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# A symmetric determinantal lower bound for diagonal power sums via polar degree

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**Abstract**

The symmetric determinantal complexity $\mathrm{sdc}(f)$ of a polynomial $f$ is the least $m$ such that $f = \mathrm{Det}(M)$ for an $m \times m$ symmetric matrix $M$ of affine-linear forms. We prove, over $\mathbb{C}$, that

$$\mathrm{sdc}\left(\sum_{i=1}^{n} x_i^n\right) \geq \left(\frac{1}{2e} - o(1)\right) n^2.$$

The result is a symmetric companion to the author's non-symmetric polar-degree preprint [8]. The method parallels that work, but the proof below is self-contained and redoes the load-bearing local incidence analysis in the symmetric setting. The general theorem is the following. If $X = V(f) \subset \mathbb{P}^{N-1}$ is a smooth degree-$d$ hypersurface, $N \geq 3$, and $f = \mathrm{Det}(A_0 + \sum_{i=1}^{N} x_i A_i)$ with all $A_i$ symmetric of size $m$, then

$$\delta_{\mathrm{top}}(X) = d(d-1)^{N-2} \leq 2^{N-2}\binom{m}{N-1}.$$

The proof uses the symmetric rank-one kernel incidence $\mathcal{M}(z,x)u = 0$, where $\mathcal{M} = zA_0 + \sum_i x_i A_i$. At a genuine polar point, $\mathcal{M}$ has rank $m-1$, and the symmetric local normal form

$$\mathcal{M} = \begin{pmatrix} B & c \\ c^{\mathsf{T}} & s \end{pmatrix}, \qquad \det B \in \mathcal{O}^{\times},$$

eliminates the unique projective kernel line scheme-theoretically: $u = (-B^{-1}c, 1)$ and $\det \mathcal{M} = (\det B)(s - c^{\mathsf{T}} B^{-1} c)$. On this local graph, $\mathrm{adj}(\mathcal{M}) = (\det B) u u^{\mathsf{T}}$ along the determinant hypersurface, so the lifted conormal forms $u^{\mathsf{T}} A_i u$ are a common unit multiple of the ordinary partial derivatives $\partial_i f$. Hence the lifted polar equations cut the ordinary polar slice, up to units, and every genuine lifted polar point is a zero-dimensional scheme-theoretic isolated solution. Multihomogeneous Bezout on $\mathbb{P}^{N}_{[z:x]} \times \mathbb{P}^{m-1}_{[u]}$ then gives

$$[H^N U^{m-1}]\, H(H+U)^m (2U)^{N-2} = 2^{N-2}\binom{m}{N-1}.$$

For $F_n = \sum_i x_i^n$ this bounds $n(n-1)^{n-2}$ and yields the stated constant $1/(2e)$. More generally, for $F_{N,d} = \sum_{i=1}^{N} x_i^d$ the same theorem gives $\mathrm{sdc}(F_{N,d}) \geq (1/(2e) - o_N(1))N(d-1)$ as $N \to \infty$,

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uniformly for $d \geq 2$. We also give an explicit symmetric determinantal representation of $F_{N,d}$ of size $2N(d+1)+1$, showing that the diagonal lower bounds are non-vacuous and tight up to a constant factor. The result is for exact symmetric determinantal complexity in characteristic zero; it is not a border-complexity statement and it is not a uniform positive-characteristic theorem.

**Keywords:** symmetric determinantal complexity; polar degree; Gauss map; dual variety; algebraic complexity; multihomogeneous Bezout; Valiant's hypothesis.

**Subject classification:** 68Q17; 14N05; 14Q20; 15A15.

**Status of this work.** Three points should be stated plainly. *(i) Provenance.* This paper is a symmetric companion to the author's non-symmetric polar-degree preprint [8]. The method parallels that preprint, but the proof below is self-contained: in particular, the symmetric isolatedness step is reproved directly by a Schur-complement normal form rather than imported from the non-symmetric argument. The present statement arose from a follow-up adversarial prompt asking whether the symmetric specialization of the incidence argument is valid or whether the coincidence of the left and right kernels introduces a new failure mode. The prompt protocol is described in Section 7 and Appendix A. *(ii) What is proved.* The main inequality, the isolatedness lemma, the multidegree computation, and the $1/(2e)$ asymptotic extraction are proved in full below. The proof is not a heuristic specialization of the non-symmetric case: the symmetric Schur-complement normal form is used directly to identify the completed local incidence ring at a genuine polar point with the completed local ring of the ordinary polar slice, up to units. *(iii) Scope.* Everything is over $\mathbb{C}$, or equivalently any algebraically closed field of characteristic zero for the arguments used here. The theorem is false if stated uniformly over all fields of characteristic different from two; see Remark 3.

