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# Deterministic, Oblivious Isolation for Space-Bounded Computation Requires Large Weights

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**Abstract**

For a directed graph $G = (V, E)$, we say that a weight function $\rho\colon E \to [M]$ is *min-isolating* if the minimum-weight path from $u$ to $v$ is unique for each pair of vertices $u, v \in V$ such that $v$ is reachable from $u$. If we could efficiently construct a polynomially-bounded min-isolating weight function for any given digraph (or even just for *layered* digraphs), it would follow that $\mathsf{NL} = \mathsf{UL}$, thanks to work by Reinhardt and Allender (SICOMP 2000).

Van Melkebeek and Prakriya constructed an explicit, deterministic, min-isolating weight function for layered digraphs (SICOMP 2019). Their weight function is *oblivious*, i.e., it is a single weight function that works for all width-$w$ length-$n$ layered digraphs simultaneously, assigning a weight to each possible edge in the digraph without needing to know which edges are actually present. However, they use weights of magnitude $M = 2^{\Theta(\log n \cdot \log w)}$ instead of the desired $M = \mathrm{poly}(wn)$.

In this work, we prove that every deterministic, oblivious weight function that is min-isolating for width-$w$ length-$n$ layered digraphs must use weights of magnitude $2^{\Omega(\log w \cdot \log n)}$. This rules out one possible method of proving $\mathsf{NL} = \mathsf{UL}$. Our proof is based on a connection with coding theory. A *solid burst error* is a vector $e \in \mathbb{Z}_q^n$ such that the support of $e$ is an interval. Using spectral graph theory methods, we prove that every code $\mathcal{C} \subseteq \mathbb{Z}_q^n$ that can detect solid burst errors with Hamming weight up to $d$ must satisfy $|\mathcal{C}| \leq q^{n-\Omega(\log d)}$, which is optimal. The best prior bound, by Das (BEEI 2012), says $|\mathcal{C}| \leq q^n/(d+1)$ assuming the code is linear.



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# 1 Introduction

## 1.1 The NL vs. UL Problem

What is the nature of nondeterministic computation? Is it more powerful than deterministic computation? And if it is more powerful, *what is the source of its power?*

We can approach the latter question by studying restricted forms of nondeterministic computation. We tie the nondeterministic algorithm's hands behind its back, as it were, in order to investigate how much of the algorithm's power lies in its "hands."

In this paper, we study an appealing model called *unambiguous computation*. We say that a nondeterministic algorithm is *unambiguous* if the number of accepting computational paths is always either zero or one. We focus on unambiguous space-bounded computation. The class of languages that can be decided by unambiguous log-space algorithms is denoted UL.

Clearly, $\mathsf{L} \subseteq \mathsf{UL} \subseteq \mathsf{NL}$. Evidence suggests that $\mathsf{UL} = \mathsf{NL}$. This evidence is based on the concept of *isolation*.

**Definition 1.1** (Min-isolating weight function)**.** Let $G = (V, E)$ be a digraph and let $M \in \mathbb{N}$. We say that a weight function $\rho\colon E \to [M]$ is *min-isolating* for $G$ if, for all vertices $u, v \in V$ such that $v$ is reachable from $u$, there is a unique path of minimal weight from $u$ to $v$ with respect to $\rho$.

Reinhardt and Allender designed an unambiguous algorithm to decide reachability, given an $N$-vertex digraph $G = (V, E)$ and a min-isolating weight function $\rho\colon E \to [M]$, using $O(\log N + \log M)$ bits of space [RA00]. Reachability in general digraphs is complete for NL. Consequently, to prove $\mathsf{NL} = \mathsf{UL}$, it would suffice to design a deterministic, log-space algorithm that constructs polynomially-bounded min-isolating weights for any given digraph. Such a weight function always exists, thanks to Mulmuley, Vazirani, and Vazirani's famous Isolation Lemma [MVV87]; the challenge is to find it. This problem has been solved for various special classes of digraphs, such as planar graphs, bounded-genus graphs, and bounded-treewidth graphs [ADR05; ABCDR09; BTV09; TV12; DKTV12; AGGT16; DKMTVZ20; DGJMST21; CGT26], but the general case remains open.

## 1.2 Oblivious Isolation

One appealing approach for constructing min-isolating weights is to design a single "oblivious" weight function that assigns a weight to each "potential edge," without knowing which edges are actually present.

**Definition 1.2** (Oblivious weight function)**.** Let $\mathcal{G}$ be a class of digraphs on the vertex set $V$ without parallel edges and let $M \in \mathbb{N}$. A (deterministic) *oblivious weight function* is a function $\rho\colon V^2 \to [M]$. We say that $\rho$ is *min-isolating* for $\mathcal{G}$ if, for every digraph $G = (V, E) \in \mathcal{G}$, the weight function $\rho|_E$ is min-isolating for $G$.

For example, Bourke, Tewari, and Vinodchandran proved that the following oblivious weight function is min-isolating for directed *grid graphs* with $n$ rows and $n$ columns [BTV09]:

$$\rho(u,v) = \begin{cases} n^4 & \text{if } (u,v) \text{ points east or west} \\ n^4 + i & \text{if } (u,v) \text{ points north and is in column } i \\ n^4 - i & \text{if } (u,v) \text{ points south and is in column } i. \end{cases}$$

They remark, "Although we are unable to come up with a weight function for more general class of graphs that works for solving $\mathsf{NL} = \mathsf{UL}$ problem, our result indicates that such a task may not be all that difficult" [BTV09].

Another example comes from Tewari and Vinodchandran's work on *planar digraphs* [TV12], which generalize directed grid graphs. Let $\mathcal{G}$ be the class of planar digraphs with straight-line embeddings into $[n] \times [n] \subseteq \mathbb{R}^2$. Tewari and Vinodchandran proved that the following oblivious weight function is min-isolating for $\mathcal{G}$ [TV12]:

$$\rho((x, y), (x', y')) = 2n^4 + (y' - y) \cdot (x' + x).$$

### 1.3 Layered Digraphs and Our Contribution

In this work, we focus on *layered digraphs*. A width-$w$ length-$n$ layered digraph is a digraph on the vertex set $\{0, 1, \ldots, n\} \times [w]$ in which each edge has the form $((i-1, a), (i, a'))$ for some $i \in [n]$ and some $a, a' \in [w]$. For convenience, we represent such an edge by the triple $(i, a, a')$, so an oblivious weight function for width-$w$ length-$n$ layered digraphs can be represented by a function $\rho\colon [n] \times [w]^2 \to [M]$.

The reachability problem for layered digraphs is complete for $\mathsf{NL}$, essentially because layer $i$ in the graph can correspond to time step $i$ in the nondeterministic computation. Van Melkebeek and Prakriya studied min-isolation for layered digraphs [vMP19], and one of their "warm-up" constructions fits into our "oblivious weight function" framework. They construct an oblivious weight function $\rho\colon [n] \times [w]^2 \to [M]$ that is min-isolating for width-$w$ length-$n$ layered digraphs, using weights of quasipolynomial magnitude $M = 2^{O(\log n \cdot \log(wn))}$. A slight refinement of their analysis shows that the following oblivious weight function works, with the improved bound $M = 2^{O(\log n \cdot \log w)}$:

$$\rho(i, a, a') = a' \cdot (5w)^{\max\{r \geq 0 \,:\, 2^r \text{ divides } i\}}.$$

(See Appendix A.) If Van Melkebeek and Prakriya's quasipolynomial weight bound could be improved further to $M = \mathrm{poly}(wn)$, it would follow that $\mathsf{NL} = \mathsf{UL}$. Unfortunately, our main result says that their simple weight function is essentially optimal.

