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Optimal PSPACE-hardness of Approximating q-CSP Reconfiguration

Shuichi Hirahara
National Institute of Informatics, Japan
s_hirahara@nii.ac.jp

Naoto Ohsaka
CyberAgent, Inc., Japan
ohsaka_naoto@cyberagent.co.jp

Abstract

In the MAXMIN q-CSP RECONFIGURATION problem, given a satisfiable q-CSP instance and a pair of its satisfying assignments, we are asked to transform one assignment into the other by repeatedly changing the value assigned to a single variable. The objective is to find such a transformation that maximizes the minimum fraction of satisfied constraints along the transformation. In this paper, we prove that for any $q \geqslant 2$ and $\varepsilon > 0$, MAXMIN q-CSP RECONFIGURATION is PSPACE-hard to approximate within a factor of $\frac{1}{2^{q-1}} + \varepsilon$. To complement this hardness result, we prove that a $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation for MAXMIN q-CSP RECONFIGURATION is in NP in the perfect completeness case. These results establish the optimal PSPACE-hardness of approximating MAXMIN q-CSP RECONFIGURATION for every $q \geqslant 2$ under NP $\neq$ PSPACE.

Contents

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1 Introduction

q-CSP RECONFIGURATION is a canonical reconfiguration problem defined as follows: Let G be a satisfiable instance of the q-ary constraint satisfaction problem (q-CSP) with n variables and alphabet size σ. A sequence of assignments for G, denoted by $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$, is called a ***reconfiguration sequence*** if every adjacent pair $f^{(t)}$ and $f^{(t+1)}$ differ in a single variable. In the q-CSP RECONFIGURATION problem, for a pair of satisfying assignments f_{start} and f_{end} for G, we are asked to decide if there exists a reconfiguration sequence $\vec{f}$ from f_{start} to f_{end} consisting only of satisfying assignments for G. In other words, q-CSP RECONFIGURATION asks the st-connectivity question over the ***solution space*** of G, which is defined as the subgraph of the Hamming graph $H(n, \sigma)$ induced by all satisfying assignments for G. Special cases of q-CSP RECONFIGURATION include the following reconfiguration problems:

- In the k-SAT RECONFIGURATION problem [GKMP09], for a satisfiable k-CNF formula φ and a pair of its satisfying assignments, we seek a path from one assignment to the other in the Boolean hypercube that passes only through satisfying assignments for φ. This problem is in P if $k \leqslant 2$ and PSPACE-complete if $k \geqslant 3$ [GKMP09]. Studying k-SAT RECONFIGURATION and its relatives was originally motivated by applications to analyzing the structure of the solution space of Boolean formulas [GKMP09].
- In the k-COLORING RECONFIGURATION problem [Cer07, CvdHJ08], for a k-colorable graph G and a pair of its proper k-colorings, we wish to transform one k-coloring into the other by recoloring a single vertex at a time. This problem is in P if $k \leqslant 3$ [CvdHJ11] and PSPACE-complete if $k \geqslant 4$ [BC09]. The solution space of k-COLORING RECONFIGURATION is related to the Glauber dynamics [DFFV06, Jer95, Mol04]; see also [vdHeu13, §5].

In general, q-CSP RECONFIGURATION with alphabet size σ is in P if $q \leqslant 2$ and $\sigma \leqslant 2$ [GKMP09, HIZ18] while it is PSPACE-complete if ($q = 2$ and $\sigma = 3$) [HIZ18] or ($q = 3$ and $\sigma = 2$) [GKMP09, HD05, HD09]. See Section 3 for related work on other reconfiguration problems.

In this paper, we study the *approximability* of q-CSP RECONFIGURATION. Recently, approximability of reconfiguration problems has been studied from both hardness and algorithmic perspectives [GKMRW25, GMWZ26, HO24a, HO24b, HO25a, HO25b, HOST26, Ohs23, Ohs24a, Ohs24b, Ohs24c, Ohs25a, Ohs25b, OM22] (see also Section 3). The approximate version of q-CSP RECONFIGURATION is called MAXMIN q-CSP RECONFIGURATION [IDHPSUU11, Ohs23]. Given a satisfiable q-CSP instance G and a pair of its satisfying assignments f_{start} and f_{end}, this problem asks for a reconfiguration sequence $\vec{f}$ from f_{start} to f_{end} consisting of any (not necessarily satisfying) assignments for G. The objective is to maximize the minimum fraction of satisfied constraints of G, where the minimum is taken over all assignments appearing in $\vec{f}$.

MAXMIN q-CSP RECONFIGURATION

Input:	a satisfiable q-CSP instance G and a pair of its satisfying assignments f_{start} and f_{end}.
Output:	a reconfiguration sequence $\vec{f}$ from f_{start} to f_{end}.
Goal:	maximize the minimum fraction of satisfied constraints of G over all assignments in $\vec{f}$.

Solving this problem approximately, we may obtain a reasonable reconfiguration sequence consisting of almost-satisfying assignments, which may help us handle NO-instances of q-CSP RECONFIGURATION. Note that an "NP analogue" of MAXMIN q-CSP RECONFIGURATION is the MAX q-CSP problem.

We review known results on the complexity of MAXMIN 2-CSP RECONFIGURATION, which has been studied intensively. MAXMIN 2-CSP RECONFIGURATION is PSPACE-hard to solve exactly, which follows from the PSPACE-hardness of 2-CSP RECONFIGURATION, e.g., [BC09, HIZ18]. The Probabilistically Checkable Reconfiguration Proof (PCRP) theorem [GKMRW25, HO24b] implies that MAXMIN 2-CSP RECONFIGURATION is PSPACE-hard to approximate within some constant factor. Since the PCRP theorem does not provide an explicit hardness factor, gap amplifiability for MAXMIN 2-CSP RECONFIGURATION has been investigated. Note that, unlike the parallel repetition theorem for MAX 2-CSP [Raz98], parallel repetition does not reduce the soundness error of MAXMIN 2-CSP RECONFIGURATION [Ohs25a]. Ohsaka [Ohs24b] proved that a 0.9942-factor approximation is PSPACE-hard, and a $(0.75+\varepsilon)$-factor approximation is NP-hard. Ohsaka [Ohs25a] developed a $(0.25-\varepsilon)$-factor approximation algorithm for sparse 2-CSP instances. Guruswami, Karthik C. S., Manurangsi, Ren, and Wu [GKMRW25] proved the NP-hardness of a $(0.5+\varepsilon)$-factor approximation, improving upon [Ohs24b], and developed a $(0.5-\varepsilon)$-factor approximation algorithm, improving upon [Ohs25a]. These results are tight with respect to NP-hardness. Very recently, Gur, Minzer, Weissenberg, and Zheng [GMWZ26] proved the PSPACE-hardness of a $(0.9+\varepsilon)$-factor approximation. These hardness results leave open the intriguing possibility that a 0.51-factor approximation could be in NP. Indeed, Guruswami, Karthik C. S., Manurangsi, Ren, and Wu [GKMRW25] posed determining the optimal PSPACE-hardness of approximating MAXMIN 2-CSP RECONFIGURATION as an open problem.[1]

1.1 Our Results

In this paper, we prove that for each arity $q \geqslant 2$, MAXMIN q-CSP RECONFIGURATION is PSPACE-hard to approximate within a factor of $\frac{1}{2^{q-1}} + \varepsilon$.

Theorem 1.1 (informal; see Theorem 6.1)**.** *For any integer $q \geqslant 2$ and any real $\varepsilon > 0$, there exists a positive integer σ such that it is* PSPACE*-hard, given a satisfiable q-CSP instance G with alphabet size σ and a pair of its satisfying assignments f_{start} and f_{end}, to distinguish between the following two cases:*

(Completeness) *There exists a reconfiguration sequence from f_{start} to f_{end} consisting of satisfying assignments for G.*[2]

(Soundness) *Every reconfiguration sequence from f_{start} to f_{end} contains an assignment that violates more than a $\left(1 - \frac{1}{2^{q-1}} - \varepsilon\right)$-fraction of the constraints of G.*

In particular, for any integer $q \geqslant 2$ and any small real $\varepsilon > 0$, MAXMIN q-CSP RECONFIGURATION *is* PSPACE*-hard to approximate within a factor of $\frac{1}{2^{q-1}} + \varepsilon$. Moreover, the same hardness result holds even if G is regular (i.e., each variable appears in the same number of constraints).*

Theorem 1.1 identifies an *approximation threshold* of 0.5 for MAXMIN 2-CSP RECONFIGURATION; i.e., achieving an approximation factor above 0.5 is PSPACE-hard, whereas below 0.5 is in P [GKMRW25]. This resolves the open problem posed by [GKMRW25] and rules out the possibility that a 0.51-factor approximation could be in NP (unless NP = PSPACE). Theorem 1.1 also improves the PSPACE-hardness of approximation for MAXMIN q-CSP RECONFIGURATION for every $q \geqslant 2$ due to [GMWZ26] and the NP-hardness

[1] Specifically, [GKMRW25] noted the following: "the tight PSPACE-hardness threshold for GapMaxMin-2-CSP$_q$ remains open. [...] a negative answer—by placing the gap version in a complexity class believed to be a strict subset of PSPACE (e.g., Σ_2^P)—would also be very interesting (indeed, arguably more so than a PSPACE-hardness result)."

[2] This is a YES-instance of q-CSP RECONFIGURATION.

Table 1: Summary of approximability results for Maxmin q-CSP Reconfiguration for $2 \leqslant q \leqslant 5$, where $\varepsilon > 0$ is any small real, and "regular" means that each variable appears in the same number of constraints. The best results for PSPACE-hardness, NP-membership, NP-hardness, and P-membership are highlighted.

arity q	PSPACE-hard (this paper)	NP (this paper)	P (regular) (this paper)	PSPACE-hard (existing results)	NP-hard (existing results)	P (existing results)
2	$0.5+\varepsilon$	$0.5-\varepsilon$	$0.5-\varepsilon$	$0.9+\varepsilon$ [GMWZ26]	$0.5+\varepsilon$ [GKMRW25]	$0.5-\varepsilon$ [GKMRW25]
3	$0.25+\varepsilon$	$0.25-\varepsilon$	$0.25-\varepsilon$	$0.9+\varepsilon$ [GMWZ26]	$0.5+\varepsilon$ [GKMRW25]	—
4	$0.125+\varepsilon$	$0.125-\varepsilon$	$0.125-\varepsilon$	$0.81+\varepsilon$ [GMWZ26] †	ε [Ohs24b]	—
5	$0.0625+\varepsilon$	$0.0625-\varepsilon$	$0.0625-\varepsilon$	$0.5+\varepsilon$ [GMWZ26]	ε [Ohs24b]	—

† The PSPACE-hardness of a $(0.81+\varepsilon)$-factor approximation for $q=4$ follows by applying standard sequential repetition to the PSPACE-hardness of a $(0.9+\varepsilon)$-factor approximation for the case of $q=2$.

of a $(0.5+\varepsilon)$-factor approximation for Maxmin 3-CSP Reconfiguration due to [GKMRW25]. See Table 1 for a comparison of Theorem 1.1 with the existing results.

To complement Theorem 1.1, we prove that for each arity $q \geqslant 2$, a $\left(\frac{1}{2^{q-1}}-\varepsilon\right)$-factor approximation for Maxmin q-CSP Reconfiguration is in NP in the perfect completeness case.

Theorem 1.2 (informal; see Theorem 7.1)**.** *For a satisfiable q-CSP instance G with n variables and alphabet size σ and a pair of its satisfying assignments f_{start} and f_{end}, suppose that there exists a reconfiguration sequence from f_{start} to f_{end} consisting of satisfying assignments for G. Then, for any real $\varepsilon > 0$, there exists an $n^{O_{q,\sigma,\varepsilon}(1)}$-length reconfiguration sequence from f_{start} to f_{end} such that every assignment satisfies at least a $\left(\frac{1}{2^{q-1}}-\varepsilon\right)$-fraction of the constraints of G. In particular, for any positive integers $q \geqslant 2$ and σ, and any small real $\varepsilon > 0$, a $\left(\frac{1}{2^{q-1}}-\varepsilon\right)$-factor approximation for* Maxmin q-CSP Reconfiguration *is in* NP *in the perfect completeness case.*

Theorems 1.1 and 1.2 establish the optimal PSPACE-hardness of approximating Maxmin q-CSP Reconfiguration for every arity $q \geqslant 2$ assuming NP $\neq$ PSPACE.

Combining Theorems 1.1 and 1.2 and [GKMRW25, Ohs24b], we obtain the following threshold result for the approximability of Maxmin q-CSP Reconfiguration. For any reals c,s with $0 < s \leqslant c \leqslant 1$, Gap$_{c,s}$ q-CSP Reconfiguration is defined as a promise problem that asks whether the optimal value of Maxmin q-CSP Reconfiguration is at least c or less than s.

Corollary 1.3 (from Theorems 1.1 and 1.2 and [GKMRW25, Ohs24b])**.** *Let $q \geqslant 4$ be any integer and $s \in (0,1]$ be any real. For any sufficiently large alphabet size σ, the following hold:*

- *If $s > \frac{1}{2}$, then* Gap$_{1,s}$ 2-CSP Reconfiguration *is* PSPACE*-complete.*
- *If $s < \frac{1}{2}$, then* Gap$_{1,s}$ 2-CSP Reconfiguration *is in* P *[GKMRW25].*
- *If $s > \frac{1}{4}$, then* Gap$_{1,s}$ 3-CSP Reconfiguration *is* PSPACE*-complete.*
- *If $s < \frac{1}{4}$, then* Gap$_{1,s}$ 3-CSP Reconfiguration *is in* NP*.*
- *If $s > \frac{1}{2^{q-1}}$, then* Gap$_{1,s}$ q-CSP Reconfiguration *is* PSPACE*-complete.*
- *If $s < \frac{1}{2^{q-1}}$, then* Gap$_{1,s}$ q-CSP Reconfiguration *is* NP*-complete, where* NP*-hardness is shown in [Ohs24b].*

Corollary 1.3 reveals an intriguing complexity-theoretic contrast between the regimes of $q = 2$ and $q \geqslant 4$. For Maxmin 2-CSP Reconfiguration, the approximation threshold $\frac{1}{2}$ separates P-membership from PSPACE-completeness. By contrast, for Maxmin 4-CSP Reconfiguration, the approximation threshold $\frac{1}{8}$ separates NP-completeness from PSPACE-completeness. To the best of our knowledge, this is the first PSPACE-complete reconfiguration problem whose approximate version is NP-complete (rather than lying in P, as in the case of $q = 2$). See also Section 2.3 for the discussion on the approximability of Maxmin 3-CSP Reconfiguration.

We finally develop a deterministic $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation algorithm for Maxmin q-CSP Reconfiguration on ***regular*** instances; i.e., each variable appears in the same number of constraints.

Theorem 1.4 (informal; see Theorem 8.1)**.** *For a satisfiable regular q-CSP instance G, a pair of its satisfying assignments f_{start} and f_{end}, and any real $\varepsilon > 0$, there exists a polynomial-length reconfiguration sequence $\vec{f}$ from f_{start} to f_{end} such that every assignment satisfies at least a $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-fraction of the constraints of G. Moreover, such $\vec{f}$ can be found in deterministic polynomial time. In particular, for any integer $q \geqslant 2$ and any small real $\varepsilon > 0$, there exists a deterministic $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation algorithm for* Maxmin *q-*CSP Reconfiguration *on regular instances.*

1.2 Organization

The rest of this paper is organized as follows. In Section 2, we present an overview of the proofs of Theorems 1.1 and 1.2. In Section 3, we review related work on reconfiguration problems and direct product testing. In Section 4, we formally define Maxmin q-CSP Reconfiguration and its gap version. In Section 5, we introduce and analyze the tolerant q-query direct product tester. In Section 6, we prove the PSPACE-hardness of a $\left(\frac{1}{2^{q-1}} + \varepsilon\right)$-factor approximation. In Section 7, we prove the NP-membership of a $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation. In Section 8, we develop a deterministic $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation algorithm for regular instances. Some technical proofs are deferred to Appendices A and B.

2 Proof Overview

2.1 PSPACE-hardness of $\left(\frac{1}{2^{q-1}} + \varepsilon\right)$-factor Approximation (Sections 5 and 6)

First, we outline the proof of Theorem 1.1, i.e., PSPACE-hardness of a $\left(\frac{1}{2^{q-1}} + \varepsilon\right)$-factor approximation for Maxmin q-CSP Reconfiguration. Hereafter, a ***q-CSP instance*** is specified by a quadruple $G = (V, E, \Sigma, \Psi)$ such that (V, E) is a q-uniform hypergraph called the ***underlying hypergraph***, Σ is a finite set called the ***alphabet***, and $\Psi = (\psi_e)_{e \in E}$ is a collection of ***q-ary constraints*** $\psi_e \colon \Sigma^e \to \{0,1\}$ associated with hyperedges e in E. For an assignment $f \colon V \to \Sigma$, the ***value*** $\mathsf{val}_G(f)$ is defined as the fraction of constraints satisfied by f. For a reconfiguration sequence $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$, the ***value*** $\mathsf{val}_G(\vec{f})$ is defined as the minimum value of $\mathsf{val}_G(f^{(t)})$ over all assignments $f^{(t)}$ in $\vec{f}$. For a pair of satisfying assignments $f_{\mathsf{start}}, f_{\mathsf{end}} \colon V \to \Sigma$, the ***optimal value*** $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}})$ is defined as the maximum value of $\mathsf{val}_G(\vec{f})$ over all possible reconfiguration sequences $\vec{f}$ from f_{start} to f_{end}. For reals c, s with $0 < s \leqslant c \leqslant 1$, Gap$_{c,s}$ q-CSP Reconfiguration is defined as a promise problem that asks whether $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) \geqslant c$ or $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) < s$. See Section 4 for the formal definition.