# 1 Introduction

## 1.1 Symmetric determinantal complexity

Let $f \in \mathbb{C}[x_1, \ldots, x_N]$. Its *symmetric determinantal complexity*, denoted $\mathrm{sdc}(f)$, is the least integer $m$ for which there exist symmetric matrices $A_0, A_1, \ldots, A_N \in \mathbb{C}^{m \times m}$ such that

$$f(x) = \mathrm{Det}\left( A_0 + \sum_{i=1}^{N} x_i A_i \right).$$

This is a restriction of ordinary affine determinantal complexity $\mathrm{dc}(f)$, in which the representing matrix need not be symmetric. Determinantal complexity is one of the central algebraic models in Valiant's program [9]; quadratic lower bounds for the permanent are known [6], but superquadratic and superpolynomial lower bounds remain major open problems.

The ordinary determinant model has two kernel variables at a smooth point of the determinant hypersurface: a left kernel line and a right kernel line. The symmetric model has only one. At rank $m-1$ the left and right kernels coincide, so the conormal direction of a symmetric determinant is controlled by quadratic forms

$$u^{\mathsf{T}} A_i u$$

in one projective kernel variable $[u] \in \mathbb{P}^{m-1}$ rather than by bilinear forms $u^{\mathsf{T}} A_i v$ in two projective kernel variables. This reduction is exactly what improves the Bezout constant from $1/(4e)$ in the ordinary polar-degree count of the companion preprint [8] to $1/(2e)$ here.

The improvement is model-specific. Since $\mathrm{sdc}(f) \geq \mathrm{dc}(f)$, any lower bound for $\mathrm{dc}(f)$ is automatically a lower bound for $\mathrm{sdc}(f)$, but the argument in this paper is sharper because it uses the symmetric geometry directly.

### 1.2 Main results

The key geometric input is the top polar degree of a smooth hypersurface. If $X = V(f) \subset \mathbb{P}^{N-1}$ is smooth of degree $d \geq 2$, then the gradient map

$$\gamma_X : X \longrightarrow (\mathbb{P}^{N-1})^\vee, \qquad [x] \longmapsto [\partial_1 f(x) : \cdots : \partial_N f(x)]$$

is a morphism and $\gamma_X^* \mathcal{O}(1) = \mathcal{O}_X(d-1)$. We define $\delta_{\mathrm{top}}(X)$ as the length of the inverse image of a general codimension-$(N-2)$ linear subspace of the dual projective space. Equivalently, for a smooth hypersurface,

$$\delta_{\mathrm{top}}(X) = \int_X c_1(\mathcal{O}_X(d-1))^{N-2} = d(d-1)^{N-2}.$$

This is also the classical dual-class degree in characteristic zero, but the proof uses the polar intersection length, not any closedness property of dual varieties.

**Theorem 1** (Symmetric polar-degree bound)**.** *Let $N \geq 3$, and let $f \in \mathbb{C}[x_1, \ldots, x_N]$ be homogeneous of degree $d \geq 2$ such that $X = V(f) \subset \mathbb{P}^{N-1}$ is smooth. Suppose*

$$f(x) = \mathrm{Det}\left( A_0 + \sum_{i=1}^{N} x_i A_i \right)$$

*for symmetric $m \times m$ complex matrices $A_i$. Then*

$$\delta_{\mathrm{top}}(X) = d(d-1)^{N-2} \leq 2^{N-2} \binom{m}{N-1}.$$

**Corollary 1** (Degree-$n$ diagonal power sums)**.** *For $F_n = x_1^n + \cdots + x_n^n$ over $\mathbb{C}$,*

$$\mathrm{sdc}(F_n) \geq \left( \frac{1}{2e} - o(1) \right) n^2.$$

**Corollary 2** (General diagonal power sums)**.** *Let*

$$F_{N,d} = x_1^d + \cdots + x_N^d$$

*over $\mathbb{C}$, with $N \geq 3$ and $d \geq 2$. Then*

$$d(d-1)^{N-2} \leq 2^{N-2} \binom{\mathrm{sdc}(F_{N,d})}{N-1},$$

*and hence*

$$\mathrm{sdc}(F_{N,d}) \geq \left( \frac{d(d-1)^{N-2}(N-1)!}{2^{N-2}} \right)^{1/(N-1)}.$$

*In particular, uniformly for $d \geq 2$ as $N \to \infty$,*

$$\mathrm{sdc}(F_{N,d}) \geq \left( \frac{1}{2e} - o_N(1) \right) N(d-1).$$

The $n$-variable degree-$n$ case is stated separately because it is the most compact asymptotic form and parallels the companion non-symmetric result.