**Theorem 1.3** (Lower bound for min-isolating oblivious weight functions)**.** *Let $n, w \geq 2$ and $M \in \mathbb{N}$. For every oblivious weight function $\rho\colon [n] \times [w]^2 \to [M]$, if $\rho$ is min-isolating for width-$w$ length-$n$ layered digraphs, then $M \geq 2^{\Omega(\log n \cdot \log w)}$.*

We hope that Theorem 1.3 helps to explain why the $\mathsf{NL}$ vs. $\mathsf{UL}$ problem has not been resolved in the years since Bourke, Tewari, and Vinodchandran made their optimistic remark [BTV09].

### 1.4 Implications for Seeded Isolation

Theorem 1.3 rules out one approach for proving $\mathsf{NL} = \mathsf{UL}$, but there are others. One way to evade Theorem 1.3 would be to work with a different class of graphs $\mathcal{G}$ instead of layered digraphs. A second approach would be to assign weights in a non-oblivious way, i.e., take the full description of the graph into account when computing the edge weights. A third approach, which we feel is most promising, would be to design a *weight assignment generator* [vMP19]. A weight assignment generator can be defined as a collection of oblivious weight functions. Precisely:

**Definition 1.4** (Min-isolating weight assignment generator [vMP19])**.** Let $\mathcal{G}$ be a class of digraphs on the vertex set $V$ without parallel edges and let $s, M \in \mathbb{N}$. A *weight assignment generator* is a function $\mathsf{GEN}\colon \{0,1\}^s \times V^2 \to [M]$. The parameter $s$ is called the *seed length* of the generator. We say that $\mathsf{GEN}$ is *min-isolating* for $\mathcal{G}$ if, for every digraph $G = (V, E) \in \mathcal{G}$, there exists a seed $x \in \{0,1\}^s$ such that the weight function $\rho_x$ is min-isolating for $G$, where $\rho_x(u, v) = \mathsf{GEN}(x, u, v)$ for every edge $(u, v) \in E$.

Non-explicitly, there exists a weight assignment generator $\mathsf{GEN}\colon \{0,1\}^s \times [N]^2 \to [M]$ that is min-isolating for all $N$-vertex digraphs, with $s = O(\log N)$ and $M = \mathrm{poly}(N)$. If we could construct an explicit generator with the same parameters, it would follow that $\mathsf{NL} = \mathsf{UL}$, thanks to Reinhardt and Allender's work [RA00].

Van Melkebeek and Prakriya introduced the concept of a weight assignment generator and used it to prove $\mathsf{NL} \subseteq \mathsf{UL}^{3/2}$ [vMP19]. The concept is also implicit in earlier proofs, such as Reinhardt and Allender's proof that $\mathsf{NL} \subseteq \mathsf{UL/poly}$ [RA00] and Allender, Reinhardt, and Zhou's proof that $\mathsf{NL} = \mathsf{UL}$ assuming plausible circuit lower bounds [ARZ99].

Our main result (Theorem 1.3) concerns generators with seed length zero. As it turns out, there is a simple reduction from the problem of constructing a generator with seed length $o(\log(wn))$ to the problem of constructing a generator with seed length zero. Consequently, Theorem 1.3 implies that every min-isolating weight assignment generator for width-$w$ length-$n$ layered digraphs with seed length $o(\log(wn))$ uses weights of magnitude $2^{\Omega(\log w \cdot \log n)}$. (See Theorem 5.3.) We hope that our work helps to clarify the role of the seed in weight assignment generators.

## 1.5 Proof Overview

### 1.5.1 Coding Theory

The proof of our main result (Theorem 1.3) is based on a connection with coding theory. Let $\rho$ be an oblivious weight function that is min-isolating for width-$w$ length-$n$ layered digraphs. Let $P$ and $P'$ be two distinct paths that both start at vertex $u$ in layer 0 and end at vertex $v$ in layer $n$. If the set of layers in which $P$ and $P'$ differ forms an interval, then the weights of $P$ and $P'$ under $\rho$ must be unequal, because there exists a graph in which $P$ and $P'$ are the only paths from $u$ to $v$. Consequently, if a collection of paths from $u$ to $v$ all have equal weight, then the internal vertices of those paths define a *code* $\mathcal{C} \subseteq \mathbb{Z}_w^{n-1}$ that can detect so-called "solid burst errors," defined below.

**Definition 1.5** (Solid burst error)**.** A vector $e \in \mathbb{Z}_q^n$ is a *solid burst error* if there exist $1 \le i \le j \le n$ such that $\mathrm{supp}(e) = \{i, i+1, \ldots, j\}$, where $\mathrm{supp}(e)$ denotes the support of $e$.

**Definition 1.6** (Error detection)**.** Let $\mathcal{C} \subseteq \mathbb{Z}_q^n$ and $\mathcal{E} \subseteq \mathbb{Z}_q^n \setminus \{0\}$. We say that $\mathcal{C}$ *can detect errors in* $\mathcal{E}$ if, for every codeword $x \in \mathcal{C}$ and every error $e \in \mathcal{E}$, we have $x + e \notin \mathcal{C}$.

For context, a *burst error* of length at most $d$ is a vector $e \in \mathbb{Z}_q^n$ such that $\mathrm{supp}(e)$ is a subset of some interval $\{i, i+1, \ldots, i+d-1\}$. The problem of detecting/correcting burst errors is natural and well-motivated. After all, when data is transmitted or stored in practice, it is perfectly reasonable to expect that errors will tend to be localized to a small region in time or space. See, e.g., McEliece's textbook [McE02] for an overview of the subject. However, in this paper, we focus on the lesser-known *solid* burst error model, i.e., whenever there are errors in positions $i, j \in [n]$, we assume there are also errors in all positions between $i$ and $j$. The solidness assumption is arguably unnatural, but still, the coding community has shown some interest in solid burst errors. For example, see the references [Sch64; SS69; ML86; Jai02; HMH08; LB12; Das12].

It turns out that the assumption of solid burst errors is extremely helpful for error detection. We show that there is a code $\mathcal{C} \subseteq \mathbb{Z}_q^n$ that can detect solid burst errors with Hamming weight up to $d$ with cardinality $|\mathcal{C}| = q^{n - O(\log d)}$. (See Appendix B.) Remarkably, we can even take $d = n$, so $\mathcal{C}$ can detect *all* solid burst errors, regardless of Hamming weight. Our code construction is closely related to Van Melkebeek and Prakriya's oblivious weight function [vMP19]. The key to proving Theorem 1.3 is a matching impossibility result.

**Theorem 1.7** (Bound on codes detecting solid burst errors)**.** *Let $d \le n$ and $q \ge 2$. Let $\mathcal{C} \subseteq \mathbb{Z}_q^n$, and suppose $\mathcal{C}$ can detect all solid burst errors with Hamming weight up to $d$. Then $|\mathcal{C}| \le q^{n - \Omega(\log d)}$.*

Our code construction and impossibility result improve prior work by Das [Das12]. For each prime power $q$ and each $n \in \mathbb{N}$, Das presented a code $\mathcal{C} \subseteq \mathbb{F}_q^n$ that can detect solid burst errors; when $q = 2$, his code has cardinality $2^{n-\Theta(\log d)}$, which matches our construction, but when $q > 2$, his code has cardinality $q^{n-\Theta(d)}$, which is worse than our construction. On the impossibility side, Das proved that every linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$ that detects solid burst errors with weight up to $d$ must satisfy $|\mathcal{C}| \leq q^n/(d+1)$. Our bound is quantitatively stronger when $q$ is large, and our bound does not assume linearity.