The proof of Theorem 1.1 is based on the following gap-amplifying reduction from Maxmin 2-CSP Reconfiguration to Maxmin q-CSP Reconfiguration.

Lemma 2.1 (informal; see Lemma 6.2). *For any reals $s \in (0,1)$ and $\varepsilon > 0$, and any positive integers $q \geqslant 2$ and σ, there exists a positive integer k and a polynomial-time reduction from* $\text{Gap}_{1,s}$ 2-CSP Reconfiguration *with alphabet size σ to* $\text{Gap}_{1,\frac{1}{2q-1}+\varepsilon}$ q-CSP Reconfiguration *with alphabet size σ^{2k}.*

By the PCRP theorem [GKMRW25, HO24b], $\text{Gap}_{1,s}$ 2-CSP Reconfiguration with alphabet size $\mathrm{O}(1)$ is PSPACE-hard for some real $s \in (0,1)$. Combining this with Lemma 2.1, we obtain that $\text{Gap}_{1,\frac{1}{2q-1}+\varepsilon}$ q-CSP Reconfiguration is PSPACE-hard for any real $\varepsilon > 0$, which implies Theorem 1.1. In the remainder of this subsection, we outline the proof of Lemma 2.1.

For the sake of simplicity, we first show the imperfect-completeness version in the case of $q = 2$.

Lemma 2.2. *For any reals $s \in (0,1)$ and $\varepsilon > 0$, and any positive integer σ, there exists a positive integer k and a polynomial-time reduction from* $\text{Gap}_{1,s}$ 2-CSP Reconfiguration *with alphabet size σ to* $\text{Gap}_{1-o(1),\frac{1}{2}+\varepsilon}$ 2-CSP Reconfiguration *with alphabet size σ^{2k}.*

Remark 2.3. *We note that the* PSPACE*-hardness result of [GMWZ26] is based on [GKMRW25, Corollary 4], which does not inherently derive Theorem 1.1 in the case of $q = 2$. Specifically, [GKMRW25, Corollary 4] states that if there exists a q-query Probabilistically Checkable Proof of Proximity (PCPP) [BGHSV06, DR06] with randomness $\mathrm{O}(\log n)$, soundness $s < 1$, and alphabet size $\mathrm{O}_s(1)$, then* Maxmin 2-CSP Reconfiguration *is* PSPACE*-hard to approximate within a factor of $1 - \frac{1-s}{q+1}$. Since any such PCPP must have query complexity $q \geqslant 2$ (under $\mathsf{P} \neq \mathsf{NP}$),*[3] *even the best possible PCPP would yield only the $\frac{2}{3}$-factor hardness. To obtain the stronger $\left(\frac{1}{2}+\varepsilon\right)$-factor hardness, we develop a gap-amplifying reduction different from the one in [GKMRW25].*

2.1.1 Proof Overview of Lemma 2.2

Our general strategy for proving Lemma 2.2 is to encode assignments for Maxmin 2-CSP Reconfiguration using the direct product function. For a function $f\colon V \to \Sigma$, its ***k*-wise direct product** is defined as a k-set function $f^k\colon \binom{V}{k} \to \Sigma^k$ such that

$$f^k\left(\{x_1,\ldots,x_k\}\right) \coloneqq \left(f(x_1),\ldots,f(x_k)\right) \text{ for each set } \{x_1,\ldots,x_k\} \in \binom{V}{k}. \tag{2.1}$$

Goldreich and Safra [GS00] introduced the problem of direct product testing, which has found several applications to hardness of approximation [BMVY25, Din07, DM11, DR06, IKW12] and to hardness amplification. We first recapitulate the canonical 2-query direct product tester [DG08, DR06, IKW12]. Let $F\colon \binom{V}{k} \to \Sigma^k$ be a k-set function that we wish to test for closeness to the direct product function. The ***2-query direct product tester*** $\mathcal{T}_2^{\text{std}}$ first samples $\boldsymbol{I}_1 \in \binom{V}{\ell}$ and $\boldsymbol{X}_1, \boldsymbol{X}_2 \in \binom{V}{k}$ conditioned on $\boldsymbol{X}_1 \cap \boldsymbol{X}_2 \supset \boldsymbol{I}_1$,[4] where $\ell \leqslant k$ is the ***intersection parameter***. Then, $\mathcal{T}_2^{\text{std}}$ accepts if $F(\boldsymbol{X}_1)$ and $F(\boldsymbol{X}_2)$ agree on the intersection $\boldsymbol{I}_1$; namely, $F(\boldsymbol{X}_1)|_{\boldsymbol{I}_1} = F(\boldsymbol{X}_2)|_{\boldsymbol{I}_1}$. Dinur and Goldenberg [DG08, Theorem 1.3] proved that if $\mathcal{T}_2^{\text{std}}$ with $\ell = \Theta(\sqrt{k})$ accepts F with probability $p \geqslant k^{-\Omega(1)}$, then there exists a function $f\colon V \to \Sigma$ such that $F(\boldsymbol{X})$ agrees with $f^k(\boldsymbol{X})$ on all but a $k^{-\Omega(1)}$-fraction of coordinates with probability $p(1-o(1))$; namely,

$$\Pr_{\boldsymbol{X} \in \binom{V}{k}}\left[F(\boldsymbol{X}) \underset{k^{-\Omega(1)}}{\approx} f^k(\boldsymbol{X})\right] \geqslant p(1-o(1)), \tag{2.2}$$

[3] If such a 1-query PCPP existed, then there exists a 1-query PCP for NP with randomness $\mathrm{O}(\log n)$ and alphabet size $\mathrm{O}_s(1)$. Calculating its maximum acceptance probability in polynomial time, one would obtain $\mathsf{P} = \mathsf{NP}$.

[4] Throughout this paper, we write random variables in boldface.

where "$\alpha \underset{\delta}{\approx} \beta$" means that α and β are δ-close.

We now describe a gap-amplifying reduction from MAXMIN 2-CSP RECONFIGURATION to itself, which is based on the reduction from MAX 2-CSP to itself due to [IKW12, Theorem 1.4]. Let $s \in (0,1)$ and $\varepsilon > 0$ be any reals, and σ be any positive integer. Let k be a sufficiently large integer. Let $(G, f_{\text{start}}, f_{\text{end}})$ be an instance of $\text{GAP}_{1,s}$ 2-CSP RECONFIGURATION, where $G = (V, E, \Sigma, \Psi)$ is a satisfiable 2-CSP instance with $|\Sigma| = \sigma$, and $f_{\text{start}}, f_{\text{end}} \colon V \to \Sigma$ are a pair of its satisfying assignments. Our reduction is described in the language of PCP verifiers. Specifically, we design a 2-query verifier along with a pair of its accepting proofs, which define a new instance of MAXMIN 2-CSP RECONFIGURATION. The ***proof*** is represented by a k-set function $F \colon \binom{E}{k} \to (\Sigma^2)^k$; namely, for an edge set $Z \in \binom{E}{k}$, a pair of symbols in Σ^2 are assigned to the endpoints of each edge in Z. Since the domains of the assignment $f \colon V \to \Sigma$ and the proof $F \colon \binom{E}{k} \to (\Sigma^2)^k$ are different (i.e., V and E), we introduce the ***induced edge function*** of f, which is a function $f_E \colon E \to \Sigma^2$ defined as

$$f_E(xy) \coloneqq \big(f(x), f(y)\big) \text{ for each edge } xy \in E. \tag{2.3}$$

Intuitively, F is supposed to be the k-wise direct product of the induced edge function of some assignment $f \colon V \to \Sigma$. Specifically, we consider the function $f_E^k \colon \binom{E}{k} \to (\Sigma^2)^k$ such that

$$f_E^k\big(\{e_1, \ldots, e_k\}\big) \coloneqq \big(f_E(e_1), \ldots, f_E(e_k)\big) \text{ for each set } \{e_1, \ldots, e_k\} \in \binom{E}{k}. \tag{2.4}$$

Consider the 2-query verifier $\mathcal{V}$ for MAXMIN 2-CSP RECONFIGURATION, which has oracle access to $F \colon \binom{E}{k} \to (\Sigma^2)^k$ and performs the following two tests:

(Direct product test) This test aims to determine whether F is close to f_E^k for some assignment $f \colon V \to \Sigma$. For this purpose, we run $\mathcal{T}_2^{\text{std}}$ on the following ***randomized oracle*** $C^F \colon \binom{V}{k} \to \Sigma^k$ [IKW12]: For a vertex set $X \in \binom{V}{k}$, the oracle samples k random edges $\boldsymbol{Z} \in \binom{E}{k}$ incident to the vertices in X and returns $F(\boldsymbol{Z})|_X$. If $\mathcal{T}_2^{\text{std}}$ rejects, then we immediately reject; otherwise, we advance to the satisfiability test. This test makes two queries $\boldsymbol{Z}_1, \boldsymbol{Z}_2 \in \binom{E}{k}$ to F.

(Satisfiability test) This test aims to simulate $\Omega(k)$ runs of the 2-CSP verifier for G. Since F passed the direct product test, the query results obtained in the direct product test (i.e., $F(\boldsymbol{Z}_1)$ and $F(\boldsymbol{Z}_2)$) are expected to contain consistent information about assignments to $\Omega(k)$ edges of $\boldsymbol{Z}_1 \cup \boldsymbol{Z}_2$. One can thus verify multiple constraints without any additional queries. Specifically, for each $i \in [2]$, we check whether every edge $\boldsymbol{e}$ in $\boldsymbol{Z}_i$ is satisfied by the corresponding assignment $F(\boldsymbol{Z}_i)|_{\boldsymbol{e}}$.

The starting and ending proofs $F_{\text{start}}, F_{\text{end}} \colon \binom{E}{k} \to (\Sigma^2)^k$ are defined as the direct product of the induced edge functions of f_{start} and f_{end}, respectively; namely, $F_{\text{start}} \coloneqq (f_{\text{start}})_E^k$ and $F_{\text{end}} \coloneqq (f_{\text{end}})_E^k$. This defines a new instance $(\mathcal{V}, F_{\text{start}}, F_{\text{end}})$ of MAXMIN 2-CSP RECONFIGURATION, completing the description of the reduction. For a function F, the ***value*** $\mathsf{val}_{\mathcal{V}}(F)$ is defined as the probability that $\mathcal{V}$ accepts F. For a reconfiguration sequence $\vec{F} = (F^{(1)}, \ldots, F^{(T)})$, the ***value*** $\mathsf{val}_{\mathcal{V}}(\vec{F})$ is defined as the minimum value of $\mathsf{val}_{\mathcal{V}}(F^{(t)})$ over all functions $F^{(t)}$ in $\vec{F}$. The ***optimal value*** $\mathsf{opt}_{\mathcal{V}}(F_{\text{start}} \leftrightsquigarrow F_{\text{end}})$ is defined as the maximum value of $\mathsf{val}_{\mathcal{V}}(\vec{F})$ over all possible reconfiguration sequences $\vec{F}$ from F_{start} to F_{end}.

We shall prove the following properties.

(Completeness) If $\mathsf{opt}_G(f_{\text{start}} \leftrightsquigarrow f_{\text{end}}) = 1$, then $\mathsf{opt}_{\mathcal{V}}(F_{\text{start}} \leftrightsquigarrow F_{\text{end}}) \geqslant 1 - o(1)$.

(Soundness) If $\mathsf{opt}_G(f_{\text{start}} \leftrightsquigarrow f_{\text{end}}) < s$, then $\mathsf{opt}_{\mathcal{V}}(F_{\text{start}} \leftrightsquigarrow F_{\text{end}}) < \frac{1}{2} + \varepsilon$.

The completeness is almost immediate from the definition of $\mathcal{V}$, and thus we focus on proving the soundness. Assume that $\mathsf{opt}_G(f_{\mathrm{start}} \leftrightsquigarrow f_{\mathrm{end}}) < s$. Let $\vec{F} = (F^{(1)}, \ldots, F^{(T)})$ be any reconfiguration sequence from F_{start} to F_{end}. We would like to show that $\mathsf{val}_{\mathcal{V}}(\vec{F}) < \frac{1}{2} + \varepsilon$. Suppose first that $\mathcal{T}_2^{\mathrm{std}}$ accepts the randomized oracle $C^{F^{(t)}}$ with probability at most $\frac{1}{2} + \varepsilon$ for some $t \in [T]$. By definition, $\mathcal{V}$ also accepts $F^{(t)}$ with probability at most $\frac{1}{2} + \varepsilon$; i.e., $\mathsf{val}_{\mathcal{V}}(\vec{F}) < \frac{1}{2} + \varepsilon$, as desired. Hereafter, we can assume that $\mathcal{T}_2^{\mathrm{std}}$ accepts $C^{F^{(t)}}$ with probability at least $\frac{1}{2} + \varepsilon$ for every $t \in [T]$. By applying [DG08, Theorem 1.3] (see also Eq. (2.2)) to each function $F^{(t)}$, there exists an assignment $f^{(t)} \colon V \to \Sigma$ such that

$$\mathop{\mathbf{Pr}}_{\boldsymbol{X} \in \binom{V}{k}}\left[C^{F^{(t)}}(\boldsymbol{X}) \underset{k^{-\Omega(1)}}{\approx} (f^{(t)})^k(\boldsymbol{X}) \right] \geqslant \frac{1}{2} + \Omega(\varepsilon). \tag{2.5}$$

By [IKW12, Proof of Theorem 6.1] (see also Claim 6.14), we further obtain

$$\mathop{\mathbf{Pr}}_{\boldsymbol{Z} \in \binom{E}{k}}\left[F^{(t)}(\boldsymbol{Z}) \underset{k^{-\Omega(1)}}{\approx} (f^{(t)})^k_E(\boldsymbol{Z}) \right] \geqslant \frac{1}{2} + \Omega(\varepsilon). \tag{2.6}$$

Consider now the assignment sequence $\vec{f} \coloneqq (f^{(1)}, \ldots, f^{(T)})$.[5] If $\vec{f}$ were a valid reconfiguration sequence, the soundness assumption would imply that $\mathsf{val}_G(f^{(t)}) < s$ for some assignment $f^{(t)}$. However, $\vec{f}$ is not necessarily a reconfiguration sequence since $f^{(t)}$ and $f^{(t+1)}$ may differ in two or more vertices. To address this issue, we first use the following claim.

Claim 2.4 (informal; see Claim 6.6)**.** *The distance between $f^{(t)}$ and $f^{(t+1)}$ is $k^{-\Omega(1)}$.*

The crux of the proof of Claim 2.4 is that we can perform the *unique decoding* on $F^{(t)}$ and $F^{(t+1)}$ to obtain $f^{(t)}$ and $f^{(t+1)}$. Since $F^{(t)}$ and $F^{(t+1)}$ differ only in a single coordinate and Eq. (2.6) holds, $(f^{(t)})^k_E(\boldsymbol{Z})$ and $(f^{(t+1)})^k_E(\boldsymbol{Z})$ approximately agree with non-negligible probability $\Omega(\varepsilon)$, which implies that $f^{(t)}$ and $f^{(t+1)}$ must become closer as k increases. We then show that if every adjacent pair of assignments in $\vec{f}$ are sufficiently close, then some assignment $f^{(t)}$ violates an $\Omega(1)$-fraction of the constraints.

Claim 2.5 (informal; see Claim 6.7)**.** *If $f^{(t)}$ and $f^{(t+1)}$ are $\frac{1-s}{4}$-close for every $t \in [T-1]$, then there exists some assignment $f^{(t)}$ such that*

$$\mathsf{val}_G(f^{(t)}) < \frac{1+s}{2} = 1 - \Omega(1). \tag{2.7}$$

In the proof of Claim 2.5, we reconstruct a valid reconfiguration sequence from $\vec{f}$ by "interpolating" between each adjacent pair $f^{(t)}$ and $f^{(t+1)}$. Since we can assume that the underlying graph (V,E) is regular [Ohs23], all intermediate assignments connecting $f^{(t)}$ and $f^{(t+1)}$ have nearly the same value.

By applying Claims 2.4 and 2.5 for sufficiently large k, we obtain $F^{(t)}$ and $f^{(t)}$ satisfying both Eqs. (2.6) and (2.7). It remains to bound the acceptance probability of $\mathcal{V}$ from above.

Claim 2.6 (informal; see Claim 6.8)**.** *$\mathcal{V}$ accepts $F^{(t)}$ with probability at most $\frac{1}{2} + \varepsilon$.*

To provide intuition for the proof of Claim 2.6, let us ignore the correlation between the two queries $\mathbf{Z}_1, \mathbf{Z}_2 \in \binom{E}{k}$ and consider a thought experiment in which they are independent. By Eq. (2.6), $F^{(t)}(\mathbf{Z}_i) \underset{k^{-\Omega(1)}}{\approx} (f^{(t)})^k_E(\mathbf{Z}_i)$

[5] We can safely assume that $f^{(1)} = f_{\mathrm{start}}$ and $f^{(T)} = f_{\mathrm{end}}$.