### 1.3 Why the symmetric isolatedness step is the whole proof

The intended incidence count is visually simple. Homogenize the representing matrix:

$$\mathcal{M}(z,x) = zA_0 + \sum_{i=1}^{N} x_i A_i, \qquad \operatorname{Det}\mathcal{M} = z^{m-d} f(x).$$

A genuine polar point of $X$ is lifted to a point $([z:x],[u]) \in \mathbb{P}^N \times \mathbb{P}^{m-1}$ satisfying

$$\mathcal{M}(z,x)u = 0, \qquad q_j(u^{\mathsf{T}}A_1u,\dots,u^{\mathsf{T}}A_Nu) = 0, \qquad h(z,x) = 0.$$

The $m$ kernel equations have class $H+U$, the $N-2$ polar equations have class $2U$, and the slice has class $H$. Thus a square multihomogeneous Bezout count would give

$$[H^N U^{m-1}]\, H(H+U)^m (2U)^{N-2} = 2^{N-2}\binom{m}{N-1}.$$

But Bezout bounds only isolated zero-dimensional contributions. The load-bearing issue is therefore not the coefficient extraction; it is whether a genuine lifted polar point can lie on a positive-dimensional component of the symmetric incidence scheme. In particular, one must rule out the possibility that the kernel equation acquires extra fiber dimension near a rank-drop point, or that the quadratic forms $u^{\mathsf{T}}A_iu$ vanish in a way that creates a spurious component through a genuine point. Section 3 proves precisely this local isolatedness statement by a symmetric Schur-complement normal form. This is the point at which the symmetry constraint must be checked rather than assumed.

## 2 Preliminaries

We work over $\mathbb{C}$. Let

$$M(x) = A_0 + \sum_{i=1}^{N} x_i A_i$$

with all $A_i$ symmetric. Its homogenization is

$$\mathcal{M}(z,x) = zA_0 + \sum_{i=1}^{N} x_i A_i.$$

Since $\operatorname{Det} M(x) = f(x)$ is homogeneous of degree $d$, one has

$$\operatorname{Det}\mathcal{M}(z,x) = z^{m-d} f(x). \tag{1}$$

Here necessarily $m \geq d$ unless $f = 0$, because the determinant of an affine $m \times m$ matrix has degree at most $m$.

We use the identity

$$\frac{\partial}{\partial x_i}\operatorname{Det}\mathcal{M} = \operatorname{tr}\big(\operatorname{adj}(\mathcal{M})A_i\big). \tag{2}$$

At a symmetric rank-$(m-1)$ matrix, the kernel is a line and the adjugate is a nonzero symmetric rank-one matrix with image equal to that kernel line.

**Lemma 1** (Rank at genuine smooth points)**.** *Let $[z_0 : x_0] \in \mathbb{P}^N$ satisfy $z_0 \neq 0$ and $f(x_0) = 0$, with $[x_0] \in X$ smooth. Then*

$$\operatorname{rank} \mathcal{M}(z_0, x_0) = m - 1.$$

*Consequently the projective kernel line $[u_0] \in \mathbb{P}^{m-1}$ is unique.*

*Proof.* Since $\operatorname{Det} \mathcal{M}(z_0, x_0) = 0$, the rank is at most $m-1$. If the rank were at most $m-2$, then all $(m-1) \times (m-1)$ cofactors would vanish, so $\operatorname{adj}(\mathcal{M}(z_0, x_0)) = 0$. By (2), every partial derivative of $\operatorname{Det} \mathcal{M}$ with respect to the $x_i$ would vanish at $[z_0 : x_0]$. But by (1), and because $z_0 \neq 0$,

$$\frac{\partial}{\partial x_i} \operatorname{Det} \mathcal{M}(z_0, x_0) = z_0^{m-d} \partial_i f(x_0).$$

The point $[x_0] \in X$ is smooth, so not all $\partial_i f(x_0)$ vanish. This contradiction proves the rank is $m-1$. A rank-$(m-1)$ matrix has one-dimensional kernel. □

**Lemma 2** (Symmetric conormal identity)**.** *At a point as in Lemma 1, let $0 \neq u_0 \in \ker \mathcal{M}(z_0, x_0)$. Then there is a scalar $\alpha \neq 0$ such that*

$$\partial_i f(x_0) = \alpha \, u_0^{\mathsf{T}} A_i u_0 \qquad (1 \leq i \leq N).$$

*In particular,*

$$[\partial_1 f(x_0) : \cdots : \partial_N f(x_0)] = [u_0^{\mathsf{T}} A_1 u_0 : \cdots : u_0^{\mathsf{T}} A_N u_0].$$

*Proof.* At a symmetric rank-$(m-1)$ matrix, the adjugate is a nonzero scalar multiple of $u_0 u_0^{\mathsf{T}}$. Thus

$$\operatorname{adj}(\mathcal{M}(z_0, x_0)) = \beta u_0 u_0^{\mathsf{T}}$$

for some $\beta \neq 0$. By (2),

$$\frac{\partial}{\partial x_i} \operatorname{Det} \mathcal{M}(z_0, x_0) = \beta \, u_0^{\mathsf{T}} A_i u_0.$$

Using (1) and $z_0 \neq 0$ gives $z_0^{m-d} \partial_i f(x_0) = \beta u_0^{\mathsf{T}} A_i u_0$. Take $\alpha = \beta z_0^{d-m}$. □

## 3 The symmetric local normal form

This section proves the isolatedness statement needed for Bezout. We state it in the form used later.

