pairwise rank is at least $\xi_k^2(\delta_U)$, by either Lemma 5.11 or Lemma 5.12.

Let $\mathcal{H}'$ be the union of $\mathcal{H}^{i,j}$ over those i, j such that $\left|\mathcal{H}^{i,j}\right| \leqslant \xi_k^2(\delta_U)$. The size of $\mathcal{H}'$ is at most $\binom{k}{2}\xi_k^1(\delta_U)$, and by the choice of β_k, we can pick $H \in \mathcal{H} \setminus \mathcal{H}'$. We claim that this H has the required property. The pairs i, j with $\mathcal{H}^{i,j} \geqslant \xi_k^2(\delta_U)+1$ have already been handled. If i, j is such that $\mathcal{H}^{i,j} \leqslant \xi_k^2(\delta_U)$, then by construction either H belongs to one of T_i, T_j or H is core preserving for $T_i \Delta T_j$.

6 Putting it all together

In this final section, we will combine the results from the previous section and prove our rank bounds. The proof proceeds by induction on the top fan-in k.

Theorem 6.1. *For each integer $k \geqslant 2$, there exists functions $\mu_k : \mathbb{N}^2 \to \mathbb{N}^2$ and $\xi : \mathbb{N}^2 \to \mathbb{N}$ with the following property: Let $S = \mathbb{K}[x_1, \cdots, x_N]$ be a polynomial ring. If $U \subseteq S_{\leqslant 2}$ is a graded vector space that is μ_k-lifted strong, and $\mathcal{C}$ is any simple and minimal $\Sigma^k\Pi\Sigma\Pi^2$ identity in $R := S/(U)$ then the rank of $\mathcal{C}$ is at most $\xi_k(\delta_U)$.*

Proof. We prove the theorem by induction on k. The base case is $k = 2$. In any ring R which is a UFD, there are no simple and minimal depth four identities with fan-in two. We define $\mu_2(d_1, d_2) := (A(3,1) + 3(d_1 + d_2), A(3,2) + 3(d_1 + d_2)$, where A is the function from Corollary 3.13. By the corollary, if U is any vector space that is μ_2-strong, then $S/(U)$ is a UFD, thus we can define $\xi_2(d_1, d_2) := 0$. This completes the base case.

We assume ξ_{k-1} and μ_{k-1} have been defined. Define $\xi_k^2 : \mathbb{N}^2 \to \mathbb{N}$ as $\xi_k^1(\delta) := \xi_{k-1}\left(\left(C_{\mu_{k-1}} \circ t_1\right)(\delta)\right) + 1$ and $\xi_k^2 : \mathbb{N}^2 \to \mathbb{N}$ as $\xi_k^2(\delta) := \xi_{k-1}\left(\left(C_{2\mu_{k-1}} \circ t_1\right)(\delta)\right) + 1$. Define the function $M_k(\delta) : \mathbb{N}^2 \to \mathbb{N}$ as

$$M_k(\delta) := \max_{\delta' \in \mathbb{N}^2, \|\delta'\| = \binom{k}{2}\xi_k^1(\delta)} \mu_{k-1}(\delta + \delta' + (0, k^2\xi_k^2(\delta))).$$

Note that these are the same functions defined in Lemma 4.1, instantiated with our inductive bounds. Define $b_k(\delta) := k^2\left(\xi_k^1(\delta) + 1\right)$. Define $g_k^{dim}(\delta) := 4^{b_k(\delta)+1} + b_k(\delta)^3 + b_k(\delta) + 16$. Define $g_k^{ps}(\delta) := 3^{b_k(\delta)} M_k(\delta)$. Finally, let μ_k be the function μ_k^{low} defined in Lemma 5.3, instantiated with the above parameters. This completes the definition of μ_k.

We now define $\xi_k(\delta)$. This will be the max of three terms, one coming from Lemma 4.1, one coming from Lemma 5.3, and one coming from the rank bound of Kayal and Saraf [KS09]. The first of these terms is $\xi_{k-1}\left(C_{2\mu_{k-1}}(\delta + b_k(\delta))\right) \cdot 72 \cdot 2^{|\delta|_1} \cdot 6^{2 \cdot b_k(\delta)} \cdot C_{2\mu_{k-1}}(\delta + b_k(\delta))^2$. The second of these terms is $\xi_{k-1}\left(C_{2\mu_{k-1}}(\delta + \rho_k(\delta))\right) \cdot 72 \cdot 2^{|\delta|_1} \cdot 6^{2 \cdot \rho_k(\delta)} \cdot C_{2\mu_{k-1}}(\delta + \rho_k(\delta))^2$, where ρ_k is the function defined in Lemma 5.3 instantiated with the parameters above. The third term is $\xi_k^{KS}\left(C_{2\mu_{k-1}}(\delta + \rho_k(\delta))\right) \cdot 72 \cdot 2^{|\delta|_1} \cdot 6^{2 \cdot \rho_k(\delta)} \cdot C_{2\mu_{k-1}}(\delta + \rho_k(\delta))^2$, where ξ_k^{KS} is the rank bound for $\Sigma^k\Pi\Sigma$ identities over $\mathbb{C}$ proved in [KS09].