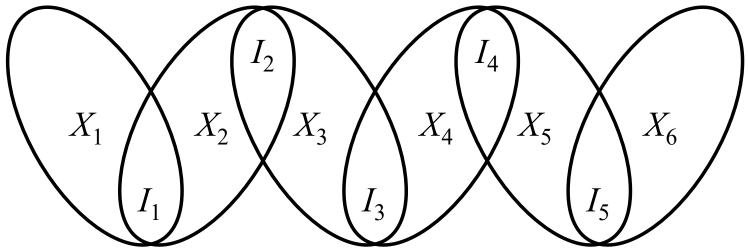

Figure 1: Illustration of the query sequence of the q-query direct product tester. For each $i \in [q-1]$, the consecutive query sets X_i and X_{i+1} both contain I_i, yielding the "zig-zag" pattern.

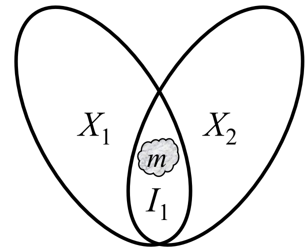

Figure 2: Illustration of the acceptance condition of the tolerant direct product tester, which allows up to m disagreements between $F(\boldsymbol{X}_1)|_{\boldsymbol{I}_1}$ and $F(\boldsymbol{X}_2)|_{\boldsymbol{I}_1}$.

holds for some $i \in [2]$ with probability $1 - \frac{1}{4}$. Moreover, by Eq. (2.7), $f^{(t)}$ violates an $\Omega(1)$-fraction of the edges in such $\boldsymbol{Z}_i$ with probability $1 - 2^{-\Omega(k)}$. Combining these events, with probability $1 - \frac{1}{4} - 2^{-\Omega(k)}$, at least one edge of $\boldsymbol{Z}_i$ is violated by $F^{(t)}(\boldsymbol{Z}_i)$; i.e., $F^{(t)}$ would fail the satisfiability test. Consequently, $\mathsf{val}_{\mathcal{V}}(\vec{F}) \leqslant \mathsf{val}_{\mathcal{V}}(F^{(t)}) \leqslant \frac{1}{4} + 2^{-\Omega(k)} < \frac{1}{2} + \varepsilon$, which completes the proof of Lemma 2.2. □

2.1.2 Highlight of the Proof of Lemma 2.1

Consider now proving Lemma 2.1. To this end, we extend the 2-query direct product tester $\mathcal{T}_2^{\mathrm{std}}$ used in Section 2.1.1 in two directions. Let $F \colon \binom{V}{k} \to \Sigma^k$ be a k-set function.

(q-query tester) To reduce MAXMIN 2-CSP RECONFIGURATION to MAXMIN q-CSP RECONFIGURATION, we modify $\mathcal{T}_2^{\mathrm{std}}$ to make q queries for an arbitrary integer $q \geqslant 2$. Specifically, we first sample $\boldsymbol{I}_1, \ldots, \boldsymbol{I}_{q-1} \in \binom{V}{\ell}$ and $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_q \in \binom{V}{k}$ conditioned on $\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1} \supset \boldsymbol{I}_i$ for every $i \in [q-1]$. Then, we accept if $F(\boldsymbol{X}_i)$ and $F(\boldsymbol{X}_{i+1})$ agree on $\boldsymbol{I}_i$ for every $i \in [q-1]$; namely, $F(\boldsymbol{X}_i)|_{\boldsymbol{I}_i} = F(\boldsymbol{X}_{i+1})|_{\boldsymbol{I}_i}$. See Figure 1 for an illustration of the "zig-zag" query pattern of this test. Note that the cases of $q = 2$ and $q = 3$ coincide with the 2-query direct product tester (called the "V-test") [DG08, DR06, IKW12] and the 3-query direct product tester (called the "Z-test") [DL23, IKW12],[6] respectively. Hence, this q-query tester can be thought of as a natural generalization of these canonical direct product testers.

(Tolerant tester) To ensure perfect completeness, we relax the acceptance condition of $\mathcal{T}_2^{\mathrm{std}}$ so as to accept "approximate" direct product functions. Similarly to $\mathcal{T}_2^{\mathrm{std}}$, we first sample $\boldsymbol{I}_1 \in \binom{V}{\ell}$ and $\boldsymbol{X}_1, \boldsymbol{X}_2 \in \binom{V}{k}$ conditioned on $\boldsymbol{X}_1 \cap \boldsymbol{X}_2 \supset \boldsymbol{I}_1$. Then, we accept if $F(\boldsymbol{X}_1)$ and $F(\boldsymbol{X}_2)$ agree on at least $\ell - m$ coordinates of $\boldsymbol{I}_1$; namely, $F(\boldsymbol{X}_1)|_{\boldsymbol{I}_1} \underset{m/\ell}{\approx} F(\boldsymbol{X}_2)|_{\boldsymbol{I}_1}$. Here, m is the ***tolerance parameter*** such that $1 \leqslant m \leqslant \ell$. See Figure 2 for an illustration of the acceptance condition of this test. Observe that if there exists a function $f \colon V \to \Sigma$ such that $F(X) \underset{m/(2k)}{\approx} f^k(X)$ for each set $X \in \binom{V}{k}$, then F is accepted with probability 1. In this sense, we always accept approximate direct product functions.

Combining these extensions, we obtain the ***tolerant q-query direct product tester*** $\mathcal{T}_q^{\mathrm{tol}}$ described below.

[6] Strictly speaking, the case of $q = 3$ is identical to [DL23, Test 1].

Tolerant q-query direct product tester $\mathcal{T}_q^{\text{tol}}$

Input: an intersection parameter ℓ and a tolerance parameter m with $1 \leqslant m \leqslant \ell \leqslant k$.
Oracle access: a k-set function $F\colon \binom{V}{k} \to \Sigma^k$.

1: sample $\boldsymbol{I}_1, \ldots, \boldsymbol{I}_{q-1}$ from $\binom{V}{\ell}$ uniformly.
2: sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_q$ from $\binom{V}{k}$ uniformly conditioned on $\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1} \supset \boldsymbol{I}_i$ for every $i \in [q-1]$.
3: **if** $F(\boldsymbol{X}_i)|_{\boldsymbol{I}_i} \underset{m/\ell}{\approx} F(\boldsymbol{X}_{i+1})|_{\boldsymbol{I}_i}$ for every $i \in [q-1]$ **then**
4: **return** 1.
5: **else**
6: **return** 0.

The following theorem provides the completeness and soundness guarantees of $\mathcal{T}_q^{\text{tol}}$, which may be of independent interest.

Theorem 2.7 (informal; see Theorem 5.1)**.** *For a k-set function $F\colon \binom{V}{k} \to \Sigma^k$, the following hold:*

(Completeness) *Suppose that there exists a function $f\colon V \to \Sigma$ such that*

$$\Pr_{\boldsymbol{X} \in \binom{V}{k}} \left[F(\boldsymbol{X}) \underset{m/(2k)}{\approx} f^k(\boldsymbol{X}) \right] \geqslant p. \tag{2.8}$$

Then, $\mathcal{T}_q^{\text{tol}}$ accepts F with probability at least $p^q(1-o(1))$. Moreover, if $p = 1$, then $\mathcal{T}_q^{\text{tol}}$ accepts F with probability 1.

(Soundness) *Suppose that $\mathcal{T}_q^{\text{tol}}$ accepts F with probability $p \geqslant k^{-\Omega(1)}$. Then, there exists a function $f\colon V \to \Sigma$ such that*

$$\Pr_{\boldsymbol{X} \in \binom{V}{k}} \left[F(\boldsymbol{X}) \underset{\delta}{\approx} f^k(\boldsymbol{X}) \right] \geqslant p^{\frac{1}{q-1}}(1-o(1)), \text{ where } \delta = \max\left\{ \frac{\ell}{k}, \frac{m}{\ell} \right\}^{\Omega(1)}. \tag{2.9}$$

Some remarks are in order. First, approximate agreement in the soundness guarantee becomes more likely as q increases. Suppose, for example, that $\mathcal{T}_q^{\text{tol}}$ accepts F with probability $\frac{1}{2^{q-1}} + \varepsilon$; then, there exists a function $f\colon V \to \Sigma$ such that $F(\boldsymbol{X}) \underset{\delta}{\approx} f^k(\boldsymbol{X})$ with probability $\frac{1}{2} + \Omega_q(\varepsilon)$. Second, we have a good approximate agreement parameter $\delta = k^{-\Omega(1)}$ as long as $\frac{\ell}{k} = k^{-\Omega(1)}$ and $\frac{m}{\ell} = k^{-\Omega(1)}$. For example, setting $\ell := \Theta(\sqrt{k})$ and $m := \Theta(\sqrt{\ell})$ is sufficient. Theorem 2.7 can be proved by extending the analysis of $\mathcal{T}_2^{\text{std}}$ due to [DG08, IKW12]. Our insight behind the proof is that the query sequence $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_q)$ of $\mathcal{T}_q^{\text{tol}}$ can be approximated by the length-q random walk on the Johnson graph $J(n,k,\ell)$.[7] Since $J(n,k,\ell)$ is an $\mathrm{O}\left(\frac{\ell}{k}\right)$-expander [DG08], we can apply the expander hitting property [AFWZ95, AKS87] to extend the argument of [DG08] to the q-query regime. See Section 5 for details.

Our gap-amplifying reduction from Maxmin 2-CSP Reconfiguration to Maxmin q-CSP Reconfiguration is obtained basically from the reduction in Section 2.1.1 by replacing the standard 2-query direct product tester $\mathcal{T}_2^{\text{std}}$ with the tolerant q-query direct product tester $\mathcal{T}_q^{\text{tol}}$. See Section 6 for details.

[7]The vertex set of $J(n,k,\ell)$ is $\binom{[n]}{k}$ and the edge set contains an edge X_1X_2 if $|X_1 \cap X_2| = \ell$.

2.2 NP-membership of $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor Approximation (Section 7)

Second, we outline the proof of Theorem 1.2, i.e., NP-membership of a $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation in the perfect completeness case. Let $G = (V, E, \Sigma, \Psi)$ be a satisfiable q-CSP instance, and $f_{\mathrm{start}}, f_{\mathrm{end}} \colon V \to \Sigma$ be a pair of its satisfying assignments. Define $n := |V|$, $m := |E|$, and $\sigma := |\Sigma|$. Let $\varepsilon > 0$ be any real. Assume that $\mathsf{opt}_G(f_{\mathrm{start}} \leftrightsquigarrow f_{\mathrm{end}}) = 1$. To prove Theorem 1.2, it is sufficient to show that there exists a polynomial-length reconfiguration sequence whose value is at least $\frac{1}{2^{q-1}} - \varepsilon$.

Our main idea is to partition the vertex set V into low-degree and high-degree vertices. A similar strategy was used to approximate Maxmin 2-CSP Reconfiguration [GKMRW25] and Maxmin k-Cut Reconfiguration [HO25b]. Define

$$\Delta := \Theta\left(\frac{\varepsilon^2 m}{q \log n}\right). \tag{2.10}$$

For a vertex x of G, let $d(x)$ denote the ***degree*** of v. We say that a vertex is ***low degree*** if its degree is at most Δ and ***high degree*** otherwise. Let L and H denote the sets of low-degree and high-degree vertices, respectively; namely, $L := \{x \in V \mid d(x) \leqslant \Delta\}$ and $H := \{x \in V \mid d(x) > \Delta\}$. The number of high-degree vertices is bounded by $|H| \leqslant \frac{qm}{\Delta} = \mathrm{O}_{q,\varepsilon}(\log n)$.

We first consider the restricted case where f_{start} and f_{end} disagree only on low-degree vertices. Then, there always exists a polynomial-length reconfiguration sequence from f_{start} to f_{end} with value at least $\frac{1}{2^{q-1}} - \varepsilon$.

Lemma 2.8 (informal; see Lemma 7.2)**.** *Let D denote the set of vertices on which f_{start} and f_{end} disagree; namely, $D := \{x \in V \mid f_{\mathrm{start}}(x) \neq f_{\mathrm{end}}(x)\}$. If every vertex $x \in D$ is low degree, then there exists a reconfiguration sequence $\vec{f}$ from f_{start} to f_{end} of length $|D|+1$ such that $\mathsf{val}_G(\vec{f}) \geqslant \frac{1}{2^{q-1}} - \varepsilon$.*

Proof Sketch. Consider a random reconfiguration sequence $\vec{\boldsymbol{f}} = (\boldsymbol{f}^{(1)}, \ldots, \boldsymbol{f}^{(|D|+1)})$ from f_{start} to f_{end} obtained by changing the assignments to the vertices of D in a random order. Simple calculation yields that each intermediate assignment $\boldsymbol{f}^{(t)}$ satisfies each hyperedge $e \in E$ with probability at least $\frac{1}{2^{q-1}}$. Since the degree of every vertex in D is at most Δ, McDiarmid's inequality [McD89] derives that with probability $1 - n^{-\Omega(1)}$, all functions in $\vec{\boldsymbol{f}}$ simultaneously satisfy at least a $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-fraction of the constraints. Therefore, $\mathsf{val}_G(\vec{\boldsymbol{f}}) \geqslant \frac{1}{2^{q-1}} - \varepsilon$ with high probability, as desired. □

Lemma 2.8 implies that we can ignore assignments to the low-degree vertices, which exponentially reduces the number of assignments to consider.

We now consider the general case where f_{start} and f_{end} may disagree on high-degree vertices. Let $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$ be a reconfiguration sequence from f_{start} to f_{end} consisting only of satisfying assignments for G. Since $\vec{f}$ may be exponentially long, we shall sparsify it to reduce the number of assignments. Specifically, we extract a subsequence of $\vec{f}$ by the following procedure.

Sparsifying $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$

1: let $I := [T]$.
2: **for each** index $t_\ell \in I$ in ascending order **do**
3: find the largest index $t_r \in I$ such that $f^{(t_\ell)}|_H = f^{(t_r)}|_H$.
4: **if** $t_\ell + 1 \leqslant t_r - 1$ **then**
5: delete $t_\ell + 1, \ldots, t_r - 1$ from I.
6: **return** the subsequence $(f^{(t)})_{t \in I}$.

Intuitively, we consider the Hamming graph whose vertices are partial assignments $\alpha\colon H \to \Sigma$ to the high-degree vertices, and compress subpaths connecting the same vertex of this graph. Let $\vec{g} = (g^{(1)}, \ldots, g^{(T')})$ denote the subsequence of $\vec{f}$ obtained by the sparsification procedure. Note that $g^{(1)} = f^{(1)}$ and $g^{(T')} = f^{(T)}$. Consider first bounding the length T' of $\vec{g}$. Let $\alpha\colon H \to \Sigma$ be a partial assignment for the high-degree vertices. By the sparsification procedure, α appears in $g^{(1)}|_H, \ldots, g^{(T')}|_H$ at most twice.[8] The total number of such partial assignments is thus $|\Sigma^H|$. Therefore, $T' \leqslant 2\sigma^{|H|} \leqslant n^{\mathrm{O}_{q,\sigma,\varepsilon}(1)}$.

We are now ready to construct a polynomial-length reconfiguration sequence with value at least $\frac{1}{2^{q-1}} - \varepsilon$. Observe that for each $t \in [T'-1]$, either $g^{(t)}|_H = g^{(t+1)}|_H$, or $g^{(t)}$ and $g^{(t+1)}$ differ in a single vertex. In the former case, by Lemma 2.8, there exists a reconfiguration sequence $\vec{h}_t$ from $g^{(t)}$ to $g^{(t+1)}$ of length at most $n+1$ with value at least $\frac{1}{2^{q-1}} - \varepsilon$. In the latter case, $\vec{h}_t := (g^{(t)}, g^{(t+1)})$ is already a valid reconfiguration sequence with value 1. By concatenating $\vec{h}_t$ for every $t \in [T'-1]$, we obtain a reconfiguration sequence $\vec{h}$ from $g^{(1)}$ to $g^{(T')}$ with value at least $\frac{1}{2^{q-1}} - \varepsilon$. Moreover, the length of $\vec{h}$ is at most $T'(n+1) \leqslant n^{\mathrm{O}_{q,\sigma,\varepsilon}(1)}$. Consequently, there exists an $n^{\mathrm{O}_{q,\sigma,\varepsilon}(1)}$-length reconfiguration sequence from f_{start} to f_{end} whose value is at least $\frac{1}{2^{q-1}} - \varepsilon$, which completes the proof of Theorem 1.2.

2.3 Discussion and Open Problem

In this paper, we establish the optimal PSPACE-hardness of approximating MAXMIN q-CSP RECONFIGURATION; namely, a $\left(\frac{1}{2^{q-1}} + \varepsilon\right)$-factor approximation is PSPACE-hard, whereas a $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation is in NP. In particular, we discover that the approximability of MAXMIN 4-CSP RECONFIGURATION can be NP-complete. Prior to this work, researchers had implicitly assumed that the approximability of reconfiguration problems falls into either P or PSPACE-complete. Our results exemplify the first reconfiguration problem whose approximation is both NP-complete and PSPACE-complete, suggesting a more intricate complexity landscape for the approximability of reconfiguration problems. Specifically, to fully understand the complexity of approximating a reconfiguration problem, one may need to resolve not only (1) polynomial-time approximability and (2) PSPACE-hardness of approximation, but also (3) NP-membership of approximation and (4) NP-hardness of approximation.