Choose general linear forms $q_1, \ldots, q_{N-2}$ on the dual coordinates so that

$$P_q = \{[x] \in X : q_j(\partial_1 f(x), \ldots, \partial_N f(x)) = 0 \text{ for } 1 \leq j \leq N-2\}$$

is a reduced zero-dimensional scheme of length $\delta_{\mathrm{top}}(X)$. This is possible because the partial derivatives define a base-point-free linear system $|\mathcal{O}_X(d-1)|$ on the smooth variety $X$; iterated Bertini gives a reduced complete intersection of degree $d(d-1)^{N-2}$.

Choose a linear form $\ell(x)$ nonzero on the finite set $P_q$ and set

$$h(z, x) = \ell(x) - z.$$

Each $[\xi] \in P_q$ determines the affine-slice representative $[z : x] = [\ell(\xi) : \xi]$, which has $z \neq 0$.

**Lemma 3** (Symmetric isolatedness)**.** *Let $p = ([z_0 : x_0], [u_0])$ be the lift of a point of $P_q$ to the symmetric incidence scheme*

$$\mathcal{M}(z, x)u = 0, \qquad q_j(u^{\mathsf{T}} A_1 u, \ldots, u^{\mathsf{T}} A_N u) = 0 \ (1 \leq j \leq N-2), \qquad h(z, x) = 0 \tag{3}$$

*in $\mathbb{P}^N_{[z:x]} \times \mathbb{P}^{m-1}_{[u]}$. Then $p$ is a zero-dimensional scheme-theoretically isolated solution of (3).*

*Proof.* By Lemma 1, $\mathcal{M}(z_0, x_0)$ has rank $m-1$. Since it is symmetric, its kernel is a one-dimensional radical. After a constant congruence change of basis $\mathcal{M} \mapsto P^\mathsf{T}\mathcal{M}P$, which gives an isomorphic local incidence problem via $u = Pu'$ and preserves the quadratic forms since $u'^\mathsf{T}(P^\mathsf{T}A_iP)u' = u^\mathsf{T}A_iu$, we may assume that the kernel line at $p$ is spanned by the last coordinate vector. The determinant is multiplied only by the nonzero scalar $(\det P)^2$, which is harmless in all local unit comparisons below. Because the kernel is the radical of the symmetric bilinear form, the restriction to any complementary $(m-1)$-dimensional subspace is nondegenerate. Thus, in a Zariski neighborhood of $[z_0 : x_0]$, write

$$\mathcal{M} = \begin{pmatrix} B & c \\ c^\mathsf{T} & s \end{pmatrix}, \qquad \det B \in \mathcal{O}^\times.$$

Work on the affine chart $u_m = 1$ and write $u = (u', 1)^\mathsf{T}$. The kernel equations are

$$Bu' + c = 0, \qquad c^\mathsf{T}u' + s = 0.$$

Since $B$ is invertible in the local ring, the first $m-1$ equations are equivalent to

$$u' = -B^{-1}c.$$

With

$$g = s - c^\mathsf{T}B^{-1}c,$$

the last equation becomes

$$c^\mathsf{T}u' + s = c^\mathsf{T}(u' + B^{-1}c) + g.$$

Therefore the ideal generated by $\mathcal{M}u$ is exactly

$$\left(u' + B^{-1}c,\ g\right)$$

in the local ring. Scheme-theoretically, the kernel incidence is the graph

$$u = \nu := \begin{pmatrix} -B^{-1}c \\ 1 \end{pmatrix}$$

over the hypersurface $g = 0$. The determinant is

$$\operatorname{Det}\mathcal{M} = (\det B)g. \tag{4}$$

On $g = 0$, the matrix $\mathcal{M}$ has kernel spanned by $\nu$ and rank $m-1$ because $B$ remains invertible. Its adjugate is therefore a scalar multiple of $\nu\nu^\mathsf{T}$. The lower-right cofactor equals $\det B$, while the lower-right entry of $\nu\nu^\mathsf{T}$ is 1, so

$$\operatorname{adj}(\mathcal{M}) = (\det B)\nu\nu^\mathsf{T} \qquad \text{on } g = 0. \tag{5}$$

Hence, in the local quotient by the kernel equations,

$$\frac{\partial}{\partial x_i}\operatorname{Det}\mathcal{M} = (\det B)\nu^\mathsf{T}A_i\nu.$$

Using (1), we get

$$\nu^\mathsf{T}A_i\nu = \frac{z^{m-d}}{\det B}\,\partial_i f(x) \qquad \text{on the local kernel graph.} \tag{6}$$

The factor $z^{m-d}/\det B$ is a unit near $p$, since $z_0 \neq 0$ and $\det B$ is a unit.