### 1.5.2 Spectral Graph Theory

The proof of our code bound (Theorem 1.7) uses spectral graph theory. In general, for any symmetric set $\mathcal{E} \subseteq \mathbb{Z}_q^n \setminus \{0\}$, a code $\mathcal{C}$ can detect errors in $\mathcal{E}$ if and only if $\mathcal{C}$ is an independent set in the Cayley graph $\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})$. To bound the independence number of $\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})$ where $\mathcal{E}$ is the set of solid burst errors, we use *Hoffman's ratio bound* [Hof74]. Hoffman's ratio bound is a bound on the independence number of any regular undirected graph in terms of the smallest eigenvalue of the graph's adjacency matrix. Crucially, Hoffman's ratio bound holds even for weighted graphs, so we have the freedom to re-weight the edges of $\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})$ before applying the bound. The key technical step in the proof is the construction of a good distribution over intervals that we can use to re-weight the edges of $\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})$. In particular, using an inductive argument, we show that for every $\alpha \in [0, 1/2]$ and every power of two $n$, there is a distribution over nonempty intervals $I \subseteq [n]$ such that for every $A \subseteq [n]$, we have $\mathbb{E}[(-\alpha)^{|A \cap I|}] \geq -(2\alpha)^{\log n}$ (Lemma 3.2).

There is prior work in coding theory that uses Hoffman's ratio bound and generalizations thereof [AA19; AKR24; ANR25; AAR25; APR25]. Our contribution is to show how to apply Hoffman's bound to the specific case of solid burst errors.

The spectral graph theory argument described above works well when $q \geq 4$. For $q = 2, 3$, we use a different and simpler argument: we construct a clique of size $\Omega(n)$ in $\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})$ and then apply the clique-coclique bound.

### 1.5.3 The Constant-Width Case

Theorem 1.7 readily implies that oblivious min-isolation for width-$w$ length-$n$ layered digraphs requires weights of magnitude $M \geq \frac{1}{n} \cdot 2^{\Omega(\log n \cdot \log w)}$. If $w$ is sufficiently large, then the $\frac{1}{n}$ factor has no asymptotic effect, completing the proof of our main result (Theorem 1.3). However, when $w$ is a small constant, we need a separate argument. We use elementary Fourier analysis and the pigeonhole principle to prove that every oblivious weight function that is min-isolating for width-2 length-$n$ digraphs must use weights of magnitude $\Omega(n)$.

## 1.6 Organization

After some preliminaries in Section 2, we prove our bound on codes detecting solid burst errors in Section 3. Next, in Section 4, we use our code bound to prove our main result: a lower bound on the weights of oblivious weight functions that are min-isolating for layered digraphs. In Section 5, we extend our main result to the case of a weight assignment generator with a short seed. In Appendix A, we present Van Melkebeek and Prakriya's oblivious weight function [vMP19], and in Appendix B, we present a construction of a code that detects solid burst errors.

## 2 Preliminaries

In this section, we review standard graph theory concepts.

**Definition 2.1** (Independence number and clique number)**.** Let $G$ be an undirected graph. We define $\alpha(G)$ to be the size of the largest independent set in $G$, and we define $\omega(G)$ to be the size of the largest clique in $G$. If $X$ is a set of vertices in a graph with edge weights, then $X$ is considered "independent" if all edges between vertices in $X$ have weight zero, and $X$ is considered to be a "clique" if there is an edge with nonzero weight between every two vertices in $X$.

**Definition 2.2** (Cayley graphs)**.** Let $\Gamma$ be a finite abelian group and let $Z\colon \Gamma \to \mathbb{R}$. We define $\mathrm{Cay}(\Gamma, Z)$ to be a weighted digraph on the vertex set $\Gamma$. For every $x, e \in \Gamma$, there is an edge $(x, x+e)$ with weight $Z(e)$. We say that $Z$ is *symmetric* if $Z(e) = Z(-e)$ for every $e \in \Gamma$; in this case, $\mathrm{Cay}(\Gamma, Z)$ is undirected. If $Z$ is the indicator function for a set $\mathcal{E}$, then we will also write $\mathrm{Cay}(\Gamma, \mathcal{E})$ instead of $\mathrm{Cay}(\Gamma, Z)$.

One can define Cayley graphs in greater generality – for example, one can allow nonabelian groups – but Definition 2.2 is sufficient for this paper. For us, the group $\Gamma$ will always be $\mathbb{Z}_q^n$, where $\mathbb{Z}_q$ denotes the integers modulo $q$. Undirected Cayley graphs satisfy the *clique-coclique bound*:

**Lemma 2.3** (Clique-coclique bound)**.** *Let $\Gamma$ be a finite abelian group and let $\mathcal{E}$ be a symmetric subset of $\Gamma$. Then*

$$\omega(\mathrm{Cay}(\Gamma, \mathcal{E})) \cdot \alpha(\mathrm{Cay}(\Gamma, \mathcal{E})) \le |\Gamma|.$$

*Proof.* Let $\{x_1, x_2, \dots, x_k\}$ be a clique. For each $T \subseteq \Gamma$, we can let $T_i = T + x_i - x_1$ for $i \in [k]$. If $T_1, \dots, T_k$ are not all disjoint, say $x \in T_i \cap T_j$, then $x = a + x_i - x_1 = a' + x_j - x_1$ for some $a, a' \in T$, hence $a - a' = x_j - x_i \in \mathcal{E}$, demonstrating that $T$ is not an independent set. Therefore, if $T$ is an independent set, then $|T| \cdot k \le |\Gamma|$. □

The clique-coclique bound holds more generally for all "vertex-transitive" graphs. See, e.g., the textbook by Godsil and Meagher [GM15].

Hoffman's ratio bound is a different way to upper bound the independence number of a regular graph. See Haemers' work [Hae21] for the history of the bound. We include a proof of the bound for convenience.