An immediate open problem arising from Corollary 1.3 is to determine the complexity of a $\left(\frac{1}{4} - \varepsilon\right)$-factor approximation for MAXMIN 3-CSP RECONFIGURATION, which is currently only known to lie in NP. There are two possibilities. One is that a $\left(\frac{1}{4} - \varepsilon\right)$-factor approximation is in P, analogous to the case of MAXMIN 2-CSP RECONFIGURATION, which admits a $\left(\frac{1}{2} - \varepsilon\right)$-factor approximation [GKMRW25]. This would require finding a reconfiguration sequence achieving a $\left(\frac{1}{4} - \varepsilon\right)$-factor approximation in polynomial time. The other is that a $\left(\frac{1}{4} - \varepsilon\right)$-factor approximation is NP-hard. By Theorem 1.4, MAXMIN 3-CSP

[8]Suppose that there exist $t_1, t_2, t_3 \in I$ such that $t_1 < t_2 < t_3$ and $f^{(t_1)}|_H = f^{(t_2)}|_H = f^{(t_3)}|_H$. Then, the sparsification procedure would have removed t_2.

Reconfiguration on regular instances admits a $\left(\frac{1}{4} - \varepsilon\right)$-factor approximation. Therefore, proving such NP-hardness would require a reduction that produces highly non-regular instances. This is a different trend from Max q-CSP, which exhibits almost the same inapproximability behavior on regular and non-regular instances, see, e.g., [MZ24, §7], [MZ25, §6], and [Sta22]. Indeed, to show that Maxmin 4-CSP Reconfiguration is NP-hard to approximate within any constant factor, Ohsaka [Ohs24b] constructed a reduction from Max 2-CSP to Maxmin 4-CSP Reconfiguration such that a few variables appear in all constraints. It is unclear how to adapt this reduction to Maxmin 3-CSP Reconfiguration.

3 Related Work

3.1 Reconfiguration Problems

Combinatorial reconfiguration aims to study algorithmic problems and structural properties in the space of feasible solutions. In the unified framework [IDHPSUU11], a ***reconfiguration problem*** is defined with respect to a combinatorial problem Π and a transformation rule R on the feasible solutions of Π. For an instance $\mathcal{I}$ of Π and a pair of its feasible solutions S_{start} and S_{end}, the reconfiguration problem asks whether S_{start} can be transformed into S_{end} by repeatedly applying the transformation rule R while always preserving the feasibility of any intermediate solution. Speaking differently, the reconfiguration problem concerns the st-connectivity in the ***configuration graph***, which is a graph $G_{\mathcal{I},R}$ where each node corresponds to a feasible solution of $\mathcal{I}$ and each link represents that its endpoints can be transformed into each other by applying R. The instance $(\mathcal{I}, S_{\text{start}}, S_{\text{end}})$ is a Yes-instance of the reconfiguration problem if and only if there is a path from S_{start} to S_{end} in $G_{\mathcal{I},R}$. Such a sequence of feasible solutions that forms a path in the configuration graph is called a ***reconfiguration sequence***. Over the past two decades, many combinatorial problems have given rise to reconfiguration problems. For example, reconfiguration problems of 3-SAT [GKMP09], 4-Coloring [BC09], Independent Set [HD05, HD09, KMM12], and Shortest Path [Bon13] are PSPACE-complete, whereas those of 2-SAT [GKMP09], 3-Coloring [CvdHJ11], Matching [IDHPSUU11], and Spanning Tree [IDHPSUU11] belong to P. We refer the reader to the surveys [BMNS24, MN19, Nis18, vdHeu13] as well as the Combinatorial Reconfiguration wiki [Hoa24].

3.2 Approximability of Reconfiguration Problems

For a reconfiguration problem, its approximate version allows infeasible feasible solutions, but requires optimizing the "worst" feasibility along the reconfiguration sequence. Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, and Uno [IDHPSUU11] showed that several reconfiguration problems are NP-hard to approximate. Since many reconfiguration problems are PSPACE-complete, NP-hardness results are not optimal. [IDHPSUU11] posed the PSPACE-hardness of approximation for reconfiguration problems as an open problem. Ohsaka [Ohs23] postulated a reconfiguration analogue of the PCP theorem [ALMSS98, AS98], called the Reconfiguration Inapproximability Hypothesis (RIH), and proved that assuming RIH, several reconfiguration problems are PSPACE-hard to approximate, including those of 3-SAT, Independent Set, Vertex Cover, Clique, and Set Cover. Guruswami, Karthik C. S., Manurangsi, Ren, and Wu [GKMRW25] and Hirahara and Ohsaka [HO24b] independently proved RIH by establishing the Probabilistically Checkable Reconfiguration Proof (PCRP) theorem, which provides a PCP-type characterization of PSPACE. The PCRP theorem, along with a series of gap-preserving reductions, implies the PSPACE-hardness of approximating the reconfiguration problems listed above, thereby resolving the open problem of [IDHPSUU11] affirmatively.

Since the PCRP theorem itself only implies PSPACE-hardness of approximation within some constant factor, subsequent work has investigated explicit hardness factors. In the NP regime, the parallel repetition theorem of Raz [Raz98] can be used to derive many strong inapproximability results, e.g., [BGS98, Fei98, Hås01, Hås99]. In particular, MAX 2-CSP is NP-hard to approximate within any constant factor. However, for MAXMIN 2-CSP RECONFIGURATION, parallel repetition does not reduce the soundness error and a $(0.25-\varepsilon)$-factor approximation algorithm exists for sparse instances [Ohs25a]. Using Dinur's gap amplification [Din07], Ohsaka [Ohs24b] showed that MAXMIN 2-CSP RECONFIGURATION and MINMAX SET COVER RECONFIGURATION are PSPACE-hard to approximate within factors of 0.9942 and 1.0029, respectively. Guruswami, Karthik C. S., Manurangsi, Ren, and Wu [GKMRW25] proved the NP-hardness of a $(0.5+\varepsilon)$-factor approximation for MAXMIN 2-CSP RECONFIGURATION and of a $(2-\varepsilon)$-factor approximation for MINMAX SET COVER RECONFIGURATION. These results are tight with respect to NP-hardness because MAXMIN 2-CSP RECONFIGURATION admits a $(0.5-\varepsilon)$-factor approximation algorithm [GKMRW25] and MINMAX SET COVER RECONFIGURATION admits a 2-factor approximation algorithm [IDHPSUU11]. Hirahara and Ohsaka [HO24a] showed that MINMAX SET COVER RECONFIGURATION is PSPACE-hard to approximate within a factor of $2-o(1)$, improving upon [GKMRW25, Ohs24b]. This is the first optimal PSPACE-hardness result for approximability of reconfiguration problems. Hirahara and Ohsaka [HO25b] showed that the approximation threshold of MAXMIN k-CUT RECONFIGURATION is $1-\Theta\left(\frac{1}{k}\right)$. Hirahara and Ohsaka [HO25a] showed that the approximation threshold of MAXMIN Ek-SAT RECONFIGURATION is $1-\Theta\left(\frac{1}{k}\right)$. Since the approximation threshold of its NP analogue, i.e., MAX Ek-SAT, is $1-\frac{1}{2^k}$ [Hås01], this is the first reconfiguration problem whose approximation threshold is (asymptotically) worse than that of its NP analogue. Very recently, Gur, Minzer, Weissenberg, and Zheng [GMWZ26] proved the PSPACE-hardness of a $(0.9+\varepsilon)$-factor approximation for MAXMIN 2-CSP RECONFIGURATION and of a $(0.5+\varepsilon)$-factor approximation for MAXMIN 5-CSP RECONFIGURATION, improving upon [GKMRW25, Ohs24b].

3.3 Direct Product Testing

Goldreich and Safra [GS00] introduced direct product testing as a combinatorial analogue of low-degree testing [AS03, RS97]. For a function $f\colon V\to\Sigma$, its k-wise direct product is a function $f^k\colon \binom{V}{k}\to\Sigma^k$ such that $f^k(X)=f|_X$ for each $X\in\binom{V}{k}$. In direct product testing, we aim to design an efficient test that, given oracle access to a function $F\colon\binom{V}{k}\to\Sigma^k$, determines whether F is close to some direct product function. There are two regimes of interest with respect to the acceptance probability p of the test. The high-soundness (or "99%") regime considers the case where $p=1-\varepsilon$ for a small real $\varepsilon\approx 0$. One wishes to show that there exists a function $f\colon V\to\Sigma$ such that $F(X)\approx f|_X$ for a $(1-\mathrm{O}(\varepsilon))$-fraction of $X\in\binom{V}{k}$. The low-soundness (or "1%") regime in which p may be very small is more challenging. One is expected to show that $F(X)\approx f|_X$ for a $\mathrm{poly}(p)$-fraction of $X\in\binom{V}{k}$. This regime is particularly important for applications to constructing PCPs with small soundness.

In the 99% regime, Goldreich and Safra [GS00] gave the first direct product tester with a constant number of queries. Dinur and Reingold [DR06] later reduced the query complexity to 2. In the 1% regime, Dinur and Goldenberg [DG08] first proved that if a 2-query direct product tester (called the "V-test") accepts F with probability $p\geqslant k^{-\Omega(1)}$, then $F(X)\approx f|_X$ for an $\Omega(p^6)$-fraction of $X\in\binom{V}{k}$. [DG08] also showed that the V-test does not work when $p\ll k^{-1}$. Impagliazzo, Kabanets, and Wigderson [IKW12] introduced a 3-query direct product test (called the "Z-test"), which works even when $p\leqslant 2^{-\mathrm{poly}(k)}$. Moreover, [IKW12] showed how to apply the V-test to obtain a 2-query PCP whose soundness error decreases exponentially in k, giving an alternative proof of the parallel repetition theorem [Raz98]. Subsequent work further investigated both the V-test and the Z-test for various parameter settings [DL23, DS14].

One drawback of the direct product encoding is its size: encoding a function $f\colon V \to \Sigma$ by its k-wise direct product $f^k\colon \binom{V}{k} \to \Sigma^k$ incurs the exponential size in k. Such a blow-up is problematic in some applications, e.g., when constructing length-efficient PCPs with small soundness error. This motivates the study of derandomized direct product testing, which aims to identify a small subfamily $\mathcal{K} \subset \binom{V}{k}$ such that direct product testing remains possible when F is defined only on $\mathcal{K}$. Early derandomization results were obtained in [GS00, IKW12]. Recently, Dinur and Kaufman [DK17] proposed derandomized direct product testing on high-dimensional expanders, which led to significant progress [BLM24, BM24, DD19, DD24a, DD24b, DDL24, KOW25].

Our focus differs from the above studies in that for each $q \geqslant 2$, we need a q-query direct product tester that accepts approximate direct product functions and captures the fine-grained relation between the acceptance probability and the approximate agreement with direct product functions.

4 Preliminaries

Notations. For a nonnegative integer n, let $[n] := \{1,2,3,\dots,n\}$. Unless otherwise specified, the base of logarithms is 2. The symmetric group on $[n]$ (i.e., the set of all permutations of $[n]$) is denoted by $\mathfrak{S}_n$. For nonnegative integers n and k with $k \leqslant n$, let $n^{\underline{k}}$ denote the falling factorial; namely, $n^{\underline{k}} := \binom{n}{k}k! = \prod_{0 \leqslant i \leqslant k-1}(n-i)$. For a finite set V and a positive integer k, we write $\binom{V}{k}$ for the family of all size-k subsets of V, and write $V^{\underline{k}}$ for the set of all k-tuples with distinct elements in V; namely,

$$\binom{V}{k} := \left\{\{x_1,\dots,x_k\} \subseteq V \;\middle|\; \forall i \neq j,\, x_i \neq x_j\right\}, \tag{4.1}$$

$$V^{\underline{k}} := \left\{(x_1,\dots,x_k) \in V^k \;\middle|\; \forall i \neq j,\, x_i \neq x_j\right\}. \tag{4.2}$$

Note that $\left|\binom{V}{k}\right| = \binom{|V|}{k}$, $|V^{\underline{k}}| = |V|^{\underline{k}}$, and $V^{\underline{k}} \subset V^k$. A sequence of a finite number of elements $a^{(1)},\dots,a^{(T)}$ is denoted by $\vec{a} = (a^{(1)},\dots,a^{(T)})$. We use the Iverson bracket $[\![\cdot]\!]$; i.e., for a statement P, we define $[\![P]\!]$ as 1 if P is true and 0 otherwise. We write random variables in boldface (e.g., $\boldsymbol{x}$ and $\boldsymbol{X}$). For a set V, we write $\boldsymbol{x} \in V$ to mean that $\boldsymbol{x}$ is a random variable uniformly sampled from V.

Let $f,g\colon \mathcal{X} \to \mathcal{Y}$ be two functions. We shall use the function notation $f\colon \mathcal{X} \to \mathcal{Y}$ and the string notation $f \in \mathcal{Y}^{\mathcal{X}}$ interchangeably. For a subset $\mathcal{I} \subset \mathcal{X}$, we use $f|_{\mathcal{I}}\colon \mathcal{I} \to \mathcal{Y}$ to denote the restriction of f to $\mathcal{I}$. The ***relative Hamming distance*** between f and g, denoted by $\mathrm{dist}(f,g)$, is defined as the fraction of coordinates on which f and g are different; namely,

$$\mathrm{dist}(f,g) := \frac{1}{|\mathcal{X}|}\left|\left\{x \in \mathcal{X} \;\middle|\; f(x) \neq g(x)\right\}\right| = \mathop{\mathbf{Pr}}_{\boldsymbol{x} \in \mathcal{X}}\left[f(\boldsymbol{x}) \neq g(\boldsymbol{x})\right]. \tag{4.3}$$

We say that f is ***δ-close*** to g if $\mathrm{dist}(f,g) \leqslant \delta$, and f is ***δ-far*** from g if $\mathrm{dist}(f,g) > \delta$. We denote $f \underset{\delta}{\approx} g$ to mean that f is δ-close to g, and $f \underset{\delta}{\not\approx} g$ to mean that f is δ-far from g. The ***mixture*** of f and g, denoted by $\mathsf{mix}(f,g)$, is defined as the set of all functions $h\colon \mathcal{X} \to \mathcal{Y}$ such that $h(x)$ is equal to $f(x)$ or $g(x)$ for each $x \in \mathcal{X}$; namely,

$$\mathsf{mix}(f,g) := \left\{h\colon \mathcal{X} \to \mathcal{Y} \;\middle|\; \forall x \in \mathcal{X},\, h(x) \in \left\{f(x),g(x)\right\}\right\}. \tag{4.4}$$

Consider a k-set function $F\colon \binom{V}{k} \to \Sigma^k$ for two finite sets V and Σ and a positive integer k. For notational convenience, we view $F(S)$ for each set $S \in \binom{V}{k}$ as a k-tuple indexed by the elements of S, i.e., $F(S) \in \Sigma^S$. Specifically, if $F(S) = (\alpha_1, \ldots, \alpha_k)$, then we write $F(S)|_{x_i} := \alpha_i$ for a canonical ordering $S = \{x_1, \ldots, x_k\}$. For a positive integer k and a function $f\colon V \to \Sigma$, the ***k-wise direct product*** of f is defined as a k-set function $f^k\colon \binom{V}{k} \to \Sigma^k$ such that

$$f^k\bigl(\{x_1, \ldots, x_k\}\bigr) := \bigl(f(x_1), \ldots, f(x_k)\bigr) \text{ for each } k\text{-set } \{x_1, \ldots, x_k\} \in \tbinom{V}{k}. \tag{4.5}$$

By abuse of notation, we also think of the k-wise direct product as a k-tuple function $f^k\colon V^k \to \Sigma^k$ such that

$$f^k(x_1, \ldots, x_k) := \bigl(f(x_1), \ldots, f(x_k)\bigr) \text{ for each } k\text{-tuple } (x_1, \ldots, x_k) \in V^k. \tag{4.6}$$

For a graph $G = (V,E)$, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We write xy for an edge between a pair of vertices x and y. For a vertex set $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. The ***size*** of G, denoted by $|G|$, is defined as the number of edges in G; namely, $|G| := |E(G)|$. The ***degree*** of v, denoted by $d(v)$, is defined as the number of edges that are incident to v. We use the same notations for hypergraphs.

Definition of MAXMIN q-CSP RECONFIGURATION. We formulate MAXMIN q-CSP RECONFIGURATION and its gap version. The notion of q-CSP instance is introduced below.

Definition 4.1. A ***q-CSP instance*** is defined as a quadruple $G = (V, E, \Sigma, \Psi)$ such that

- (V,E) is a q-uniform hypergraph called the ***underlying hypergraph***,
- Σ is a finite set called the ***alphabet***, and
- $\Psi = (\psi_e)_{e \in E}$ is a collection of ***q-ary constraints***, where each constraint $\psi_e\colon \Sigma^e \to \{0,1\}$ is a circuit.

An ***assignment*** for a q-CSP instance $G = (V, E, \Sigma, \Psi = (\psi_e)_{e \in E})$ is a function $f\colon V \to \Sigma$ that assigns a symbol of Σ to each vertex of V. We say that f ***satisfies*** a hyperedge $e = \{x_1, \ldots, x_q\} \in E$ (or a constraint ψ_e) if $\psi_e(f(x_1), \ldots, f(x_q)) = 1$, and f ***satisfies*** G if it satisfies all hyperedges of G. For a pair of assignments $f_{\text{start}}, f_{\text{end}}\colon V \to \Sigma$ for G, a ***reconfiguration sequence*** from f_{start} to f_{end} is defined as a sequence $(f^{(1)}, \ldots, f^{(T)})$ over Σ^V such that $f^{(1)} = f_{\text{start}}$, $f^{(T)} = f_{\text{end}}$, and every adjacent pair $f^{(t)}$ and $f^{(t+1)}$ differ in at most a single vertex. We sometimes call f_{start} and f_{end} the ***starting*** and ***ending*** assignments, respectively. In the q-CSP RECONFIGURATION problem, for a satisfiable q-CSP instance G and a pair of its satisfying assignments f_{start} and f_{end}, we are asked to decide if there exists a reconfiguration sequence from f_{start} to f_{end} consisting of satisfying assignments for G. Hereafter, the suffix "q-CSP$_\sigma$" indicates that the alphabet size is σ.