It follows that, after eliminating the kernel variables, each lifted polar equation

$$q_j(\nu^\top A_1\nu, \dots, \nu^\top A_N\nu) = 0$$

is the ordinary polar equation

$$q_j(\partial_1 f, \dots, \partial_N f) = 0$$

multiplied by the same local unit. By (4) and (1), the equation $g = 0$ is also the equation $f = 0$ up to a unit on $z \neq 0$.

Thus the completed local ring of the incidence scheme at $p$ is isomorphic to

$$\widehat{\mathcal{O}}_{\mathbb{P}^N,[z_0:x_0]} \Big/ \big(f,\ q_1(\nabla f), \dots, q_{N-2}(\nabla f),\ h\big),$$

up to multiplication of the displayed equations by units. The chosen $q_j$ cut out the finite reduced polar set $P_q$ on $X$, and the slice $h$ meets the corresponding cone line at exactly one point. Hence this local ring is zero-dimensional. Since the schemes are of finite type over $\mathbb{C}$, it is Artinian and has finite length. □

**Remark 1** (No new degeneracy from coincident kernels)**.** The coincidence of the left and right kernels in the symmetric model removes a projective variable; it does not create an additional fiber. Near a genuine point the kernel line is unique and is given scheme-theoretically by the graph $u = (-B^{-1}c, 1)$. Moreover $(u^\top A_1 u, \dots, u^\top A_N u)$ is a unit multiple of $\nabla f$ on this graph. Since $X$ is smooth, $\nabla f$ is not zero at a genuine point. Therefore the quadratic conormal forms cannot vanish identically on a positive-dimensional branch through such a point. Spurious components elsewhere, including components over rank-drop loci or over $z = 0$, do not affect the local isolated contribution at the genuine polar points.

# 4 The multihomogeneous count

Let $H$ denote the hyperplane class on $\mathbb{P}^N_{[z:x]}$ and $U$ the hyperplane class on $\mathbb{P}^{m-1}_{[u]}$. The ambient product has dimension

$$N + (m-1) = N + m - 1.$$

The incidence system (3) has

$$m + (N-2) + 1 = N + m - 1$$

equations, so it is square.

The $m$ equations $\mathcal{M}u = 0$ are linear in $[z : x]$ and linear in $[u]$, hence have class $H + U$. For a linear form $q(Y) = \sum_i \lambda_i Y_i$ on the dual coordinates,

$$q(u^\top A_1 u, \dots, u^\top A_N u) = u^\top \left( \sum_i \lambda_i A_i \right) u.$$

This is homogeneous of degree 2 in $u$ and independent of $[z : x]$. It is a genuine section of $\mathcal{O}(0,2)$ for every nonzero $q$ in the polar linear system. Indeed, if $u^\top(\sum_i \lambda_i A_i)u$ vanished identically, then, because the matrix $\sum_i \lambda_i A_i$ is symmetric and the characteristic is zero, one would have $\sum_i \lambda_i A_i = 0$. Then $\sum_i \lambda_i \partial_i f = 0$ identically by differentiating the determinant. After a linear change of the $x$-coordinates, $f$ would be independent of one variable; since $f$ is homogeneous of degree at least two, the corresponding coordinate point would be singular on $V(f)$, contradicting smoothness. Hence the polar equations have class $2U$.

There is no missing analogue of the non-symmetric redundant-left-equation lemma. In the ordinary determinant model one has both $\mathcal{M}v = 0$ and $u^{\mathsf{T}}\mathcal{M} = 0$; one left equation is locally redundant after imposing the right-kernel equations. In the symmetric model there is only the single kernel condition $\mathcal{M}u = 0$. Locally, $m-1$ of these equations solve for the affine kernel coordinates $u'$ and the remaining equation is the Schur complement $g = 0$. Thus the factor is $(H+U)^m$, not $(H+U)^{m-1}$ and not a product involving a second kernel variable.

The slice $h(z,x) = 0$ has class $H$. Therefore multihomogeneous Bezout gives that the sum of isolated local solution multiplicities is at most

$$[H^N U^{m-1}]\, H(H+U)^m (2U)^{N-2}. \tag{7}$$

This upper bound remains valid even if the full incidence scheme has positive- dimensional excess components; isolated local intersection multiplicities are bounded by the corresponding Chow-ring coefficient.