**Theorem 2.4** (Hoffman's ratio bound [Hof74])**.** *Let $G$ be an undirected graph on $N$ vertices with real edge weights in which the (weighted) degree of each vertex is 1. Let $W$ be the (weighted) adjacency matrix of $G$. Let $\varepsilon > 0$, and assume that every eigenvalue of $W$ is at least $-\varepsilon$. Then $\alpha(G) \le \frac{\varepsilon}{1+\varepsilon} \cdot N$.*

*Proof.* Let $1_S$ be the indicator vector of an independent set of size $\mu N$, and let 1 be the all-ones vector. On the one hand,

$$\begin{aligned}(1_S - \mu 1)^\top W (1_S - \mu 1) &= 1_S^\top W 1_S - \mu 1_S^\top W 1 - \mu 1^\top W 1_S + \mu^2 1^\top W 1 \\ &= 0 \qquad - \mu^2 N \qquad - \mu^2 N \qquad + \mu^2 N \\ &= -\mu^2 N.\end{aligned}$$

On the other hand, by the spectral theorem for real symmetric matrices, $W$ has an orthonormal eigenbasis. By expressing $1_S - \mu 1$ with respect to this basis, we get

$$\begin{aligned}(1_S - \mu 1)^\top W (1_S - \mu 1) \ge -\varepsilon \cdot \|1_S - \mu 1\|_2^2 &= -\varepsilon \cdot (\mu N \cdot (1-\mu)^2 + (1-\mu) N \cdot \mu^2) \\ &= -\varepsilon \cdot (1-\mu) \cdot \mu N.\end{aligned}$$

Consequently, $\varepsilon \ge \frac{\mu}{1-\mu}$, which implies $\mu \le \frac{\varepsilon}{1+\varepsilon}$. □

There is a clean formula for the eigenvalues of a Cayley graph. Recall that if $\Gamma$ is a finite abelian group, without loss of generality $\Gamma$ has the form $\Gamma = \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_n}$ for some $q_1, \ldots, q_n \in \mathbb{N}$. For each $x \in \Gamma$, the *character function* $\chi_x \colon \Gamma \to \mathbb{C}$ is given by

$$\chi_x(e) = \exp\left(2\pi\sqrt{-1} \cdot \sum_{i=1}^{n} \frac{x_i \cdot e_i}{q_i}\right).$$

**Theorem 2.5** (Eigenvalues of Cayley graphs [Lov75])**.** *Let $\Gamma$ be a finite abelian group and let $Z \colon \Gamma \to \mathbb{R}$. Let $W$ be the (weighted) adjacency matrix of $\mathrm{Cay}(\Gamma, Z)$. Then the eigenvalues of $W$ are given by*

$$\lambda_x = \sum_{e \in \Gamma} Z(e) \cdot \chi_x(e)$$

*for $x \in \Gamma$. If $Z$ is symmetric, then these eigenvalues are all real.*

If $Z$ is a probability distribution, then the formula becomes $\lambda_x = \mathbb{E}_{e \sim Z}[\chi_x(e)]$.

# 3 Solid Burst Error Detection

In this section, we prove our bound on the cardinality of a code that can detect solid burst errors with Hamming weight up to $d$ (Theorem 1.7). We mainly focus on the case $d = n$, and handle smaller $d$ at the end. We use two distinct techniques. First, in Section 3.1, we present a simple argument based on the clique-coclique bound. This method works well when $q$ is constant, but it provably does not give good bounds when $q$ is large. Then, in Section 3.2, we use Hoffman's ratio bound to get good bounds when $q \geq 4$.

## 3.1 The Clique-Coclique Method

**Proposition 3.1** (Clique number bounds)**.** *Let $n, q \geq 2$ and let $\mathcal{E}$ be the set of solid burst errors in $\mathbb{Z}_q^n$. Then $\omega(\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})) = \Theta(n \cdot q^2)$ and $\alpha(\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})) \leq O(q^{n-2}/n)$.*

We split the proof into three parts: the clique number lower bound, the clique number upper bound, and the independence number upper bound.

*Proof that the clique number is $\Omega(n \cdot q^2)$.* Define

$$X = \{0^i ab 1^{n-i-2} : a \text{ is even, } b \text{ is odd, and } 0 \leq i \leq n-2\},$$

where $0^i$ or $1^i$ denotes $i$ consecutive zeroes or ones respectively. Then $|X| = (n-1) \cdot \lfloor \frac{q}{2} \rfloor \cdot \lceil \frac{q}{2} \rceil = \Omega(n \cdot q^2)$. Now consider two vectors in $X$, say $x = 0^i ab 1^{n-i-2}$ and $y = 0^j cd 1^{n-j-2}$, where $i \leq j$. Then:

- $x$ and $y$ agree in positions $1, 2, \ldots, i$, because they both have zeroes in these positions.
- $x$ and $y$ disagree in positions $i+2, i+3, \ldots, j, j+1$, because $x$ has odd values in these positions and $y$ has even values in these positions.
- $x$ and $y$ agree in positions $j+3, j+4, \ldots, n$, because they both have ones in these positions.

Therefore, regardless of whether $x$ and $y$ agree in positions $i+1$ and $j+2$, the set of positions where they disagree is an interval, hence $X$ is a clique. □

*Proof that the clique number is $O(n \cdot q^2)$.* Let $X$ be a set of vertices. We will prove that either $|X| \leq (n-1) \cdot q^2$ or else $X$ is not a clique.

First, suppose that for every $x \in X$, there is some $i \in [n-1]$ such that for every $y \in X \setminus \{x\}$, we have $(x_i, x_{i+1}) \neq (y_i, y_{i+1})$. In this case, we get an injective map $x \mapsto (i, x_i, x_{i+1})$, demonstrating $|X| \leq (n-1) \cdot q^2$.

Otherwise, fix some counterexample $x \in X$. If there is some $y \in X \setminus \{x\}$ that is not adjacent to $x$, then $X$ is not a clique and we are done. Assume now that $x$ is adjacent to every $y \in X \setminus \{x\}$. Then for every $i \in [n-1]$, there is some $y \in X \setminus \{x\}$ such that $x - y \in \mathcal{E}$ and $(x_i, x_{i+1}) = (y_i, y_{i+1})$. It follows that either $\mathrm{supp}(x-y) \subseteq [i-1]$, or else $\mathrm{supp}(x-y) \subseteq \{i+2, \ldots, n\}$. In the first case, we refer to $y$ as a "left witness" for $i$, and in the second case, we refer to $y$ as a "right witness" for $i$. Trivially, there cannot be a left witness for $i = 1$, and there cannot be a right witness for $i = n-1$. Therefore, there is some $i \in \{2, 3, \ldots, n-1\}$ such that there is a right witness $y$ for $i-1$ and there is a left witness $z$ for $i$. This means $\mathrm{supp}(x-y) \subseteq \{i+1, \ldots, n\}$ and $\mathrm{supp}(x-z) \subseteq [i-1]$. Therefore:

- $y$ and $z$ disagree somewhere to the left of position $i$.
- $y$ and $z$ agree in position $i$.
- $y$ and $z$ disagree somewhere to the right of position $i$.

This demonstrates that $y - z \notin \mathcal{E}$, hence $X$ is not a clique, completing the proof that the clique number is $O(n \cdot q^2)$. □

*Proof that the independence number is $O(q^{n-2}/n)$.* This follows from the clique-coclique bound (Lemma 2.3). □

If $q$ is constant, then Proposition 3.1 tells us that $\alpha(\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})) \leq q^{n - \Omega(\log n)}$, which is what we are aiming for. However, when $q$ is large, we need a different approach.

### 3.2 The Ratio Bound Method

In this subsection, we use Hoffman's ratio bound (Theorem 2.4) to bound the cardinality of codes that can detect solid burst errors. To apply Hoffman's ratio bound, we would like to design a good distribution over solid burst errors. The first step is to design a good distribution over intervals.

**Lemma 3.2** (A good distribution over intervals)**.** *Let $n$ be a power of two and let $\alpha \in [0, 1/2]$. There exists a distribution $I_{\alpha,n}$ over subsets of $[n]$ such that for every $A \subseteq [n]$, we have*

$$\mathop{\mathbb{E}}_{I \sim I_{\alpha,n}} [(-\alpha)^{|A \cap I|}] \geq -(2\alpha)^{\log n}. \tag{1}$$

*Furthermore, every set in the support of $I_{\alpha,n}$ is a nonempty interval.*

*Proof.* If $n = 1$, we define $I_{\alpha,n}$ to be $[1]$ with probability 1. If $n \geq 2$, we define $I_{\alpha,n}$ by the following recursive sampling procedure.