We now formulate an approximate version of q-CSP RECONFIGURATION. Let $G = (V, E, \Sigma, \Psi)$ be a q-CSP instance. The ***value*** of an assignment $f\colon V \to \Sigma$ for G, denoted by $\mathsf{val}_G(f)$, is defined as the fraction of hyperedges of G satisfied by f; namely,

$$\mathsf{val}_G(f) := \frac{1}{|E|}\bigl|\bigl\{e \in E \bigm| f \text{ satisfies } e\bigr\}\bigr| = \mathop{\mathbf{Pr}}_{\boldsymbol{e} \in E}\bigl[f \text{ satisfies } \boldsymbol{e}\bigr]. \tag{4.7}$$

The ***value*** of a reconfiguration sequence $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$, denoted by $\mathsf{val}_G(\vec{f})$, is defined as the minimum fraction of satisfied hyperedges of G, where the minimum is taken over all assignments $f^{(t)}$ in $\vec{f}$; namely,

$$\mathsf{val}_G(\vec{f}) := \min_{1 \leqslant t \leqslant T}\bigl\{\mathsf{val}_G(f^{(t)})\bigr\}. \tag{4.8}$$

The MAXMIN q-CSP RECONFIGURATION problem [IDHPSUU11, Ohs23] is defined as follows.

Problem 4.2. Given a satisfiable q-CSP instance G and a pair of its satisfying assignments f_{start} and f_{end}, MAXMIN q-CSP RECONFIGURATION asks for a reconfiguration sequence $\vec{f}$ from f_{start} to f_{end} such that $\mathsf{val}_G(\vec{f})$ is maximized.

Let $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}})$ denote the ***optimal value*** of MAXMIN q-CSP RECONFIGURATION, which is defined as the maximum value of $\mathsf{val}_G(\vec{f})$ over all possible reconfiguration sequences $\vec{f}$ from f_{start} to f_{end}; namely,

$$\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) \coloneqq \max_{\vec{f}=(f_{\mathsf{start}},\ldots,f_{\mathsf{end}})} \left\{\mathsf{val}_G(\vec{f})\right\}. \tag{4.9}$$

The gap version of MAXMIN q-CSP RECONFIGURATION is defined as follows.

Problem 4.3. For any reals c,s with $0 < s \leqslant c \leqslant 1$, GAP$_{c,s}$ q-CSP RECONFIGURATION is a promise problem that asks, given a satisfiable q-CSP instance G and a pair of its satisfying assignments f_{start} and f_{end}, to distinguish whether $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) \geqslant c$ or $\mathsf{opt}_G(f_{\mathsf{start}} \leftrightsquigarrow f_{\mathsf{end}}) < s$.

Note that the case of $c = s = 1$ is equivalent to q-CSP RECONFIGURATION. By the PCRP theorem [GKMRW25, HO24b], GAP$_{1,s}$ 2-CSP$_\sigma$ RECONFIGURATION is PSPACE-complete for some real $s \in (0,1)$ and some positive integer σ.

Expander Graphs and Random Walk. Let G be a d-regular graph and A be its normalized adjacency matrix; i.e., $A_{x,y} \coloneqq \frac{1}{d}[\![xy \in E(G)]\!]$ for each $x,y \in V(G)$. The ***second largest eigenvalue*** of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) \coloneqq \max_{x \in \mathbb{R}^{V(G)}: x \perp \vec{1}} \frac{\|Ax\|_2}{\|x\|_2}, \tag{4.10}$$

where $\vec{1}$ is the all-ones vector. For a real $\lambda \in (0,1)$, a ***λ-expander graph*** is defined as a regular graph such that $\lambda(G) \leqslant \lambda$.

The ***length-q random walk*** on a graph G is defined as a sequence of random q vertices $(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_q)$ such that $\boldsymbol{x}_1$ is selected from $V(G)$ uniformly and $\boldsymbol{x}_{i+1}$ is selected from the neighbors of $\boldsymbol{x}_i$ uniformly. We write $(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_q) \sim \mathrm{RW}_q(G)$ for the length-q random walk on G. The hitting property of expander walks [AFWZ95, AKS87] says that for a λ-expander graph G and a vertex set S of size $p|V(G)|$, the length-q random walk on G is entirely contained in S with probability $(p \pm \mathrm{O}(\lambda))^q$ (see also [HLW06, §3]).

Theorem 4.4 (Expander hitting property [AFWZ95, AKS87])**.** *Let G be a λ-expander graph, $S \subseteq V(G)$ be a vertex set, and $p \coloneqq \frac{|S|}{|V(G)|}$. Then,*

$$\Pr_{(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_q)\sim \mathrm{RW}_q(G)}\left[\bigwedge_{1 \leqslant i \leqslant q} \boldsymbol{x}_i \in S\right] \leqslant p(p+\lambda)^{q-1}. \tag{4.11}$$

Moreover, if $\lambda < \frac{p}{6}$, then

$$\Pr_{(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_q)\sim \mathrm{RW}_q(G)}\left[\bigwedge_{1 \leqslant i \leqslant q} \boldsymbol{x}_i \in S\right] \geqslant p(p-2\lambda)^{q-1}. \tag{4.12}$$

Concentration Inequalities. We introduce Hoeffding's inequality [Hoe63], the Chernoff bound [Che52], and McDiarmid's inequality [McD89].

Lemma 4.5 (Hoeffding's inequality [Hoe63])**.** *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be n independent random variables such that $a_i \leqslant \boldsymbol{X}_i \leqslant b_i$ for every $i \in [n]$. Let $\boldsymbol{X} := \sum_{1 \leqslant i \leqslant n} \boldsymbol{X}_i$. Then, for any real $t > 0$,*

$$\mathbf{Pr}\Big[\big|\boldsymbol{X} - \mathbf{E}[\boldsymbol{X}]\big| \geqslant t\Big] \leqslant 2\exp\left(-\frac{2t^2}{\sum_{1\leqslant i\leqslant n}(b_i - a_i)^2}\right). \tag{4.13}$$

Hoeffding's inequality holds when $\boldsymbol{X}_i$'s are sampled without replacement, described as follows.

Lemma 4.6 (Hoeffding's inequality for sampling without replacement)**.** *Let $\mathcal{X} = (x_1, \ldots, x_N) \in [0,1]^N$ be a finite population of N reals, and $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be n random variables sampled without replacement from $\mathcal{X}$ uniformly. Let $\boldsymbol{X} := \sum_{1 \leqslant i \leqslant n} \boldsymbol{X}_i$. Then, for any real $t > 0$,*

$$\mathbf{Pr}\Big[\big|\boldsymbol{X} - \mathbf{E}[\boldsymbol{X}]\big| \geqslant t\Big] \leqslant 2\exp\left(-\frac{2t^2}{n}\right). \tag{4.14}$$

Lemma 4.7 (Chernoff bound [Che52])**.** *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be n independent Bernoulli random variables, $\boldsymbol{X} := \sum_{1\leqslant i\leqslant n} \boldsymbol{X}_i$, and $\mu := \mathbf{E}[\boldsymbol{X}]$. Then, for any real $\delta \in [0,1]$,*

$$\mathbf{Pr}\Big[\big|\boldsymbol{X} - \mu\big| \geqslant \delta\mu\Big] \leqslant 2\exp\left(-\frac{\delta^2}{3}\mu\right). \tag{4.15}$$

A function $g\colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathbb{R}$ satisfies the ***bounded differences property*** with bounds $c_1, \ldots, c_n$ if for each $i \in [n]$, replacing the assignment to the i^{th} coordinate changes the value of g by at most c_i; namely, for any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ such that $x_j = y_j$ for every $j \neq i$,

$$\big|g(x_1, \ldots, x_n) - g(y_1, \ldots, y_n)\big| \leqslant c_i. \tag{4.16}$$

Lemma 4.8 (McDiarmid's inequality [McD89])**.** *Let $g\colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathbb{R}$ be a function that satisfies the bounded differences property with bounds $c_1, \ldots, c_n$. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be n independent random variables with $\boldsymbol{x}_i \in \mathcal{X}_i$ for every $i \in [n]$, and $\boldsymbol{Y} := g(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. Then, for any real $t > 0$,*

$$\mathbf{Pr}\Big[\big|\boldsymbol{Y} - \mathbf{E}[\boldsymbol{Y}]\big| \geqslant t\Big] \leqslant 2\exp\left(-\frac{2t^2}{\sum_{1\leqslant i\leqslant n} c_i^2}\right). \tag{4.17}$$

5 Tolerant q-query Direct Product Tester

In this section, we introduce and analyze the tolerant q-query direct product tester. Let $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ be a k-set function, where $\mathscr{R}$ is the set of strings. We will think of F as the following randomized oracle: for a given set $X \in \binom{V}{k}$, the oracle samples a string $\boldsymbol{r} \in \mathscr{R}$ and returns $F(X; \boldsymbol{r})$. For each integer $q \geqslant 2$, the ***tolerant q-query direct product tester*** $\mathcal{T}_q$, parameterized by the ***intersection parameter*** ℓ and the ***tolerance parameter*** m with $1 \leqslant m \leqslant \ell \leqslant k$, is described below.

Tolerant q-query direct product tester $\mathcal{T}_q$ for a k-set function

Input: positive integers ℓ and m with $1 \leqslant m \leqslant \ell \leqslant k$.
Oracle access: a k-set function $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$.

```
 1: sample X_1 from (V choose k) uniformly.
 2: for each i ∈ [q−1] do
 3:     sample I_i from (X_i choose ℓ) uniformly.
 4:     sample A_{i+1} from (V\I_i choose k−ℓ) uniformly.
 5:     let X_{i+1} := I_i ∪ A_{i+1}.
 6: for each i ∈ [q] do
 7:     sample r_i from R uniformly.
 8:     read F(X_i; r_i).
 9: for each i ∈ [q−1] do
10:     if F(X_i; r_i)|_{I_i} ≉_{m/ℓ} F(X_{i+1}; r_{i+1})|_{I_i} then
11:         return 0.
12: return 1.
```

The main result of this section is the following.

Theorem 5.1. *Let $q \geqslant 2$ be an integer. Let $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ be a k-set function and $n := |V|$. Let ℓ and m be positive integers with $1 \leqslant m \leqslant \ell \leqslant k$ such that $\ell = \Theta(k^\eta)$ for a real $\eta \in (0,1)$ and $m = \ell^{1-\Theta(1)}$. Define*

$$\varepsilon_{k,\ell,m} := \widetilde{\Theta}\left(\max\left\{\frac{\ell}{k}, \frac{m^{\frac{1}{8}}}{\ell^{\frac{1}{8}}}, \frac{1}{\ell^{\frac{1}{16}}}\right\}\right), \tag{5.1}$$

where $\widetilde{\Theta}$ hides a $\mathrm{polylog}(k)$ *factor. If k and n are sufficiently large, the following hold:*

(Completeness) *Suppose that there exists a function $f\colon V \to \Sigma$ such that*

$$p := \mathop{\mathbf{Pr}}_{(\boldsymbol{X},\boldsymbol{r}) \in \binom{V}{k} \times \mathscr{R}}\left[F(\boldsymbol{X};\boldsymbol{r}) \underset{m/(2k)}{\approx} f^k(\boldsymbol{X})\right] = \omega\left(\frac{\ell}{k}\right). \tag{5.2}$$

Then, $\mathcal{T}_q(\ell,m)$ accepts F with probability at least

$$p^q - \mathrm{O}\left(\tfrac{\ell}{k}\right)p. \tag{5.3}$$

Moreover, if $p = 1$, then $\mathcal{T}_q(\ell,m)$ accepts F with probability 1.

(Soundness) *Suppose that $\mathcal{T}_q(\ell,m)$ accepts F with probability $p \geqslant \sqrt{\varepsilon_{k,\ell,m}}$. Then, there exists a function $f\colon V \to \Sigma$ such that*

$$\mathop{\mathbf{Pr}}_{(\boldsymbol{X},\boldsymbol{r}) \in \binom{V}{k} \times \mathscr{R}}\left[F(\boldsymbol{X};\boldsymbol{r}) \underset{\varepsilon_{k,\ell,m}}{\approx} f^k(\boldsymbol{X})\right] \geqslant p^{\frac{1}{q-1}}(1-k^{-\Omega(1)}). \tag{5.4}$$

Hereafter, let $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ be a k-set function with $V := [n]$. Let ℓ and m be positive integers with $1 \leqslant m \leqslant \ell \leqslant k$ such that $\ell = \Theta(k^\eta)$ for a real $\eta \in (0,1)$ and $m = \ell^{1-\Theta(1)}$. We write

$$\left(\vec{\boldsymbol{I}} = (\boldsymbol{I}_i)_{i\in[q-1]}, \vec{\boldsymbol{X}} = (\boldsymbol{X}_i)_{i\in[q]}, \vec{\boldsymbol{r}} = (\boldsymbol{r}_i)_{i\in[q]}\right) \sim \mathcal{T}_q(\ell,m) \tag{5.5}$$

for the random variables sampled by $\mathcal{T}_q(\ell, m)$.

The remainder of this section is organized as follows. In Section 5.1, we characterize the query pattern of $\mathcal{T}_q(\ell, m)$. In Section 5.2, we prove the completeness part of Theorem 5.1. In Section 5.3, we prove the soundness part of Theorem 5.1.

5.1 Characterization of the Query Pattern

We prove the following characterization of the query pattern of $\mathcal{T}_q(\ell, m)$.

Proposition 5.2. *For any sufficiently large integer k and for any set $\mathcal{S} \subseteq \binom{V}{k} \times \mathscr{R}$,*

$$\Pr_{(\vec{\boldsymbol{I}}, \vec{\boldsymbol{X}}, \vec{\boldsymbol{r}}) \sim \mathcal{T}_q(\ell, m)}\left[\bigwedge_{1 \leqslant i \leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}\right] \leqslant \left(\frac{|\mathcal{S}|}{\left|\binom{V}{k} \times \mathscr{R}\right|}\right)^q + \mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{S}|}{\left|\binom{V}{k} \times \mathscr{R}\right|}\right) + \mathrm{O}_{q,k}\left(\frac{1}{n}\right). \tag{5.6}$$

Moreover, if $\frac{|\mathcal{S}|}{\left|\binom{V}{k} \times \mathscr{R}\right|} = \omega\left(\frac{\ell}{k}\right)$, then

$$\Pr_{(\vec{\boldsymbol{I}}, \vec{\boldsymbol{X}}, \vec{\boldsymbol{r}}) \sim \mathcal{T}_q(\ell, m)}\left[\bigwedge_{1 \leqslant i \leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}\right] \geqslant \left(\frac{|\mathcal{S}|}{\left|\binom{V}{k} \times \mathscr{R}\right|}\right)^q - \mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{S}|}{\left|\binom{V}{k} \times \mathscr{R}\right|}\right) - \mathrm{O}_{q,k}\left(\frac{1}{n}\right). \tag{5.7}$$

For the purpose of proving Proposition 5.2, we introduce the Johnson graph.

Definition 5.3. For positive integers n, k, and ℓ with $n \geqslant k \geqslant \ell$, the ***Johnson graph*** is defined as a graph $J(n, k, \ell)$ such that

$$\begin{aligned} V(J(n,k,\ell)) &:= \binom{[n]}{k}, \\ E(J(n,k,\ell)) &:= \left\{ \{X_1, X_2\} \in \tbinom{V(J(n,k,\ell))}{2} \;\middle|\; |X_1 \cap X_2| = \ell \right\}. \end{aligned} \tag{5.8}$$

For a graph G and a finite set $\mathscr{R}$, let $G \times \mathscr{R}$ denote a graph such that

$$\begin{aligned} V(G \times \mathscr{R}) &:= V(G) \times \mathscr{R}, \\ E(G \times \mathscr{R}) &:= \left\{ \{(x_1, r_1), (x_2, r_2)\} \in \tbinom{V(G \times \mathscr{R})}{2} \;\middle|\; \{x_1, x_2\} \in E(G) \right\}. \end{aligned} \tag{5.9}$$

We first show that the probability of interest—the left-hand side of Eqs. (5.6) and (5.7)—can be approximated by the hitting probability of the length-q random walk on $J(n, k, \ell)$.