By Lemma 3, each of the $\delta_{\mathrm{top}}(X)$ genuine polar points contributes an isolated incidence solution. Therefore

$$\delta_{\mathrm{top}}(X) \le [H^N U^{m-1}]\, H(H+U)^m (2U)^{N-2}.$$

It remains only to extract the coefficient:

$$(2U)^{N-2} = 2^{N-2} U^{N-2}.$$

Thus one needs the coefficient of $H^{N-1}U^{m-N+1}$ in $(H+U)^m$, namely $\binom{m}{N-1}$. Hence

$$[H^N U^{m-1}]\, H(H+U)^m (2U)^{N-2} = 2^{N-2}\binom{m}{N-1}.$$

This proves Theorem 1.

# 5 Diagonal power sums

We first prove the arbitrary-degree diagonal corollary. Let

$$F_{N,d} = x_1^d + \cdots + x_N^d, \qquad N \ge 3, \qquad d \ge 2.$$

The hypersurface $X_{N,d} = V(F_{N,d}) \subset \mathbb{P}^{N-1}$ is smooth over $\mathbb{C}$: the partial derivatives are $dx_i^{d-1}$, and they have no common projective zero. Therefore

$$\delta_{\mathrm{top}}(X_{N,d}) = d(d-1)^{N-2}.$$

If $m = \mathrm{sdc}(F_{N,d})$, Theorem 1 gives

$$d(d-1)^{N-2} \le 2^{N-2}\binom{m}{N-1}. \tag{8}$$

This is the first displayed assertion of Corollary 2. Using $\binom{m}{N-1} \le m^{N-1}/(N-1)!$, we obtain

$$m \ge \left(\frac{d(d-1)^{N-2}(N-1)!}{2^{N-2}}\right)^{1/(N-1)}.$$

For the asymptotic form, Stirling's formula gives

$$((N-1)!)^{1/(N-1)} = (1+o_N(1))\frac{N}{e}.$$

The remaining degree factor satisfies, uniformly for $d \ge 2$,

$$\left(d(d-1)^{N-2}\right)^{1/(N-1)} = (d-1)\left(\frac{d}{d-1}\right)^{1/(N-1)} = (1+o_N(1))(d-1),$$

because $1 \le (d/(d-1))^{1/(N-1)} \le 2^{1/(N-1)}$. Finally,

$$2^{(N-2)/(N-1)} = 2 + o_N(1).$$

Thus

$$\operatorname{sdc}(F_{N,d}) \ge \left(\frac{1}{2e} - o_N(1)\right) N(d-1),$$

which proves Corollary 2.

Taking $N = d = n$ and writing $F_n = F_{n,n}$ gives

$$n(n-1)^{n-2} \le 2^{n-2}\binom{\operatorname{sdc}(F_n)}{n-1},$$

and hence

$$\operatorname{sdc}(F_n) \ge \left(\frac{1}{2e} - o(1)\right) n^2.$$

This proves Corollary 1.

### 5.1 Non-vacuity: an explicit symmetric representation

The lower bound is not merely a statement about an empty class of representations. We record a simple explicit construction showing $\operatorname{sdc}(F_{N,d}) = O(Nd)$ over $\mathbb{C}$.

**Proposition 1.** *For every $N, d \ge 1$,*

$$\operatorname{sdc}(F_{N,d}) \le 2N(d+1) + 1.$$

*In particular, for $F_n = F_{n,n}$,*

$$\operatorname{sdc}(F_n) \le 2n^2 + 2n + 1.$$

*Proof.* Let $r = d+1$, and let $J$ be the $r \times r$ nilpotent matrix with ones on the superdiagonal. For each $i = 1, \dots, N$, set

$$C_i = I_r - x_i J.$$

Then $\det C_i = 1$ and

$$C_i^{-1} = I_r + x_i J + x_i^2 J^2 + \cdots + x_i^d J^d.$$

Thus

$$e_1^{\mathsf{T}} C_i^{-1} e_r = x_i^d.$$

Define the symmetric $2r \times 2r$ matrix

$$D_i = \begin{pmatrix} 0 & C_i^{\mathsf{T}} \\ C_i & 0 \end{pmatrix}.$$

Then $\det D_i = (-1)^r$ and

$$D_i^{-1} = \begin{pmatrix} 0 & C_i^{-1} \\ (C_i^{\mathsf{T}})^{-1} & 0 \end{pmatrix}.$$

With

$$b_i = \begin{pmatrix} e_1 \\ e_r \end{pmatrix},$$

one has

$$b_i^{\mathsf{T}} D_i^{-1} b_i = 2e_1^{\mathsf{T}} C_i^{-1} e_r = 2x_i^d.$$

Let

$$D = \operatorname{diag}(D_1, \ldots, D_N), \qquad b = (b_1, \ldots, b_N)^{\mathsf{T}}.$$