- With probability $1 - 2\alpha$, output $[n]$.
- With probability $\alpha$, output a sample from $I_{\alpha,n/2}$.
- With probability $\alpha$, sample $I \sim I_{\alpha,n/2}$ and output $\{i + n/2 : i \in I\}$.

For convenience, define $E_{A,n} = \mathbb{E}_{I \sim I_{\alpha,n}}[(-\alpha)^{|A \cap I|}]$. By induction on $n$, we will prove

$$E_{A,n} \geq \begin{cases} -(2\alpha)^{\log n} + (1-2\alpha) \cdot \alpha^{|A|} & \text{if } |A| \text{ is even} \\ -(2\alpha)^{\log n} & \text{if } |A| \text{ is odd.} \end{cases}$$

For the base case, suppose $n = 1$. Then $I = [1]$ and $E_{A,1} = (-\alpha)^{|A|}$.

- If $A = [1]$, then $E_{A,1} = (-\alpha)^1 = -\alpha > -1 = -(2\alpha)^{\log 1}$.
- If $A = \varnothing$, then $E_{A,1} = (-\alpha)^0 = 1 > -2\alpha = -(2\alpha)^{\log 1} + (1-2\alpha) \cdot \alpha^0$.

Now, for the inductive step, assume $n \geq 2$. Let $L = A \cap [n/2]$ and let $R = \{i \in [n/2] : n/2 + i \in A\}$. Then

$$E_{A,n} = \alpha \cdot (E_{L,n/2} + E_{R,n/2}) + (1-2\alpha) \cdot (-\alpha)^{|A|}.$$

If $|A|$ is even, then

$$\begin{aligned} E_{A,n} &= \alpha \cdot (E_{L,n/2} + E_{R,n/2}) + (1-2\alpha) \cdot \alpha^{|A|} \\ &\geq \alpha \cdot (-2 \cdot (2\alpha)^{\log(n/2)}) + (1-2\alpha) \cdot \alpha^{|A|} && \text{by induction} \\ &= -(2\alpha)^{\log n} + (1-2\alpha) \cdot \alpha^{|A|}. \end{aligned}$$

On the other hand, if $|A|$ is odd, then either $|L|$ or $|R|$ is even. Assume without loss of generality that $|R|$ is even. Then

$$\begin{aligned} E_{A,n} &= \alpha \cdot (E_{L,n/2} + E_{R,n/2}) - (1-2\alpha) \cdot \alpha^{|A|} \\ &\geq \alpha \cdot (-2 \cdot (2\alpha)^{\log(n/2)} + (1-2\alpha) \cdot \alpha^{|R|}) - (1-2\alpha) \cdot \alpha^{|A|} && \text{by induction} \\ &= -(2\alpha)^{\log n} + (1-2\alpha) \cdot (\alpha^{|R|+1} - \alpha^{|A|}) \\ &\geq -(2\alpha)^{\log n} && \text{because } |R| + 1 \leq |A|. \quad \square \end{aligned}$$

Now that we have a good distribution over intervals, we extend it to get an appropriate distribution over solid burst errors.

**Lemma 3.3** (A good distribution over solid burst errors)**.** *Let $n$ be a power of two and let $q \geq 4$. There exists a symmetric distribution $Z$ over solid burst errors in $\mathbb{Z}_q^n$ such that for every $x \in \mathbb{Z}_q^n$, we have*

$$\mathop{\mathbb{E}}_{e \sim Z}[\chi_x(e)] \geq -\left(\frac{2}{q-1}\right)^{\log n}.$$

*(Recall that since $Z$ is symmetric, $\mathbb{E}_{e \sim Z}[\chi_x(e)]$ is real.)*

*Proof.* We define $Z$ by the following sampling procedure.

1. Sample $I \sim I_{\alpha,n}$, where $\alpha = 1/(q-1)$ and $I_{\alpha,n}$ is the distribution from Lemma 3.2.
2. Sample $u \in (\mathbb{Z}_q \setminus \{0\})^I$ uniformly at random.
3. Output $e$, where $e_i = u_i$ if $i \in I$ and $e_i = 0$ if $i \notin I$.

Clearly, $e$ is always a solid burst error. Furthermore, for any $x \in \mathbb{Z}_q^n$, we have

$$\begin{aligned}
\mathop{\mathbb{E}}_{e\sim Z}[\chi_x(e)] &= \mathop{\mathbb{E}}_{I\sim I_{\alpha,n}}\left[\mathop{\mathbb{E}}_{u\in(\mathbb{Z}_q\setminus\{0\})^I}\left[\exp\left(\frac{2\pi\cdot\sqrt{-1}}{q}\cdot\sum_{i\in I}x_i\cdot u_i\right)\right]\right] \\
&= \mathop{\mathbb{E}}_{I\sim I_{\alpha,n}}\left[\prod_{i\in I}\mathop{\mathbb{E}}_{u_i\in\mathbb{Z}_q\setminus\{0\}}\left[\exp\left(\frac{2\pi\cdot\sqrt{-1}\cdot x_i\cdot u_i}{q}\right)\right]\right] \\
&= \mathop{\mathbb{E}}_{I\sim I_{\alpha,n}}\left[\prod_{\substack{i\in I\\ x_i\neq 0}}\frac{-1}{q-1}\right] \\
&\geq -\left(\frac{2}{q-1}\right)^{\log n}
\end{aligned}$$

by Lemma 3.2. □

**Corollary 3.4** (Bound on codes that detect all solid burst errors, $q \geq 4$)**.** *Let $n$ be a power of two and let $q \geq 4$. Let $\mathcal{E}$ be the set of solid burst errors in $\mathbb{Z}_q^n$. Then*

$$\alpha(\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})) \leq q^n \cdot \left(\frac{2}{q-1}\right)^{\log n} = q^{n-\Omega(\log n)}.$$

*Proof.* Let $Z$ be the distribution from Lemma 3.3. We have

$$\alpha(\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E})) \leq \alpha(\mathrm{Cay}(\mathbb{Z}_q^n, Z)),$$

simply because the support of $Z$ is a subset of $\mathcal{E}$. Let $W$ be the (weighted) adjacency matrix of $\mathrm{Cay}(\mathbb{Z}_q^n, Z)$. By Theorem 2.5 (eigenvalues of Cayley graphs), every eigenvalue of $W$ has the form $\mathbb{E}_{e\sim Z}[\chi_x(e)]$ for some $x \in \mathbb{Z}_q^n$. By Lemma 3.3, each such eigenvalue satisfies

$$\mathop{\mathbb{E}}_{e\sim Z}[\chi_x(e)] \geq -\left(\frac{2}{q-1}\right)^{\log n}.$$

By Hoffman's ratio bound (Theorem 2.4), it follows that

$$\alpha(\mathrm{Cay}(\mathbb{Z}_q^n, Z)) \leq q^n \cdot \frac{(\frac{2}{q-1})^{\log n}}{1+(\frac{2}{q-1})^{\log n}} \leq q^n \cdot \left(\frac{2}{q-1}\right)^{\log n}.$$ □

### 3.3 Solid Burst Errors of Bounded Hamming Weight

To complete the proof of Theorem 1.7 (our bound on the size of codes that detect solid burst errors with Hamming weight up to $d$), we just need to tie up a couple of loose ends:

*Proof of Theorem 1.7.* If $d = n$ and $n$ is a power of two, then the theorem follows from either Proposition 3.1 (if $q < 4$) or Corollary 3.4 (if $q \geq 4$). To handle the remaining cases, let $\mathcal{E}_{n,d}$ be the set of solid burst errors in $\mathbb{Z}_q^n$ with Hamming weight at most $d$. Let $\mathcal{C}$ be an independent set in $\mathrm{Cay}(\mathbb{Z}_q^n, \mathcal{E}_{n,d})$. Let $m$ be the largest power of two such that $m \leq d$. For each $z \in \mathbb{Z}_q^{n-m}$, let $\mathcal{C}_z = \{x \in \mathbb{Z}_q^m : xz \in \mathcal{C}\}$. Then $\mathcal{C}_z$ is an independent set in $\mathrm{Cay}(\mathbb{Z}_q^m, \mathcal{E}_{m,m})$. Therefore,

$$|\mathcal{C}| \leq \sum_{z\in\mathbb{Z}_q^{n-m}} |\mathcal{C}_z| \leq q^{n-m}\cdot q^{m-\Omega(\log m)} = q^{n-\Omega(\log d)}.$$ □

# 4 Oblivious Weight Functions

We have completed the proof of our bound on the cardinality of codes that detect solid burst errors (Theorem 1.7). In this section, we use that bound to prove our main result (Theorem 1.3), which says that oblivious weight functions that are min-isolating for layered digraphs must use large weights. The connection between min-isolation and detecting solid burst errors is given by the following lemma. For convenience, we index the vertices in each layer using $\mathbb{Z}_w$ instead of $[w]$.

**Lemma 4.1** (Min-isolation $\implies$ code)**.** *Let $n, w, M \in \mathbb{N}$ where $n, w \geq 2$. Assume that there exists an oblivious weight function $\rho\colon [n] \times \mathbb{Z}_w^2 \to [M]$ that is min-isolating for width-$w$ length-$n$ layered digraphs. Then there exists a code $\mathcal{C} \subseteq \mathbb{Z}_q^{n-1}$ that can detect all solid burst errors, where $q = w$ and $|\mathcal{C}| \geq \frac{q^{n-1}}{Mn}$.*

*Proof.* For each $x \in \mathbb{Z}_q^{n-1}$, define

$$\phi(x) = \sum_{i=1}^{n} \rho(i, x_{i-1}, x_i) \in [Mn],$$

where $x_0 = x_n = 0$.[1] Let $\mathcal{E}$ be the set of solid burst errors in $\mathbb{Z}_q^{n-1}$. We claim that $\phi$ is a proper coloring of $\mathrm{Cay}(\mathbb{Z}_q^{n-1}, \mathcal{E})$, hence one of the color classes is an independent set of size at least $\frac{q^{n-1}}{Mn}$. To prove it, let $x, y \in \mathbb{Z}_q^{n-1}$ and suppose $x - y \in \mathcal{E}$. Define a width-$w$ length-$n$ layered digraph $G = (\{0, 1, \ldots, n\} \times \mathbb{Z}_w, E)$, where the edge set $E$ is given by

$$E = \{((i-1, x_{i-1}), (i, x_i)) : i \in [n]\} \cup \{((i-1, y_{i-1}), (i, y_i)) : i \in [n]\},$$

where $x_0 = y_0 = x_n = y_n = 0$. Because $x - y \in \mathcal{E}$, there are exactly two paths from $(0, 0)$ to $(n, 0)$ in $G$, one with weight $\phi(x)$ and one with weight $\phi(y)$. Since $\rho$ is min-isolating for $G$, we must have $\phi(x) \neq \phi(y)$. □

If $w$ is large enough, then we can complete the proof of our main result (Theorem 1.3) by simply combining Lemma 4.1 and Theorem 1.7. However, when $w$ is small, Lemma 4.1 is too weak. Therefore, we now present a separate argument to bound the weights in an oblivious weight function that is min-isolating for width-2 layered digraphs. For convenience, we index the vertices in each layer using $\{\pm 1\}$ instead of $[2]$.

**Proposition 4.2** (Lower bound for min-isolation of width-2 layered digraphs)**.** *Let $n, M \in \mathbb{N}$ where $n \geq 3$. Let $\rho\colon [n] \times \{\pm 1\}^2 \to [M]$ be an oblivious weight function that is min-isolating for width-2 length-$n$ layered digraphs. Then $M \geq (n-2)/24$.*

*Proof.* We prove the contrapositive. Assume $M < (n-2)/24$; we will prove that $\rho$ is not min-isolating. We use elementary Fourier analysis of Boolean functions; see, e.g., O'Donnell's textbook [O'D14]. For each $t \in [n]$, define a Boolean function $\rho_t\colon \{\pm 1\}^2 \to [M]$ by $\rho_t(b, b') = \rho(t, b, b')$. The Fourier expansion of $\rho_t$ has the form

$$\rho_t(b, b') = \frac{A_t}{4} + \frac{B_t}{4} \cdot b + \frac{C_t}{4} \cdot b' + \frac{D_t}{4} \cdot b \cdot b'$$

for some integers $A_t, B_t, C_t, D_t \in [-4M, 4M]$. For any possible path $P$, say

$$P = ((i, b_i), (i+1, b_{i+1}), \ldots, (j+1, b_{j+1})),$$

[1] Intuitively, $\phi$ is somewhat analogous to a "parity check matrix" of a linear code.

the weight of $P$ is given by

$$\begin{aligned}\rho(P) &= \sum_{t=i+1}^{j+1} \rho_t(b_{t-1}, b_t) \\ &= \left(\sum_{t=i+1}^{j+1} \frac{A_t}{4}\right) + \frac{B_{i+1} \cdot b_i}{4} + \frac{C_{j+1} \cdot b_{j+1}}{4} + \left(\sum_{t=i+1}^{j} \frac{(B_{t+1} + C_t) \cdot b_t}{4}\right) + \sum_{t=i+1}^{j+1} \frac{D_t \cdot b_{t-1} \cdot b_t}{4}.\end{aligned}$$

For such a path $P$, we define the "opposite" path $P^*$ by flipping the signs of all the internal vertices but leaving the endpoints unchanged, i.e.,

$$P^* = ((i, b_i), (i+1, -b_{i+1}), \ldots, (j, -b_j), (j+1, b_{j+1})).$$

The difference in weight between $P$ and $P^*$ is given by

$$\rho(P) - \rho(P^*) = \left(\sum_{t=i+1}^{j} \frac{(B_{t+1} + C_t) \cdot b_t}{2}\right) + \frac{D_{i+1} \cdot b_i \cdot b_{i+1}}{2} + \frac{D_{j+1} \cdot b_j \cdot b_{j+1}}{2}. \tag{2}$$

Motivated by the expression above, greedily choose signs $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \in \{\pm 1\}$ such that for every $i \in [n-1]$, we have

$$\sum_{t=1}^{i} (B_{t+1} + C_t) \cdot \sigma_t \in [-8M, 8M].$$

For each $i \in [n-1]$, define

$$Q_i = D_{i+1} + \sum_{t=1}^{i} (C_t + B_{t+1}) \cdot \sigma_t \in [-12M, 12M].$$