Lemma 5.4. *For any set $\mathcal{S} \subseteq \binom{V}{k} \times \mathscr{R}$,*

$$\left|\Pr_{(\vec{\boldsymbol{I}}, \vec{\boldsymbol{X}}, \vec{\boldsymbol{r}}) \sim \mathcal{T}_q(\ell, m)}\left[\bigwedge_{1 \leqslant i \leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}\right] - \Pr_{((\boldsymbol{X}_1, \boldsymbol{r}_1), \ldots, (\boldsymbol{X}_q, \boldsymbol{r}_q)) \sim \mathrm{RW}_q(J(n,k,\ell) \times \mathscr{R})}\left[\bigwedge_{1 \leqslant i \leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}\right]\right| \leqslant \mathrm{O}_{q,k}\left(\frac{1}{n}\right). \tag{5.10}$$

Proof. Conditioned on the event that $|\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1}| = \ell$ for every $i \in [q-1]$, we find $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_q)$ to be the length-q random walk on $J(n,k,\ell)$; thus, $((\boldsymbol{X}_1, \boldsymbol{r}_1), \ldots, (\boldsymbol{X}_q, \boldsymbol{r}_q))$ is equal to the length-q random walk on $J(n,k,\ell) \times \mathscr{R}$. Therefore,

$$\begin{aligned}
&\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S} \;\middle|\; \bigwedge_{1\leqslant i\leqslant q-1} |\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1}| = \ell\right] \\
&= \Pr_{((\boldsymbol{X}_1,\boldsymbol{r}_1),\ldots,(\boldsymbol{X}_q,\boldsymbol{r}_q))\in \mathrm{RW}_q(J(n,k,\ell)\times\mathscr{R})}\left[\bigwedge_{1\leqslant i\leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}\right].
\end{aligned} \tag{5.11}$$

By Markov's inequality, we derive for each $i \in [q-1]$,

$$\begin{aligned}
\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\Big[|\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1}| \neq \ell\Big] &= \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\Big[|(\boldsymbol{X}_i \setminus \boldsymbol{I}_i) \cap (\boldsymbol{X}_{i+1} \setminus \boldsymbol{I}_i)| \geqslant 1\Big] \\
&\leqslant \mathop{\mathbf{E}}_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\Big[|(\boldsymbol{X}_i \setminus \boldsymbol{I}_i) \cap (\boldsymbol{X}_{i+1} \setminus \boldsymbol{I}_i)|\Big] = \frac{(k-\ell)^2}{n-k},
\end{aligned} \tag{5.12}$$

where we used the fact that $\boldsymbol{X}_i \setminus \boldsymbol{I}_i$ and $\boldsymbol{X}_{i+1} \setminus \boldsymbol{I}_i$ are independent of each other and uniformly distributed in $\binom{V\setminus \boldsymbol{I}_i}{k-\ell}$. By the union bound, we obtain

$$\begin{aligned}
\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q-1} |\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1}| = \ell\right] &\geqslant 1 - \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigvee_{1\leqslant i\leqslant q-1} |\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1}| \neq \ell\right] \\
&\geqslant 1 - \sum_{1\leqslant i\leqslant q-1} \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\Big[|\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1}| \neq \ell\Big] \\
&\geqslant 1 - \sum_{1\leqslant i\leqslant q-1} \frac{(k-\ell)^2}{n-k} = 1 - \mathrm{O}_{q,k}\left(\frac{1}{n}\right),
\end{aligned} \tag{5.13}$$

On the one hand, we have

$$\begin{aligned}
\text{Eq. (5.11)} &\geqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S} \text{ and } \bigwedge_{1\leqslant i\leqslant q-1} |\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1}| = \ell\right] \\
&\geqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}\right] + \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q-1} |\boldsymbol{X}_i \cap \boldsymbol{X}_{i+1}| = \ell\right] - 1 \\
&\underbrace{\geqslant}_{\text{Eq. (5.13)}} \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}\right] - \mathrm{O}_{q,k}\left(\frac{1}{n}\right),
\end{aligned} \tag{5.14}$$

which implies the "upper-bound" part. On the other hand, we have

$$
\begin{aligned}
&\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\right]\\
&\geqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\text{ and }\bigwedge_{1\leqslant i\leqslant q-1}|\boldsymbol{X}_i\cap\boldsymbol{X}_{i+1}|=\ell\right]\\
&\geqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\;\middle|\;\bigwedge_{1\leqslant i\leqslant q-1}|\boldsymbol{X}_i\cap\boldsymbol{X}_{i+1}|=\ell\right]\cdot\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q-1}|\boldsymbol{X}_i\cap\boldsymbol{X}_{i+1}|=\ell\right]\\
&\underbrace{\geqslant}_{\text{Eq. (5.13)}}\text{Eq. (5.11)}\cdot\left(1-\mathrm{O}_{q,k}\left(\frac{1}{n}\right)\right)\geqslant\text{Eq. (5.11)}-\mathrm{O}_{q,k}\left(\frac{1}{n}\right),
\end{aligned}
\tag{5.15}
$$

which implies the "lower-bound" part, as desired. □

We then bound the hitting probability of the length-q random walk on the Johnson graph.

Lemma 5.5. *For a vertex set $\mathcal{S}\subseteq V(J(n,k,\ell)\times\mathscr{R})$ with density $p:=\frac{|\mathcal{S}|}{\binom{n}{k}|\mathscr{R}|}$,*

$$\Pr_{((\boldsymbol{X}_1,\boldsymbol{r}_1),\ldots,(\boldsymbol{X}_q,\boldsymbol{r}_q))\sim\mathrm{RW}_q(J(n,k,\ell)\times\mathscr{R})}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\right]\leqslant p^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)p. \tag{5.16}$$

Moreover, if $p=\omega\left(\frac{\ell}{k}\right)$, then

$$\Pr_{((\boldsymbol{X}_1,\boldsymbol{r}_1),\ldots,(\boldsymbol{X}_q,\boldsymbol{r}_q))\sim\mathrm{RW}_q(J(n,k,\ell)\times\mathscr{R})}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\right]\geqslant p^q-\mathrm{O}_q\left(\frac{\ell}{k}\right)p. \tag{5.17}$$

Proof. Observe that $J(n,k,\ell)\times\mathscr{R}$ is equal to the tensor product of $J(n,k,\ell)$ and the complete graph with self-loops $K^\circ_{|\mathscr{R}|}$. Since $\lambda(J(n,k,\ell))=\mathrm{O}\left(\frac{\ell}{k}\right)$ [DG08, Lemma 5.3], we have

$$\lambda\left(J(n,k,\ell)\times\mathscr{R}\right)=\lambda\left(J(n,k,\ell)\otimes K^\circ_{|\mathscr{R}|}\right)=\max\left\{\lambda(J(n,k,\ell)),\lambda(K^\circ_{|\mathscr{R}|})\right\}=\mathrm{O}\left(\tfrac{\ell}{k}\right). \tag{5.18}$$

Let $\lambda:=\lambda(J(n,k,\ell)\times\mathscr{R})$. On the one hand, by Theorem 4.4, we have

$$
\begin{aligned}
&\Pr_{((\boldsymbol{X}_1,\boldsymbol{r}_1),\ldots,(\boldsymbol{X}_q,\boldsymbol{r}_q))\sim\mathrm{RW}_q(J(n,k,\ell)\times\mathscr{R})}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\right]\\
&\leqslant p(p+\lambda)^{q-1}\leqslant p(p^{q-1}+2^{q-1}\lambda)=p^q+\mathrm{O}_q\left(\tfrac{\ell}{k}\right)p.
\end{aligned}
\tag{5.19}
$$

On the other hand, if $p=\omega\left(\frac{\ell}{k}\right)$, by Theorem 4.4, we have

$$
\begin{aligned}
&\Pr_{((\boldsymbol{X}_1,\boldsymbol{r}_1),\ldots,(\boldsymbol{X}_q,\boldsymbol{r}_q))\sim\mathrm{RW}_q(J(n,k,\ell)\times\mathscr{R})}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\right]\\
&\geqslant p(p-2\lambda)^{q-1}\geqslant p\left(p^{q-1}-2^{q-1}\cdot\max\left\{2\lambda,(2\lambda)^{q-1}\right\}\right)\geqslant p^q-\mathrm{O}_q\left(\tfrac{\ell}{k}\right)p,
\end{aligned}
\tag{5.20}
$$

as desired. □

By Lemmas 5.4 and 5.5, we obtain Proposition 5.2, as desired.

5.2 Completeness Part of Theorem 5.1

We prove the completeness part of Theorem 5.1.

Proof of the completeness part of Theorem 5.1. Suppose that there exists a function $f\colon V \to \Sigma$ such that

$$p := \Pr_{(\boldsymbol{X},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}}\left[F(\boldsymbol{X};\boldsymbol{r}) \underset{m/(2k)}{\approx} f^k(\boldsymbol{X})\right] = \omega\left(\frac{\ell}{k}\right). \tag{5.21}$$

Define

$$\mathcal{S} := \left\{(X,r)\in\tbinom{V}{k}\times\mathscr{R} \,\middle|\, F(X;r) \underset{m/(2k)}{\approx} f^k(X)\right\}. \tag{5.22}$$

Note that $|\mathcal{S}| = p\left|\binom{V}{k}\times\mathscr{R}\right|$. Observe that

$$\begin{aligned}
&\Pr\left[\mathcal{T}_q^F(\ell,m)=1\right] \\
&= \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q-1} F(\boldsymbol{X}_i;\boldsymbol{r}_i)|_{\boldsymbol{I}_i} \underset{m/\ell}{\approx} F(\boldsymbol{X}_{i+1};\boldsymbol{r}_{i+1})|_{\boldsymbol{I}_i}\right] \\
&\geqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q-1} F(\boldsymbol{X}_i;\boldsymbol{r}_i)|_{\boldsymbol{I}_i} \underset{m/(2\ell)}{\approx} f^k(\boldsymbol{X}_i)|_{\boldsymbol{I}_i} \text{ and } f^k(\boldsymbol{X}_{i+1})|_{\boldsymbol{I}_i} \underset{m/(2\ell)}{\approx} F(\boldsymbol{X}_{i+1};\boldsymbol{r}_{i+1})|_{\boldsymbol{I}_i}\right] \\
&\geqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q-1} F(\boldsymbol{X}_i;\boldsymbol{r}_i) \underset{m/(2k)}{\approx} f^k(\boldsymbol{X}_i) \text{ and } f^k(\boldsymbol{X}_{i+1}) \underset{m/(2k)}{\approx} F(\boldsymbol{X}_{i+1};\boldsymbol{r}_{i+1})\right] \\
&= \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q} (\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\right].
\end{aligned} \tag{5.23}$$

By Proposition 5.2, for sufficiently large n, we derive

$$\begin{aligned}
\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{1\leqslant i\leqslant q} (\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{S}\right] &\geqslant \left(\frac{|\mathcal{S}|}{\left|\binom{V}{k}\times\mathscr{R}\right|}\right)^q - \mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{S}|}{\left|\binom{V}{k}\times\mathscr{R}\right|}\right) - \mathrm{O}_{q,k}\left(\frac{1}{n}\right), \\
&\geqslant p^q - \mathrm{O}_q\left(\frac{\ell}{k}\right)p,
\end{aligned} \tag{5.24}$$

as desired. ☐

5.3 Soundness Part of Theorem 5.1

We prove the soundness part of Theorem 5.1. Suppose that $\mathcal{T}_q(\ell,m)$ accepts F with probability $p = \omega\left(\frac{\ell}{k}\right)$. We first show that there exists a function $f\colon V\to\Sigma$ such that $F(\boldsymbol{X};\boldsymbol{r})$ agrees with $f^k(\boldsymbol{X})$ on all but a $\max\left\{\frac{\ell}{k},\frac{m}{\ell}\right\}^{\Omega(1)}$-fraction of coordinates with probability $\Omega(p^6)$, whose proof is deferred to Appendix A.

Theorem 5.6 ($*$)**.** *Let $q\geqslant 2$ be an integer. Let $F\colon \binom{V}{k}\times\mathscr{R}\to\Sigma^k$ be a k-set function and $n := |V|$. Let ℓ and m be positive integers with $1\leqslant m\leqslant \ell\leqslant k$. If k and n are sufficiently large, the following holds. Suppose that $\mathcal{T}_q(\ell,m)$ accepts F with probability $p=\omega\left(\frac{\ell}{k}\right)$. Then, there exists a function $f\colon V\to\Sigma$ such that*

$$\Pr_{(\boldsymbol{X},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}}\left[F(\boldsymbol{X};\boldsymbol{r}) \underset{\delta_{k,\ell,m}}{\approx} f^k(\boldsymbol{X})\right] = \Omega(p^6), \tag{5.25}$$

where $\delta_{k,\ell,m}$ is defined as

$$\delta_{k,\ell,m} = \widetilde{\Theta}\left(\max\left\{\frac{\ell}{k}, \frac{m}{\ell}\right\}\right). \tag{5.26}$$

Remark 5.7. *If $\mathcal{T}_q(\ell,m)$ accepts F with probability at least p, then $\mathcal{T}_2(\ell,m)$ also accepts F with probability at least p. Therefore, it is sufficient to prove Theorem 5.6 only when $q = 2$. The proof of Theorem 5.6 with $q = 2$ is obtained by extending the analysis of the 2-query direct product tester due to [IKW12, Theorem 1.3].*

By applying Theorem 5.6, we then show that there exists a function $f\colon V \to \Sigma$ such that $F(\boldsymbol{X};\boldsymbol{r})$ agrees with $f^k(\boldsymbol{X})$ on all but a $\max\{\frac{\ell}{k}, \frac{m}{\ell}\}^{\Omega(1)}$-fraction of coordinates with probability $\gtrapprox p^{\frac{1}{q-1}}$.

Theorem 5.8. *Let $q \geqslant 2$ be an integer. Let $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ be a k-set function and $n := |V|$. Let ℓ and m be positive integers with $1 \leqslant m \leqslant \ell \leqslant k$ such that $\ell = \Theta(k^\eta)$ for a real $\eta \in (0,1)$ and $m = \ell^{1-\Theta(1)}$. Define*

$$\varepsilon_{k,\ell,m} := \widetilde{\Theta}\left(\max\left\{\frac{\ell}{k}, \frac{m^{\frac{1}{8}}}{\ell^{\frac{1}{8}}}, \frac{1}{\ell^{\frac{1}{16}}}\right\}\right). \tag{5.27}$$

If k and n are sufficiently large, the following holds. Suppose that $\mathcal{T}_q(\ell,m)$ accepts F with probability $p \geqslant \sqrt{\varepsilon_{k,\ell,m}}$. Then, there exists a function $f\colon V \to \Sigma$ such that

$$\Pr_{(\boldsymbol{X},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}}\left[F(\boldsymbol{X};\boldsymbol{r}) \underset{\varepsilon_{k,\ell,m}}{\approx} f^k(\boldsymbol{X})\right] \geqslant p^{\frac{1}{q-1}}\left(1 - k^{-\Omega(1)}\right). \tag{5.28}$$

The soundness part of Theorem 5.1 follows from Theorem 5.8. The remainder of this subsection is devoted to the proof of Theorem 5.8.

Proof of Theorem 5.8. Let $q \geqslant 2$ be an integer. Let $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ be a k-set function and $n := |V|$. Let ℓ and m be positive integers with $1 \leqslant m \leqslant \ell \leqslant k$ such that $\ell = \Theta(k^\eta)$ for some real $\eta \in (0,1)$ and $m = \ell^{1-\Theta(1)}$. Define ε and γ as

$$\begin{aligned}
\varepsilon &:= \Theta\left(\max\left\{\frac{\ell}{k}, \frac{m^{\frac{1}{8}}}{\ell^{\frac{1}{8}}}, \frac{1}{\ell^{\frac{1}{16}}}\right\}\log k\right), \\
\gamma &:= \varepsilon^8.
\end{aligned} \tag{5.29}$$

Note that $\varepsilon = k^{-\Theta(1)}$ and $\gamma = k^{-\Theta(1)}$. Hereafter, we use $\mathcal{T}_q$ to denote $\mathcal{T}_q(\ell,m)$. We assume that k and n are sufficiently large so that Proposition 5.2 and Theorem 5.6 can be applied.

For a k-set function $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$, a function $f\colon V \to \Sigma$, and a real $\delta \in (0,1)$, we define $\mathsf{supp}_\delta(F, f^k)$ as the set of pairs $(X,r) \in \binom{V}{k} \times \mathscr{R}$ such that $F(X;r)$ and $f^k(X)$ are δ-close; namely,

$$\mathsf{supp}_\delta(F, f^k) := \left\{(X,r) \in \binom{V}{k} \times \mathscr{R} \;\middle|\; F(X;r) \underset{\delta}{\approx} f^k(X)\right\}. \tag{5.30}$$

Suppose that $\mathcal{T}_q$ accepts F with probability $p \geqslant \sqrt{\varepsilon}$. By Theorem 5.6, there exists a function $f\colon V \to \Sigma$ such that

$$\Pr_{(\boldsymbol{X},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}}\left[F(\boldsymbol{X};\boldsymbol{r}) \underset{\widetilde{\Theta}\left(\max\left\{\frac{\ell}{k}, \frac{m}{\ell}\right\}\right)}{\approx} f^k(\boldsymbol{X})\right] = \Omega(p^6). \tag{5.31}$$

Consider T k-set functions $F_1, \ldots, F_T \colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$, T functions $f_1, \ldots, f_T \colon V \to \Sigma$, and T positive reals $\delta_1, \ldots, \delta_T$ generated by the following randomized procedure due to [DG08].

Construction of F_t's, f_t's, and δ_t's [DG08]

Input: a k-set function $F \colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$.

let $F_1 \coloneqq F$.
for each $t \geqslant 1$ **do**
  if $\mathcal{T}_q$ accepts F_t with probability at least ε **then**
    let $f_t \colon V \to \Sigma$ be a function such that $\left|\mathsf{supp}_\varepsilon(F_t, f_t^k)\right|$ is maximized.
    ▷ $\left|\mathsf{supp}_\varepsilon(F_t, f_t^k)\right| \geqslant \Omega(\varepsilon^6)\left|\binom{V}{k} \times \mathscr{R}\right|$. ◁
    let δ_t be the smallest real of the form $\varepsilon + i\gamma$ for a nonnegative integer i such that

$$\left|\mathsf{supp}_{\delta_t+\gamma}(F, f_t^k) \setminus \mathsf{supp}_{\delta_t}(F, f_t^k)\right| < \varepsilon^7 \left|\binom{V}{k} \times \mathscr{R}\right|. \tag{5.32}$$

    let $F_{t+1} \colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ be a k-set function determined randomly as follows:

$$F_{t+1}(X;r) \coloneqq \begin{cases} \text{a random value in } \Sigma^k & \text{if } (X,r) \in \mathsf{supp}_{\delta_t}(F_t, f_t^k), \\ F_t(X;r) & \text{otherwise.} \end{cases} \tag{5.33}$$

  else
    let $T \coloneqq t-1$.
    terminate.