Then $D$ is symmetric affine-linear of size $2N(d+1)$, $\det D = c \in \{\pm 1\}$, and

$$b^{\mathsf{T}} D^{-1} b = 2F_{N,d}.$$

Choose $\lambda \in \mathbb{C}$ satisfying $-2c\lambda^2 = 1$ and set

$$S = \begin{pmatrix} 0 & \lambda b^{\mathsf{T}} \\ \lambda b & D \end{pmatrix}.$$

This is symmetric and affine-linear, of size $2N(d+1)+1$. By the Schur complement formula,

$$\det S = \det D \left(0 - \lambda^2 b^{\mathsf{T}} D^{-1} b\right) = c(-2\lambda^2 F_{N,d}) = F_{N,d}.$$

For $N = d = n$ this gives the displayed size $2n^2 + 2n + 1$. □

**Remark 2** (Small-size verification of the explicit construction)**.** The construction in Proposition 1 was independently expanded for $n = 2$ and $n = 3$ in the specialization $F_n = F_{n,n}$. In both cases one obtains $b^{\mathsf{T}} D^{-1} b = 2F_n$ and $\det S = F_n$ exactly, with sizes 13 and 25, respectively. This check is not used in the proof, but it verifies the concrete non-vacuity construction in the first two nontrivial cases a reader is likely to test.

## 6 Scope and limitations

**Remark 3** (Positive characteristic)**.** The theorem is stated over $\mathbb{C}$ for a reason. It is false as a uniform theorem over all fields of characteristic different from two. If $\operatorname{char} K = p > 2$ and $n = p^r$, then the Frobenius identity gives

$$\sum_{i=1}^{n} x_i^n = (x_1 + \cdots + x_n)^n.$$

Hence

$$F_n = \operatorname{Det}\left((x_1 + \cdots + x_n) I_n\right),$$

so $\operatorname{sdc}(F_n) \le n$. Also the hypersurface $V(F_n)$ is not smooth in that case. Thus the smooth characteristic-zero hypothesis is essential for the stated asymptotic lower bound.

**Remark 4** (Exact, not border)**.** The argument proves an exact symmetric determinantal-complexity lower bound. It uses isolated solutions in a fixed incidence scheme associated with an exact representation. It does not prove a border symmetric determinantal-complexity lower bound, because polar degree is not a closed invariant in arbitrary degenerating families and the isolated local contributions can degenerate into excess components.

**Remark 5** (Relation to the non-symmetric companion preprint)**.** The companion preprint [8] proves, for ordinary determinantal complexity,

$$\delta_{\mathrm{top}}(X) \le \sum_{a=0}^{N-2} \binom{N-2}{a} \binom{m}{N-1-a} \binom{m-1}{a}.$$

For $F_n$, the leading coefficient of this ordinary determinant bound yields the constant $1/(4e)$. The symmetric incidence has only one kernel variable, so the polar equations have class $2U$ rather than $U+V$, and the coefficient becomes $2^{N-2}\binom{m}{N-1}$. This is exactly the source of the constant $1/(2e)$.

## 7 The human–AI methodology

We describe the process honestly, both because it produced the result and because it is itself of methodological interest.

**Roles and models.** A human author acted as orchestrator and domain arbiter. Large language models were used in separated roles: a generator proposing variants of the polar-degree method, an adversarial critic attempting to falsify the proposed symmetric specialization, and a verifier checking the local algebra, multidegrees, coefficient extraction, asymptotic constant, and non-vacuity. Convergence of independent model outputs was treated only as a signal to attempt a human-checkable proof, not as evidence.

**Trajectory.** The starting point was the non-symmetric polar-degree preprint [8]. A speculative exploration suggested that the symmetric determinant model should have a sharper incidence count because, at a rank-$(m-1)$ symmetric matrix, the left and right kernel lines coincide. The present proof, however, is written so that the symmetric local algebra and the resulting Bezout bound can be checked without importing any lemma from that preprint. A subsequent adversarial prompt forced the symmetric isolatedness step to be checked from scratch: the possible failure modes were rank-drop escape, positive-dimensional spurious components through a genuine polar point, nonreduced scheme structure, incorrect multidegree $2U$, an erroneous Bezout coefficient, and vacuity of the model. The resolution is the symmetric local normal form of Lemma 3; it proves that the incidence is locally a graph and that the lifted quadratic conormal equations are the ordinary polar equations up to units.

**Reproducibility.** Appendix A records the load-bearing verification prompt in ASCII-normalized form. The proof in Sections 2–5 is intended to be checked independently of its provenance. No symbolic computation is used as an assumption in the proof. The direct $n=2$ and $n=3$ checks of the explicit non-vacuity construction are recorded as sanity checks in Remark 2.

## Acknowledgement of AI assistance

The mathematical content of this paper was generated, challenged, and rewritten through large language models operated by the human author in the adversarial multi-model protocol described above. The author selected the problem, directed the verification prompts, chose the final framing, and takes responsibility for the text. The proofs are included in full so that the result can be judged by ordinary mathematical standards without reference to the models that helped produce it.