These integers $Q_1, Q_2, \ldots, Q_{n-1}$ all lie in a set of size $24M + 1 < n - 1$. Therefore, by the pigeonhole principle, there exist $1 \le i < j \le n-1$ such that $Q_i = Q_j$. Now define a path

$$P = ((i, b_i), (i+1, b_{i+1}), \ldots, (j, b_j), (j+1, b_{j+1})),$$

where $b_i = -\sigma_{i+1}$; $b_t = \sigma_t$ for every $i + 1 \le t \le j$; and $b_{j+1} = \sigma_j$. Let $G$ be a width-2 length-$n$ graph consisting only of the edges in $P$ and $P^*$. By construction, there are exactly two paths from $(i, b_i)$ to $(j+1, b_{j+1})$, namely $P$ and $P^*$. But by Eq. (2), we have

$$\begin{aligned}\rho(P) - \rho(P^*) &= \left(\sum_{t=i+1}^{j} \frac{(B_{t+1} + C_t) \cdot b_t}{2}\right) + \frac{D_{i+1} \cdot b_i \cdot b_{i+1}}{2} + \frac{D_{j+1} \cdot b_j \cdot b_{j+1}}{2} \\ &= \left(\sum_{t=i+1}^{j} \frac{(B_{t+1} + C_t) \cdot \sigma_t}{2}\right) - \frac{D_{i+1}}{2} + \frac{D_{j+1}}{2} \\ &= \frac{1}{2} \cdot (Q_j - Q_i) \\ &= 0.\end{aligned}$$

Thus, $\rho$ is not min-isolating for $G$. □

We are now ready to prove our main result (Theorem 1.3), which says that every oblivious weight function that is min-isolating for width-$w$ length-$n$ layered digraphs must use weights of magnitude $2^{\Omega(\log w \cdot \log n)}$.

*Proof of Theorem 1.3.* Let $C$ be a large enough constant. We split into four cases depending on how $w$ and $n$ compare to $C$.

- Suppose $w, n \geq C$. By Lemma 4.1, there is a code $\mathcal{C} \subseteq \mathbb{Z}_q^{n-1}$ that can detect all solid burst errors, where $q = w$ and $|\mathcal{C}| \geq \frac{q^{n-1}}{Mn}$. By Theorem 1.7, we have $|\mathcal{C}| \leq \frac{q^{n-1}}{q^{\Omega(\log(n-1))}}$. Therefore, $M \geq \frac{1}{n} \cdot 2^{\Omega(\log w \cdot \log(n-1))} = 2^{\Omega(\log w \cdot \log n)}$.
- Suppose $2 \leq n \leq C < w$. Let $\rho'$ be $\rho$ restricted to $[2] \times [w]^2$. Then $\rho'$ is min-isolating for width-$w$ length-2 layered digraphs. Therefore, by Lemma 4.1, there is a code $\mathcal{C} \subseteq \mathbb{Z}_q^1$ that can detect all solid burst errors, where $q = w$ and $|\mathcal{C}| \geq \frac{q^1}{2M}$. On the other hand, such a code trivially must satisfy $|\mathcal{C}| \leq 1$, so $M \geq w/2 = 2^{\Omega(\log n \cdot \log w)}$.
- Suppose $2 \leq w \leq C < n$. Let $\rho''$ be $\rho$ restricted to $[n] \times [2]^2$. Then $\rho''$ is min-isolating for width-2 length-$n$ layered digraphs. Therefore, by Proposition 4.2, we have $M \geq \frac{n-2}{24} = 2^{\Omega(\log w \cdot \log n)}$.
- Suppose $2 \leq w, n \leq C$. Then we use the trivial bound $M \geq 2$.

□

# 5 Weight Assignment Generators

We have completed the proof of our main result (Theorem 1.3). In this section, we extend our main result to the case of weight assignment generators with a short seed, as discussed in Section 1.4. The proof consists of two trivial reductions that convert weight assignment generators (with seeds) into oblivious weight functions (with no seeds).

**Lemma 5.1** (Eliminating a seed of length $o(\log w)$)**.** *Assume there exists a weight assignment generator* $\mathsf{GEN}\colon \{0,1\}^s \times [n] \times [w]^2 \to [M]$ *that is min-isolating for width-$w$ length-$n$ layered digraphs. Then there exists an oblivious weight function* $\rho\colon [n] \times [w']^2 \to [M]$ *that is min-isolating for width-$w'$ length-$n$ layered digraphs, where* $w' = \lfloor w/2^s \rfloor$.

*Proof.* Let $A_1, A_2, \ldots, A_{2^s}$ be disjoint subsets of $[w]$, each of size $\lfloor w/2^s \rfloor$. For each seed $x \in [2^s]$, define an oblivious weight function $\rho_x\colon [n] \times A_x^2 \to [M]$ by the rule $\rho_x(i, a, a') = \mathsf{GEN}(x, i, a, a')$. Suppose that for every seed $x$, there exists a width-$w'$ length-$n$ layered digraph $G_x$ (with vertices indexed by $A_x$) such that $\rho_x$ is not min-isolating for $G_x$. Define a width-$w$ length-$n$ layered digraph $G$ by taking the union of all the edge sets of the $G_x$ graphs. Then $\mathsf{GEN}$ is not min-isolating for $G$. □

**Lemma 5.2** (Eliminating a seed of length $o(\log n)$)**.** *Assume there exists a weight assignment generator* $\mathsf{GEN}\colon \{0,1\}^s \times [n] \times [w]^2 \to [M]$ *that is min-isolating for width-$w$ length-$n$ layered digraphs. Then there exists an oblivious weight function* $\rho\colon [n'] \times [w]^2 \to [M]$ *that is min-isolating for width-$w$ length-$n'$ layered digraphs, where* $n' = \lfloor n/2^s \rfloor$.

*Proof.* Let $I_1, I_2, \ldots, I_{2^s}$ be disjoint intervals that are subsets of $[n]$ of size $\lfloor n/2^s \rfloor$. For each seed $x \in [2^s]$, define an oblivious weight function $\rho_x\colon I_x \times [w]^2 \to [M]$ by the rule $\rho_x(i, a, a') = \mathsf{GEN}(x, i, a, a')$. Suppose that for every seed $x$, there exists a width-$w$ length-$n'$ layered digraph $G_x$ (with layers

indexed by $I_x$) such that $\rho_x$ is not min-isolating for $G_x$. Define a width-$w$ length-$n$ layered digraph $G$ by taking the union of the edge sets of all the $G_x$ graphs. Then GEN is not min-isolating for $G$. □

**Theorem 5.3** (Generators with short seeds must use large weights)**.** *Let $w, n \geq 2$ and let $s, M \in \mathbb{N}$. Let* $\mathsf{GEN}\colon \{0,1\}^s \times [n] \times [w]^2 \to [M]$*. Assume that* GEN *is min-isolating for width-$w$ length-$n$ layered digraphs. Then either $s \geq \Omega(\log(wn))$ or else $M \geq 2^{\Omega(\log n \cdot \log w)}$.*

*Proof.* If $2^s < \sqrt{w}$, then by Lemma 5.1 and Theorem 1.3, we have $M \geq 2^{\Omega(\log w \cdot \log n)}$. If $2^s < \sqrt{n}$, then by Lemma 5.2 and Theorem 1.3, we have $M \geq 2^{\Omega(\log w \cdot \log n)}$. Finally, if $2^s \geq \max\{\sqrt{w}, \sqrt{n}\}$, then $s \geq \Omega(\log(wn))$. □

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# A Van Melkebeek and Prakriya's Oblivious Weight Function

In this appendix, we present Van Melkebeek and Prakriya's oblivious weight function for layered digraphs [vMP19], or rather the slight refinement that we mentioned in Section 1.3.