We first show that the above procedure terminates within $\mathrm{O}(\varepsilon^{-6})$ iterations.

Observation 5.9 ([DG08])**.** *For every $t \in [T]$, $\delta_t \in [\varepsilon, 2\varepsilon]$. Moreover, $T = \mathrm{O}(\varepsilon^{-6})$ with high probability.*

To prove Observation 5.9, we use the following claim.

Claim 5.10. *Let $\boldsymbol{F} \colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ be a random function such that $\boldsymbol{F}(X;r)$ is selected from Σ^k uniformly. Then,*

$$\Pr_{\boldsymbol{F} \colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k}\left[\exists f \colon V \to \Sigma \text{ s.t. } \frac{\left|\mathsf{supp}_{2\varepsilon}(\boldsymbol{F}, f^k)\right|}{\left|\binom{V}{k} \times \mathscr{R}\right|} \geqslant \exp(-\Theta(k))\right] \leqslant \exp\left(-\Omega\left(\left|\binom{V}{k}\right|\right)\right). \tag{5.34}$$

Proof. Fix a function $f : V \to \Sigma$. For each pair $(X,r) \in \binom{V}{k} \times \mathscr{R}$, we introduce the indicator variable

$$\boldsymbol{I}_{X,r} \coloneqq \left[\!\!\left[\boldsymbol{F}(X;r) \underset{2\varepsilon}{\approx} f^k(X)\right]\!\!\right]. \tag{5.35}$$

Since $\boldsymbol{F}(X;r)$ is uniformly distributed in Σ^k, we have

$$\begin{aligned} \Pr_{\boldsymbol{F} \colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k}[\boldsymbol{I}_{X,r} = 1] &\leqslant \binom{k}{k-2\varepsilon k}\left(\frac{1}{|\Sigma|}\right)^{k-2\varepsilon k} \leqslant \left(\frac{\mathrm{e}k}{2\varepsilon k}\right)^{2\varepsilon k}\left(\frac{1}{2}\right)^{k-2\varepsilon k} \\ &\leqslant \exp\left(\Theta(\varepsilon k \log \varepsilon^{-1})\right)\exp\left(-\Theta(k)\right) \leqslant \exp\left(-\Theta(k)\right). \end{aligned} \tag{5.36}$$

Therefore, we have

$$\mathop{\mathbf{E}}_{\boldsymbol{F}\colon \binom{V}{k}\times\mathscr{R}\to\Sigma^k}\left[\left|\mathsf{supp}_{2\varepsilon}(\boldsymbol{F},f^k)\right|\right] = \sum_{(X,r)\in\binom{V}{k}\times\mathscr{R}} \mathop{\mathbf{Pr}}_{\boldsymbol{F}\colon \binom{V}{k}\times\mathscr{R}\to\Sigma^k}\left[\boldsymbol{I}_{X,r}=1\right] \leqslant \exp\bigl(-\Theta(k)\bigr)\cdot\left|\tbinom{V}{k}\times\mathscr{R}\right|. \tag{5.37}$$

Since $\boldsymbol{I}_{X,r}$'s are independent, the Chernoff bound derives

$$\begin{aligned}
&\mathop{\mathbf{Pr}}_{\boldsymbol{F}\colon \binom{V}{k}\times\mathscr{R}\to\Sigma^k}\left[\frac{\left|\mathsf{supp}_{2\varepsilon}(\boldsymbol{F},f^k)\right|}{\left|\binom{V}{k}\times\mathscr{R}\right|} > 2\exp\bigl(-\Theta(k)\bigr)\right]\\
&= \mathop{\mathbf{Pr}}_{\boldsymbol{F}\colon \binom{V}{k}\times\mathscr{R}\to\Sigma^k}\left[\sum_{(X,r)\in\binom{V}{k}\times\mathscr{R}} \boldsymbol{I}_{X,r} > 2\exp\bigl(-\Theta(k)\bigr)\cdot\left|\tbinom{V}{k}\times\mathscr{R}\right|\right]\\
&\leqslant 2\exp\Bigl(-\Omega\Bigl(\exp\bigl(-\Theta(k)\bigr)\cdot\left|\tbinom{V}{k}\times\mathscr{R}\right|\Bigr)\Bigr) \leqslant \exp\Bigl(-\Omega\bigl(\left|\tbinom{V}{k}\right|\bigr)\Bigr).
\end{aligned} \tag{5.38}$$

Taking a union bound over all possible functions $f\colon V\to\Sigma$, we derive

$$\begin{aligned}
\mathop{\mathbf{Pr}}_{\boldsymbol{F}\colon \binom{V}{k}\times\mathscr{R}\to\Sigma^k}\left[\exists f\colon V\to\Sigma \text{ s.t. } \frac{\left|\mathsf{supp}_{2\varepsilon}(\boldsymbol{F},f^k)\right|}{\left|\binom{V}{k}\times\mathscr{R}\right|} > 2\exp\bigl(-\Theta(k)\bigr)\right] &\leqslant \left|\Sigma^V\right|\cdot\exp\Bigl(-\Omega\bigl(\left|\tbinom{V}{k}\right|\bigr)\Bigr)\\
&\leqslant \exp\Bigl(-\Omega\bigl(\left|\tbinom{V}{k}\right|\bigr)\Bigr),
\end{aligned} \tag{5.39}$$

as desired. □

Proof of Observation 5.9. Suppose that $\delta_t > 2\varepsilon$, which implies

$$\left|\mathsf{supp}_{\delta_t+\gamma}(F,f_t^k)\right| = \left|\mathsf{supp}_{\varepsilon}(F,f_t^k)\right| + \sum_{0\leqslant i\leqslant\varepsilon^{-7}} \underbrace{\left|\mathsf{supp}_{\varepsilon+(i+1)\gamma}(F,f_t^k)\setminus\mathsf{supp}_{\varepsilon+i\gamma}(F,f_t^k)\right|}_{\geqslant\varepsilon^7\left|\binom{V}{k}\times\mathscr{R}\right| \text{ by assumption}} > \left|\tbinom{V}{k}\times\mathscr{R}\right|. \tag{5.40}$$

This is a contradiction. Therefore, $\delta_t\leqslant 2\varepsilon$.

Let Φ_t be the set of pairs $(X,r)\in\binom{V}{k}\times\mathscr{R}$ such that the value of $F_t(X;r)$ has been randomized so far. By definition, $\Phi_1=\emptyset$ and $\Phi_{t+1}=\Phi_t\cup\mathsf{supp}_{\delta_t}(F_t,f_t^k)$. Since $F_t(X;r)$ is uniformly distributed in Σ^k for $(X,r)\in\Phi_t$, the restriction $F_t|_{\Phi_t}\colon\Phi_t\to\Sigma^k$ is a random function. By Claim 5.10, with probability $1-\exp\Bigl(-\Omega\bigl(\left|\binom{V}{k}\right|\bigr)\Bigr)$, for every function $f\colon V\to\Sigma$,

$$\left|\mathsf{supp}_{2\varepsilon}(F_t,f^k)\cap\Phi_t\right| \leqslant \exp\bigl(-\Theta(k)\bigr)\cdot\left|\tbinom{V}{k}\times\mathscr{R}\right|. \tag{5.41}$$

This implies

$$\begin{aligned}
|\Phi_{t+1}| &= |\Phi_t\cup\mathsf{supp}_{\delta_t}(F_t,f_t^k)|\\
&= |\Phi_t| + |\mathsf{supp}_{\delta_t}(F_t,f_t^k)| - |\Phi_t\cap\mathsf{supp}_{\delta_t}(F_t,f_t^k)|\\
&\geqslant |\Phi_t| + \Omega(\varepsilon^6)\left|\tbinom{V}{k}\times\mathscr{R}\right| - \exp\bigl(-\Theta(k)\bigr)\left|\tbinom{V}{k}\times\mathscr{R}\right|\\
&\geqslant |\Phi_t| + \Omega(\varepsilon^6)\left|\tbinom{V}{k}\times\mathscr{R}\right|.
\end{aligned} \tag{5.42}$$

Consequently, the procedure must terminate within $\mathrm{O}(\varepsilon^{-6})$ iterations with high probability, as desired. □

By Observation 5.9, we can assume that $T = \mathrm{O}(\varepsilon^{-6})$ and $\delta_t \in [\varepsilon, 2\varepsilon]$ for every $t \in [T]$. We introduce the following additional notations.

Definition 5.11. For each $t \in [T]$, define

$$\mathcal{S}_t := \mathsf{supp}_{\delta_t}(F, f_t^k). \tag{5.43}$$

For each $\tau \subseteq [T]$, define

$$\begin{aligned} \mathcal{A}_\tau &:= \left(\bigcap_{t\in\tau} \mathcal{S}_t\right) \cap \left(\bigcap_{t\notin\tau} \overline{\mathcal{S}_t}\right), \text{ where } \overline{\mathcal{S}_t} := \left(\binom{V}{k} \times \mathcal{R}\right) \setminus \mathcal{S}_t, \\ a_\tau &:= \frac{|\mathcal{A}_\tau|}{\left|\binom{V}{k} \times \mathcal{R}\right|}. \end{aligned} \tag{5.44}$$

Note that $\{\mathcal{A}_\tau\}_{\tau\subseteq[T]}$ forms a partition of $\binom{V}{k} \times \mathcal{R}$ and thus $\sum_{\tau\subseteq[T]} a_\tau = 1$.

We will show the following lemma, which can be used to prove Theorem 5.8.

Lemma 5.12.

$$\sum_{\tau\neq\emptyset} a_\tau^q \geqslant p(1 - k^{-\Omega(1)}). \tag{5.45}$$

Proof of Theorem 5.8. By Lemma 5.12, we have

$$p(1 - k^{-\Omega(1)}) \leqslant \sum_{\tau\neq\emptyset} a_\tau^q \leqslant \max_{\tau^*\neq\emptyset}\{a_{\tau^*}^{q-1}\} \cdot \sum_{\tau\neq\emptyset} a_\tau \leqslant \max_{\tau^*\neq\emptyset}\{a_{\tau^*}^{q-1}\}, \tag{5.46}$$

where we used the inequality that $\sum_{\tau\neq\emptyset} a_\tau \leqslant 1$. Therefore, there exists $\tau^* \neq \emptyset$ such that $a_{\tau^*}^{q-1} \geqslant p(1-k^{-\Omega(1)})$. Since $\tau^* \neq \emptyset$, there exists $t \in \tau^*$ such that for every pair $(X, r) \in \mathcal{A}_{\tau^*}$, we have $(X, r) \in \mathsf{supp}_{\delta_t}(F, f_t^k)$; i.e., $F(X; r) \underset{\mathrm{O}(\varepsilon)}{\approx} f_t^k(X)$. Consequently, we obtain

$$\Pr_{(\boldsymbol{X},\boldsymbol{r})\in\binom{V}{k}\times\mathcal{R}}\left[F(\boldsymbol{X};\boldsymbol{r}) \underset{\mathrm{O}(\varepsilon)}{\approx} f_t^k(\boldsymbol{X})\right] \geqslant \frac{|\mathcal{A}_{\tau^*}|}{\left|\binom{V}{k} \times \mathcal{R}\right|} = a_{\tau^*} \geqslant \left(p(1-k^{-\Omega(1)})\right)^{\frac{1}{q-1}} \geqslant p^{\frac{1}{q-1}}(1-k^{-\Omega(1)}), \tag{5.47}$$

which completes the proof. $\square$

In the remainder of this subsection, we prove Lemma 5.12. We first show the following lemma.

Lemma 5.13. *For each $t \in [T]$ and each $i \in [q-1]$,*

$$\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F = 1 \text{ and } (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}_t \text{ and } (\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \notin \mathcal{S}_t\right] = \mathrm{O}(\varepsilon^7), \tag{5.48}$$

$$\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F = 1 \text{ and } (\boldsymbol{X}_i, \boldsymbol{r}_i) \notin \mathcal{S}_t \text{ and } (\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \in \mathcal{S}_t\right] = \mathrm{O}(\varepsilon^7). \tag{5.49}$$

Proof. By conditioning on whether $(\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \in \mathsf{supp}_{\delta_t+\gamma}(F, f_t^k)$ or not, we can rewrite the left-hand side of Eq. (5.48) as

$$\begin{aligned}
&(\text{left-hand side of Eq. (5.48)}) \\
&= \underbrace{\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\begin{array}{c} \mathcal{T}_q^F = 1 \\ (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}_t \\ (\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \notin \mathcal{S}_t \\ (\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \in \mathsf{supp}_{\delta_t+\gamma}(F, f_t^k) \end{array}\right]}_{\text{first term}} + \underbrace{\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\begin{array}{c} \mathcal{T}_q^F = 1 \\ (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}_t \\ (\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \notin \mathcal{S}_t \\ (\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \notin \mathsf{supp}_{\delta_t+\gamma}(F, f_t^k) \end{array}\right]}_{\text{second term}}. \qquad (5.50)
\end{aligned}$$

By the definition of δ_t, the first term of Eq. (5.50) can be bounded from above as

$$\begin{aligned}
&(\text{first term of Eq. (5.50)}) \\
&= \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F = 1 \text{ and } (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{S}_t \text{ and } (\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \in \mathsf{supp}_{\delta_t+\gamma}(F, f_t^k) \setminus \mathsf{supp}_{\delta_t}(F, f_t^k)\right] \\
&\leqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[(\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \in \mathsf{supp}_{\delta_t+\gamma}(F, f_t^k) \setminus \mathsf{supp}_{\delta_t}(F, f_t^k)\right] \qquad (5.51) \\
&= \frac{\left|\mathsf{supp}_{\delta_t+\gamma}(F, f_t^k) \setminus \mathsf{supp}_{\delta_t}(F, f_t^k)\right|}{\left|\binom{V}{k} \times \mathscr{R}\right|} < \varepsilon^7.
\end{aligned}$$

In order to bound the second term, we use the following two claims, which will be proved afterwards.

Claim 5.14. *For any pair $(X, r) \in \mathsf{supp}_{\delta_t}(F, f_t^k)$,*

$$\Pr_{\boldsymbol{I}\in\binom{[k]}{\ell}}\left[F(X;r)|_{\boldsymbol{I}} \underset{\delta_t+\gamma/3}{\approx} f_t^k(X)|_{\boldsymbol{I}}\right] \geqslant 1 - \exp\big(-\Omega(\gamma^2\ell)\big). \qquad (5.52)$$

Claim 5.15. *For any pair $(X, r) \notin \mathsf{supp}_{\delta_t+\gamma}(F, f_t^k)$,*

$$\Pr_{\boldsymbol{I}\in\binom{[k]}{\ell}}\left[F(X;r)|_{\boldsymbol{I}} \underset{\delta_t+2\gamma/3}{\not\approx} f_t^k(X)|_{\boldsymbol{I}}\right] \geqslant 1 - \exp\big(-\Omega(\gamma^2\ell)\big). \qquad (5.53)$$

Conditioned on the two events that

$$F(\boldsymbol{X}_i;\boldsymbol{r}_i)|_{\boldsymbol{I}_i} \underset{\delta_t+\gamma/3}{\approx} f_t^k(\boldsymbol{X}_i)|_{\boldsymbol{I}_i}, \qquad (5.54)$$

$$F(\boldsymbol{X}_{i+1};\boldsymbol{r}_{i+1})|_{\boldsymbol{I}_i} \underset{\delta_t+2\gamma/3}{\not\approx} f_t^k(\boldsymbol{X}_{i+1})|_{\boldsymbol{I}_i}, \qquad (5.55)$$

we must have $F(\boldsymbol{X}_i;\boldsymbol{r}_i)|_{\boldsymbol{I}_i} \underset{\gamma/3}{\not\approx} F(\boldsymbol{X}_{i+1};\boldsymbol{r}_{i+1})|_{\boldsymbol{I}_i}$. Since

$$\frac{\gamma}{3} \geqslant \frac{\varepsilon^8}{3} \underbrace{=}_{\text{Eq. (5.29)}} \Omega\left(\frac{m}{\ell}\log^8 k\right) = \omega\left(\frac{m}{\ell}\right), \qquad (5.56)$$

we also have $F(\boldsymbol{X}_i;\boldsymbol{r}_i)|_{\boldsymbol{I}_i} \underset{m/\ell}{\not\approx} F(\boldsymbol{X}_{i+1};\boldsymbol{r}_{i+1})|_{\boldsymbol{I}_i}$; i.e., $\mathcal{T}_q$ would reject. In other words, $\mathcal{T}_q$ would accept only if Eq. (5.54) or Eq. (5.55) does not hold. By Claims 5.14 and 5.15 and the union bound, the second term of Eq. (5.50) can be bounded from above as

$$\begin{aligned}
&(\text{second term of Eq. (5.50)}) \\
&= \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F = 1 \text{ and } (\boldsymbol{X}_i,\boldsymbol{r}_i) \in \mathsf{supp}_{\delta_t}(F,f_t^k) \text{ and } (\boldsymbol{X}_{i+1},\boldsymbol{r}_{i+1}) \notin \mathsf{supp}_{\delta_t+\gamma}(F,f_t^k)\right] \\
&\leqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\begin{bmatrix}\text{Eq. (5.54) or Eq. (5.55) does not hold} \\ (\boldsymbol{X}_i,\boldsymbol{r}_i) \in \mathsf{supp}_{\delta_t}(F,f_t^k) \\ (\boldsymbol{X}_{i+1},\boldsymbol{r}_{i+1}) \notin \mathsf{supp}_{\delta_t+\gamma}(F,f_t^k)\end{bmatrix} \\
&\leqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\begin{bmatrix}\text{Eq. (5.54) does not hold} \\ (\boldsymbol{X}_i,\boldsymbol{r}_i) \in \mathsf{supp}_{\delta_t}(F,f_t^k) \\ (\boldsymbol{X}_{i+1},\boldsymbol{r}_{i+1}) \notin \mathsf{supp}_{\delta_t+\gamma}(F,f_t^k)\end{bmatrix} + \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\begin{bmatrix}\text{Eq. (5.55) does not hold} \\ (\boldsymbol{X}_i,\boldsymbol{r}_i) \in \mathsf{supp}_{\delta_t}(F,f_t^k) \\ (\boldsymbol{X}_{i+1},\boldsymbol{r}_{i+1}) \notin \mathsf{supp}_{\delta_t+\gamma}(F,f_t^k)\end{bmatrix} \\
&\leqslant \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\text{Eq. (5.54) does not hold} \;\middle|\; (\boldsymbol{X}_i,\boldsymbol{r}_i) \in \mathsf{supp}_{\delta_t}(F,f_t^k)\right] \\
&\quad + \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\text{Eq. (5.55) does not hold} \;\middle|\; (\boldsymbol{X}_{i+1},\boldsymbol{r}_{i+1}) \notin \mathsf{supp}_{\delta_t+\gamma}(F,f_t^k)\right] \\
&\leqslant 2\exp\bigl(-\Omega(\gamma^2\ell)\bigr).
\end{aligned} \tag{5.57}$$