We now prove that the functions ξ_k and μ_k have the required property. Let U be any vector space with dimension sequence δ_U, and let $\mathcal{C}$ be any $\Sigma^k\Pi\Sigma\Pi^2$ identity in $R = S/(U)$. Suppose U is μ_k-strong. Note in particular that U is $\mu_{k-1} \circ t_1$-strong. By Lemma 4.1 instantiated with the above parameters, we deduce that one of three things happens (corresponding to the three cases).

- The first case is that $\dim \operatorname{span}_{\mathbb{K}}\{T_i \Delta T_j\} \leqslant b_k(\delta_U)$ for some i, j. In this case, if we define W to be the vector space obtained by starting with a basis of this span making the resulting vector space μ_{k-1}-lifted strong Corollary 3.17, then the dimension sequence of W is bounded coordinate wise by $C_{2\mu_{k-1}}(\delta + b_k(\delta))$. The image of $\mathcal{C}$ under a graded quotient of W has top fan-in $k-1$, and therefore by induction it has rank at most $\xi_{k-1}\left(C_{2\mu_{k-1}}(\delta + b_k(\delta))\right)$. We can deduce a rank bound on $\mathcal{C}$ by lifting the above rank bound using Proposition 3.41, and this rank bound is exactly the first of the two terms above that we maximised over to define ξ_k. Therefore, in this case the desired conclusion holds.

- The second case is the existence of a H that is μ_{k-1}-lifted strong, such that the image of $\mathcal{C}$ under the map $R \to R/(H)$ has pairwise rank at least $\xi^2_{k-1}(\delta)$, therefore also rank at least $\xi^2_{k-1}(\delta)$. However note that in this case, $R/(H)$ is isomorphic to a quotient of S by a μ_{k-1}-strong vector space of dimension sequence $\delta + (0, 1)$, and $\xi^2_{k-1}(\delta)$ is larger than $\xi_{k-1}(\delta + (0, 1))$. This contradicts the induction hypothesis, and therefore under the above parameters, this case cannot happen.

- The final case is the existence of a vector space V of dimension at most $g^{\dim}_k(\delta)$ such that every quadratic in $\mathcal{C}$ is $g^{ps}_k(\delta)$-close to V. In this case, by Lemma 5.3, we can deduce that one of three things happens (here we use the fact that $\mu_k \geqslant \mu^{low}_k$ to be able to invoke this lemma): either there is a vector space Y of dimension at most $\rho_k(\delta_U)$ whose algebra contains a symmetric difference, or there is a vector space Y of dimension at most $\rho_k(\delta_U)$ such that every form in $\mathcal{C}$ is absolutely reducible over $\mathbb{K}[Y]$, or there exists a form H such that a targeted quotient with respect to H kills a gate, and maintains pairwise rank at least $\xi^2_{k-1}(\delta)$. The first and the third case above are identical to the two cases above, and the same arguments apply. In the second case, the image of $\mathcal{C}$ under a graded quotient of Y is a $\Sigma^k\Pi\Sigma$ identity, and therefore a rank bound can be deduced by the proof of [KS09]. Lifting this rank bound gives us the desired conclusion. This completes this case. □

We now restate and prove the main application of our technical theorem above: rank bounds for $\Sigma^k\Pi\Sigma\Pi^2$ identities in polynomial rings over characteristic zero fields.

Theorem 1.3. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $S := \mathbb{K}[x_1, \ldots, x_n]$. There is a function $R : \mathbb{N} \to \mathbb{N}$, independent of $\mathbb{K}, n$, such that for any simple and minimal $\Sigma^k\Pi\Sigma\Pi^2$ identity Φ over S, we have $\operatorname{rank}(\Phi) \leqslant R(k)$.*

Proof. Let ξ_k, μ_k be the functions defined in Theorem 6.1. The zero vector space is arbitrarily strong, in particular it is μ_k-strong. Therefore, any simple minimal identity in S has rank bound at most $\xi_k((0, 0))$. Thus, we can set our function $R(k) := \xi_k((0, 0))$. □

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