**Proposition A.1** (Oblivious min-isolation for layered digraphs [vMP19])**.** *Let $n, w \geq 2$. Define $\rho\colon [n] \times [w]^2 \to \mathbb{N}$ by the formula*

$$\rho(i, a, a') = a' \cdot (5w)^{\max\{r \geq 0\,:\, 2^r \text{ divides } i\}}.$$

*Then $\rho$ is min-isolating for width-$w$ length-$n$ layered digraphs.*

*Proof.* Let $G$ be a width-$w$ length-$n$ layered digraph. Let $u$ and $v$ be vertices such that $v$ is reachable from $u$, say $u = (s, a_s)$ and $v = (s+\ell, a_{s+\ell})$. We will prove by induction on $\ell$ that the minimum-weight path from $u$ to $v$ is unique. The base cases $\ell = 0$ and $\ell = 1$ are trivial, so assume $\ell \geq 2$. Let $\nu_2(i)$ denote the largest $r$ such that $2^r$ divides $i$. Define

$$R = \max\{\nu_2(s+1), \nu_2(s+2), \ldots, \nu_2(s+\ell-1)\}.$$

Let $k \in \{s+1, s+2, \ldots, s+\ell-1\}$ be such that $\nu_2(k) = R$. For every $r \leq R$, there must be fewer than $2^{R+1-r}$ multiples of $2^r$ among $s+1, s+2, \ldots, s+\ell-1$, because otherwise one of them would be a multiple of $2^{R+1}$, contradicting the maximality of $R$. In particular, $k$ is unique.

Let $P$ be a path from $u$ to $v$, say $P = ((s, a_s), (s+1, a_{s+1}), \ldots, (s+\ell, a_{s+\ell}))$. Let $P'$ be another path from $u$ to $v$, and suppose $P$ and $P'$ pass through different vertices in layer $k$, i.e., $P'$ passes through $(k, a'_k)$ where $a'_k \neq a_k$. Without loss of generality, assume $a_k < a'_k$. Then

$$\begin{aligned}
\rho(P) &= \sum_{i=s+1}^{s+\ell} a_i \cdot (5w)^{\nu_2(i)} \\
&\leq a_{s+\ell} \cdot (5w)^{\nu_2(s+\ell)} + a_k \cdot (5w)^R + \sum_{r=0}^{R-1} 2^{R+1-r} \cdot w \cdot (5w)^r \\
&= a_{s+\ell} \cdot (5w)^{\nu_2(s+\ell)} + a_k \cdot (5w)^R + \sum_{\Delta=1}^{R} 2^{\Delta+1} \cdot w \cdot (5w)^{R-\Delta} \\
&< a_{s+\ell} \cdot (5w)^{\nu_2(s+\ell)} + (5w)^R \cdot \left( a_k + 2w \cdot \sum_{\Delta=1}^{\infty} \left( \frac{2}{5w} \right)^{\Delta} \right) \\
&= a_{s+\ell} \cdot (5w)^{\nu_2(s+\ell)} + (5w)^R \cdot \left( a_k + 2w \cdot \frac{2}{5w-2} \right) \\
&\leq a_{s+\ell} \cdot (5w)^{\nu_2(s+\ell)} + (5w)^R \cdot (a_k + 1) \\
&\leq \rho(P').
\end{aligned}$$

Therefore, all minimum-weight paths from $u$ to $v$ pass through the same vertex in layer $k$, say $(k, a_*)$. Consequently, every minimum-weight path from $u$ to $v$ consists of a minimum-weight path from $u$ to $(k, a_*)$, which is unique by induction, followed by a minimum-weight path from $(k, a_*)$ to $v$, which is also unique by induction. □

# B A Code that Detects Solid Burst Errors

As we explain in Section 4, oblivious weight functions that are min-isolating for layered digraphs imply codes that detect solid burst errors. If we combined that black-box reduction with Van Melkebeek and Prakriya's oblivious weight function [vMP19], we would get a code $\mathcal{C} \subseteq \mathbb{Z}_q^n$ that can detect all solid burst errors, with cardinality $|\mathcal{C}| \geq q^{n - O(\log n)}$. In this section, we generalize the construction to the case that we have a bound on the Hamming weight of the error.

**Proposition B.1** (A code that detects solid burst errors, cf. [vMP19])**.** *Let $q, d \geq 2$ and $n \in \mathbb{N}$. There exists a code $\mathcal{C} \subseteq \mathbb{Z}_q^n$ such that $\mathcal{C}$ can detect solid burst errors with Hamming weight up to $d$ and $|\mathcal{C}| \geq q^{n - O(\log d)}$.*

*Proof sketch.* Assume without loss of generality that $d$ is a power of two. For each $0 \le i < \log d$, define

$$S_i = \{k \cdot 2^i \in [d] : k \text{ is odd}\}.$$

and

$$T_i = \{j \in [n] : (j \bmod d) \in S_i\}.$$

Furthermore, define

$$T_{\log d} = \{j \in [n] : (j \bmod d) = 0\}.$$

The sets $T_0, T_1, \ldots, T_{\log d}$ are disjoint. The code is given by

$$\mathcal{C} = \left\{ x \in \mathbb{Z}_q^n : \forall i \in \{0, 1, \ldots, \log d\}, \quad \sum_{j \in T_i} x_j = 0 \right\}.$$

Clearly, $|\mathcal{C}| \ge q^{n - \log d - 1}$. To prove correctness, we first show by induction on $d$ that for every nonempty interval $I \subseteq [d-1]$, there is some $i \in \{0, 1, \ldots, \log d - 1\}$ such that $|S_i \cap I| = 1$. For the base case, if $d = 2$, this is trivial because $I = S_0 = \{1\}$. Now, for the inductive step, assume $d > 2$. If $d/2 \in I$, then we can take $i = \log d - 1$, because $S_{\log d - 1} = \{d/2\}$. Otherwise, $I$ is entirely contained in either $[d/2 - 1]$ or else $\{d/2 + 1, \ldots, d - 1\}$, so the claim follows by induction.

Now we prove that for every nonempty interval $I \subseteq [n]$ with $|I| \le d$, there is some $i \in \{0, 1, \ldots, \log d\}$ such that $|I \cap T_i| = 1$. If $I$ contains a multiple of $d$, then it contains exactly one multiple of $d$, so we can let $i = \log d$. Otherwise, $I \bmod d$ is an interval subset of $[d-1]$, so by our first claim, there is some $i \in \{0, 1, \ldots, \log d - 1\}$ such that $|(I \bmod d) \cap S_i| = 1$, hence $|I \cap T_i| = 1$.

Finally, let $x \in \mathcal{C}$ and let $e$ be a solid burst error with Hamming weight at most $d$. By the above argument, there is some $i \in \{0, 1, \ldots, \log d\}$ such that $|T_i \cap \operatorname{supp}(e)| = 1$. Consequently,

$$\sum_{j \in T_i} (x + e)_j = \left( \sum_{j \in T_i} x_j \right) + \left( \sum_{j \in T_i} e_j \right) \neq 0,$$

so $x + e \notin \mathcal{C}$. □



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