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# A Verification prompt

The following prompt was used to force a referee-level settlement of the symmetric specialization. It is reproduced in ASCII-normalized form and lightly line-wrapped; mathematical symbols such as "not equal" and "Omega" were normalized to avoid LaTeX encoding issues.

```
Establish or refute, to journal-referee rigor, the following claimed theorem in
algebraic complexity, working it out fully yourself rather than deferring to any
prior derivation.

CLAIM. Define symmetric determinantal complexity sdc(f) as the least m such that
f = det(M(x)) for a SYMMETRIC m x m matrix M of affine-linear forms. Then for
F_n = sum_i x_i^n,
    sdc(F_n) >= (1/2e - o(1)) n^2,
via the bound delta_top(X) <= 2^{N-2} C(m, N-1) for any size-m symmetric
determinantal representation of a smooth degree-d hypersurface X = V(f) in
P^{N-1}, where delta_top = d(d-1)^{N-2}.

The intended mechanism: at a smooth point the symmetric M has rank m-1 with
COINCIDING left/right kernel, so the incidence lives in P^N_[z:x] x P^{m-1}_[u]
with one kernel variable; the conormal identity is [partial_i f] =
[u^T A_i u]; the polar equations q_j(u^T A_i u)=0 have class 2U;
```

multihomogeneous Bezout then gives

    2^{N-2} C(m, N-1) = [H^N U^{m-1}] H (H+U)^m (2U)^{N-2}.

Do NOT assume the count is valid. The entire claim rests on a single load-bearing step, and your job is to settle that step, not to restate the mechanism. Treat a confirmation and a refutation with equal suspicion; prior versions of the analogous (non-symmetric) step were internally convincing and WRONG.

Settle each of the following with a definite verdict -- HOLDS (full proof), FAILS (explicit verified counterexample), or UNRESOLVED (state exactly what is missing). Do not round UNRESOLVED toward either side.

1. SYMMETRIC ISOLATEDNESS. Does each genuine polar point lift to a ZERO-DIMENSIONAL, scheme-theoretically isolated solution of the symmetric incidence system {M(x)u=0, q_j(u^T A_i u)=0, slice}? Prove it via the symmetric local normal form: with M = [[B,c],[c^T,s]], det B a unit near the point, kernel u=(-B^{-1}c, 1), and det M = (det B)(s - c^T B^{-1} c). Verify that the Schur complement identification goes through WITH THE SYMMETRY CONSTRAINT -- i.e. that restricting to symmetric M does not destroy the graph structure, the unit scaling adj M = alpha u u^T, or the elimination to the polar slice. State explicitly whether coincidence of the two kernels introduces any NEW degeneracy (e.g. u^T A_i u vanishing identically on a positive-dimensional locus) absent in the non-symmetric case.

2. THE MULTIDEGREE 2U. Confirm q_j(u^T A_i u) is genuinely class 2U and that the symmetric incidence is a SQUARE system on P^N x P^{m-1} (count equations vs dimensions). Verify no analogue of the redundant-left-equation reduction is silently needed or silently missing.

3. THE BEZOUT EXTRACTION. Verify [H^N U^{m-1}] H (H+U)^m (2U)^{N-2} = 2^{N-2} C(m, N-1) exactly, and that the root extraction yields the constant 1/(2e), not 1/(4e) or something else.

4. EXISTENCE / NONDEGENERACY. Does a smooth symmetric determinantal representation of F_n even exist for the relevant m, and is the generic polar section reduced of the right cardinality in the symmetric setting? If symmetric representations of F_n are obstructed, the theorem is vacuous or false -- check this.

Close with: (i) overall verdict on the n^2 bound and the 1/(2e) constant; (ii) the single weakest point; (iii) an explicit list of anything asserted but not fully verified.

## B Checklist of load-bearing verifications

For convenience, we spell out where each possible failure mode is addressed.

1. Rank-drop escape cannot pass through a genuine point because the local normal form is taken on the open set $z \neq 0$ and $\det B \neq 0$, and the local incidence is a graph over $g = 0$; see Lemma 3.

2. A positive-dimensional spurious component cannot pass through a genuine point because on the local graph the lifted polar equations are ordinary polar equations multiplied by units; see (6).

3. Nonreduced structure is controlled because the completed local incidence ring is Artinian. Nonreduced isolated multiplicity is allowed and is exactly what Bezout counts.

4. The polar equations have class $2U$ because $u^\mathsf{T} A_i u$ is a quadratic form in the single projective kernel variable. No left-kernel reduction is present in the symmetric system.

5. The coefficient extraction is the elementary identity

$$[H^N U^{m-1}]H(H+U)^m(2U)^{N-2} = 2^{N-2}\binom{m}{N-1}.$$

6. The model is nonempty for $F_n$ by Proposition 1.



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