Since

$$\exp\bigl(-\Omega(\gamma^2\ell)\bigr) = \exp\bigl(-\Omega(\varepsilon^{16}\ell)\bigr) \underbrace{=}_{\text{Eq. (5.29)}} \exp\left(-\Omega\left(\tfrac{\log^{16} k}{\ell}\ell\right)\right) = \exp\bigl(-\Omega(\log^{16} k)\bigr) = \mathrm{O}(\varepsilon^7), \tag{5.58}$$

we have

$$(\text{left-hand side of Eq. (5.48)}) \leqslant \varepsilon^7 + 2\exp\bigl(-\Omega(\gamma^2\ell)\bigr) \leqslant \mathrm{O}(\varepsilon^7), \tag{5.59}$$

as desired. Eq. (5.49) can be shown similarly. □

What remains to be done is to prove Claims 5.14 and 5.15.

Proof of Claim 5.14. For any pair $(X,r) \in \mathsf{supp}_{\delta_t}(F,f_t^k)$, let

$$\boldsymbol{Z} := \ell \cdot \mathrm{dist}\bigl(F(X;r)|_{\boldsymbol{I}}, f_t^k(X)|_{\boldsymbol{I}}\bigr) = \sum_{i\in\boldsymbol{I}} [\![F(X;r)|_i \neq f_t^k(X)|_i]\!] \tag{5.60}$$

for a random set $\boldsymbol{I} \in \binom{[k]}{\ell}$. Observe that $\boldsymbol{Z}$ follows a hypergeometric distribution where the population size is k, the number of successes is $k \cdot \mathrm{dist}\bigl(F(X;r), f_t^k(X)\bigr)$, and the sample size is ℓ. By assumption, $\mathbf{E}[\boldsymbol{Z}] \leqslant \delta_t\ell$. By Hoeffding's inequality, we have

$$\Pr_{\boldsymbol{I}\in\binom{[k]}{\ell}}\left[F(X;r)|_{\boldsymbol{I}} \underset{\delta_t+\gamma/3}{\not\approx} f_t^k(X)|_{\boldsymbol{I}}\right] \leqslant \Pr\left[\boldsymbol{Z} \geqslant \mathbf{E}[\boldsymbol{Z}] + \tfrac{\gamma}{3}\ell\right] \leqslant 2\exp\left(-2\left(\tfrac{\gamma}{3}\right)^2\ell\right) = \exp\bigl(-\Omega(\gamma^2\ell)\bigr), \tag{5.61}$$

as desired. □

Proof of Claim 5.15. For any pair $(X,r) \notin \mathsf{supp}_{\delta_t+\gamma}(F, f_t^k)$, let

$$\boldsymbol{Z} := \ell \cdot \mathrm{dist}\big(F(X;r)|_{\boldsymbol{I}}, f_t^k(X)|_{\boldsymbol{I}}\big) = \sum_{\boldsymbol{i}\in\boldsymbol{I}} [\![F(X;r)|_i \neq f_t^k(X)|_i]\!] \tag{5.62}$$

for a random set $\boldsymbol{I} \in \binom{[k]}{\ell}$. Observe that $\boldsymbol{Z}$ follows a hypergeometric distribution where the population size is k, the number of successes is $k \cdot \mathrm{dist}\big(F(X;r), f_t^k(X)\big)$, and the sample size is ℓ. By assumption, $\mathbf{E}[\boldsymbol{Z}] > (\delta_t + \gamma)\ell$. By Hoeffding's inequality, we have

$$\Pr_{\boldsymbol{I}\in\binom{[k]}{\ell}}\left[F(X;r)|_{\boldsymbol{I}} \underset{\delta_t+2\gamma/3}{\approx} f_t^k(X)|_{\boldsymbol{I}}\right] \leqslant \Pr\left[\boldsymbol{Z} \leqslant \mathbf{E}[\boldsymbol{Z}] - \tfrac{\gamma}{3}\ell\right] \leqslant 2\exp\left(-2\left(\tfrac{\gamma}{3}\right)^2\ell\right) = \exp\big(-\Omega(\gamma^2\ell)\big), \tag{5.63}$$

as desired. $\square$

By applying Lemma 5.13, we show that conditioned on the event that $\mathcal{T}_q$ accepts F, all q queries $(\boldsymbol{X}_1, \boldsymbol{r}_1), \ldots, (\boldsymbol{X}_q, \boldsymbol{r}_q)$ belong to the same nonempty $\mathcal{A}_\tau$ with high probability.

Lemma 5.16.

$$\Pr\left[\mathcal{T}_q^F = 1\right] \leqslant \mathrm{O}_q(\varepsilon) + \sum_{\tau\neq\emptyset} \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F = 1 \textit{ and } \bigwedge_{1\leqslant i\leqslant q} (\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathcal{A}_\tau\right], \tag{5.64}$$

Proof. For each pair $(X,r) \in \binom{V}{k} \times \mathscr{R}$, let $\tau(X,r)$ denote the unique $\tau \subseteq [T]$ such that $(X,r) \in \mathcal{A}_\tau$. The event that $\mathcal{T}_q$ accepts F can be partitioned into the following three cases:

(C1) $\tau(\boldsymbol{X}_i, \boldsymbol{r}_i) = \emptyset$ for every $i \in [q]$: Observing that $F_t(X;r) = F(X;r)$ for every pair $(X,r) \in \mathcal{A}_\emptyset$, we have

$$\begin{aligned}\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F = 1 \text{ and (C1) holds}\right] &= \Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^{F_t} = 1 \text{ and (C1) holds}\right] \\ &\leqslant \Pr\left[\mathcal{T}_q^{F_t} = 1\right] < \varepsilon,\end{aligned} \tag{5.65}$$

where the last inequality holds due to the termination condition of the procedure.

(C2) $\tau(\boldsymbol{X}_{j_1}, \boldsymbol{r}_{j_1}) \neq \tau(\boldsymbol{X}_{j_2}, \boldsymbol{r}_{j_2})$ for some $j_1 \neq j_2$: There exists $i \in [q-1]$ such that $\tau(\boldsymbol{X}_i, \boldsymbol{r}_i) \neq \tau(\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1})$. By the definition of $\mathcal{A}_\tau$, there exists some $t \in \tau(\boldsymbol{X}_i, \boldsymbol{r}_i) \triangle \tau(\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \neq \emptyset$[9] such that

- $(\boldsymbol{X}_i, \boldsymbol{r}_i) \in \mathsf{supp}_{\delta_t}(F, f_t^k)$ and $(\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \notin \mathsf{supp}_{\delta_t}(F, f_t^k)$, or
- $(\boldsymbol{X}_i, \boldsymbol{r}_i) \notin \mathsf{supp}_{\delta_t}(F, f_t^k)$ and $(\boldsymbol{X}_{i+1}, \boldsymbol{r}_{i+1}) \in \mathsf{supp}_{\delta_t}(F, f_t^k)$.

[9] $A \triangle B$ denotes the symmetric difference of A and B.

By applying Lemma 5.13 and the union bound, we have

$$\begin{aligned}
&\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F=1 \text{ and (C2) holds}\right] \\
&\leqslant \sum_{t\in[T]}\sum_{i\in[q-1]}\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\begin{bmatrix}\mathcal{T}_q^F=1\\ (\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathsf{supp}_{\delta_t}(F,f_t^k)\\ (\boldsymbol{X}_{i+1},\boldsymbol{r}_{i+1})\notin\mathsf{supp}_{\delta_t}(F,f_t^k)\end{bmatrix}+\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\begin{bmatrix}\mathcal{T}_q^F=1\\ (\boldsymbol{X}_i,\boldsymbol{r}_i)\notin\mathsf{supp}_{\delta_t}(F,f_t^k)\\ (\boldsymbol{X}_{i+1},\boldsymbol{r}_{i+1})\in\mathsf{supp}_{\delta_t}(F,f_t^k)\end{bmatrix} \\
&\leqslant T\cdot(q-1)\cdot 2\,\mathrm{O}(\varepsilon^7)\underbrace{\leqslant}_{T=\mathrm{O}(\varepsilon^{-6})}\mathrm{O}_q(\varepsilon).
\end{aligned} \tag{5.66}$$

(C3) $\tau(\boldsymbol{X}_i,\boldsymbol{r}_i)=\tau$ for every $i\in[q]$ and for some $\tau\neq\emptyset$: Since $\mathcal{A}_\tau$'s are pairwise disjoint, we have

$$\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F=1 \text{ and (C3) holds}\right]=\sum_{\tau\neq\emptyset}\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F=1 \text{ and } \bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{A}_\tau\right]. \tag{5.67}$$

Consequently,

$$\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F=1\right]\leqslant\mathrm{O}_q(\varepsilon)+\sum_{\tau\neq\emptyset}\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F=1 \text{ and } \bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{A}_\tau\right], \tag{5.68}$$

as desired. □

We are now ready to prove Lemma 5.12 by applying Proposition 5.2 and Lemma 5.16.

Proof of Lemma 5.12. By applying Proposition 5.2 to each $\mathcal{A}_\tau$, we have

$$\begin{aligned}
\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{A}_\tau\right]&\leqslant\left(\frac{|\mathcal{A}_\tau|}{\left|\binom{V}{k}\times\mathcal{R}\right|}\right)^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{A}_\tau|}{\left|\binom{V}{k}\times\mathcal{R}\right|}\right)+\mathrm{O}_{q,k}\left(\frac{1}{n}\right)\\
&=a_\tau^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)a_\tau+\mathrm{O}_{q,k}\left(\frac{1}{n}\right).
\end{aligned} \tag{5.69}$$

Therefore, by Lemma 5.16, we have

$$\begin{aligned}
\Pr\left[\mathcal{T}_q^F=1\right]&\leqslant\mathrm{O}_q(\varepsilon)+\sum_{\tau\neq\emptyset}\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\mathcal{T}_q^F=1 \text{ and } \bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{A}_\tau\right]\\
&\leqslant\mathrm{O}_q(\varepsilon)+\sum_{\tau\neq\emptyset}\Pr_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}})\sim\mathcal{T}_q}\left[\bigwedge_{1\leqslant i\leqslant q}(\boldsymbol{X}_i,\boldsymbol{r}_i)\in\mathcal{A}_\tau\right]\\
&\leqslant\mathrm{O}_q(\varepsilon)+\sum_{\tau\neq\emptyset}\left[a_\tau^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)a_\tau+\mathrm{O}_{q,k}\left(\frac{1}{n}\right)\right]\\
&=\mathrm{O}_q(\varepsilon)+\sum_{\tau\neq\emptyset}a_\tau^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)\sum_{\tau\neq\emptyset}a_\tau+\mathrm{O}_{q,k}\left(\frac{1}{n}\right)2^T\\
&\leqslant\mathrm{O}_q(\varepsilon)+\mathrm{O}_q\left(\frac{\ell}{k}\right)+\mathrm{O}_{q,k}\left(\frac{1}{n}\right)+\sum_{\tau\neq\emptyset}a_\tau^q,
\end{aligned} \tag{5.70}$$

where the last inequality holds because

$$\sum_{\tau \neq \emptyset} a_\tau \leqslant 1 \text{ and } \mathrm{O}_{q,k}\left(\frac{1}{n}\right) 2^T \underbrace{=}_{T=\mathrm{poly}(k)} \mathrm{O}_{q,k}\left(\frac{1}{n}\right). \tag{5.71}$$

Since $\mathbf{Pr}[\mathcal{T}_q^F = 1] = p \geqslant \sqrt{\varepsilon}$ and $\varepsilon = \Omega\left(\frac{\ell}{k}\right)$ as defined by Eq. (5.29), we derive that for sufficiently large k and n,

$$\begin{aligned}\sum_{\tau \neq \emptyset} a_\tau^q &\geqslant \mathbf{Pr}\left[\mathcal{T}_q^F = 1\right] - \mathrm{O}_q(\varepsilon) - \mathrm{O}_q\left(\frac{\ell}{k}\right) - \mathrm{O}_{q,k}\left(\frac{1}{n}\right) \\ &\geqslant p\left(1 - \mathrm{O}_q\left(\frac{\varepsilon}{p}\right) - \mathrm{O}_q\left(\frac{\ell}{kp}\right) - \mathrm{O}_{q,k}\left(\frac{1}{pn}\right)\right) \\ &\geqslant p\left(1 - \mathrm{O}_q(\sqrt{\varepsilon}) - \mathrm{O}_q\left(\sqrt{\frac{\ell}{k}}\right) - \mathrm{O}_{q,k}\left(\frac{1}{pn}\right)\right) \\ &\geqslant p(1 - k^{-\Omega(1)}),\end{aligned} \tag{5.72}$$

which completes the proof. □

6 PSPACE-hardness of $\left(\frac{1}{2^{q-1}} + \varepsilon\right)$-factor Approximation for MAXMIN q-CSP RECONFIGURATION

In this section, we prove Theorem 1.1; i.e., MAXMIN q-CSP RECONFIGURATION is PSPACE-hard to approximate within a factor of $\frac{1}{2^{q-1}} + \varepsilon$ for any small real $\varepsilon > 0$.

Theorem 6.1. *For any integer $q \geqslant 2$ and any real $\varepsilon > 0$, there exists a positive integer σ such that* GAP$_{1,\frac{1}{2^{q-1}}+\varepsilon}$ q-CSP$_\sigma$ RECONFIGURATION *is* PSPACE*-hard. Moreover, the same hardness result holds even if the underlying hypergraph is regular.*

The proof of Theorem 6.1 is based on the following gap-preserving reduction from MAXMIN 2-CSP RECONFIGURATION to MAXMIN q-CSP RECONFIGURATION.

Lemma 6.2. *For any reals $s \in (0,1)$ and $\varepsilon > 0$, and any positive integers $q \geqslant 2$, σ, and Δ, there exists a positive integer k such that there exists a polynomial-time reduction from* GAP$_{1,s}$ 2-CSP$_\sigma$ RECONFIGURATION *whose underlying graph is Δ-regular to* GAP$_{1,\frac{1}{2^{q-1}}+\varepsilon}$ q-CSP$_{\sigma^{2k}}$ RECONFIGURATION *whose underlying hypergraph is regular.*

Proof of Theorem 6.1. By the PCRP theorem [GKMRW25, HO24b] and gap-preserving reductions [Ohs23], GAP$_{1,s}$ 2-CSP$_3$ RECONFIGURATION on Δ-regular 2-CSP instances is PSPACE-hard for some real $s \in (0,1)$ and some positive integer Δ. By Lemma 6.2, for any integer $q \geqslant 2$ and any real $\varepsilon > 0$, there exists a positive integer k such that GAP$_{1,\frac{1}{2^{q-1}}+\varepsilon}$ q-CSP$_{3^{2k}}$ RECONFIGURATION on regular q-CSP instances is PSPACE-hard, as desired. □

The remainder of this section is devoted to the proof of Lemma 6.2.

Additional Notations. Some additional notations are introduced. Let $G = (V, E)$ be a graph. For a vertex x of G, let $E(x)$ denote the set of edges incident to x; namely, $E(x) \coloneqq \{xy \in E\}$. For a vertex k-tuple $\vec{x} = (x_1, \ldots, x_k)$ of G, let $E^k(\vec{x})$ denote the set of edge k-tuples obtained by selecting k edges incident to $x_1, \ldots, x_k$; namely,

$$E^k(\vec{x}) \coloneqq \prod_{1 \leqslant i \leqslant k} E(x_i) = \left\{ (e_1, \ldots, e_k) \in E^k \;\middle|\; \forall i \in [k],\ e_i \in E(x_i) \right\}. \tag{6.1}$$

For a vertex function $f \colon V \to \Sigma$, we write $f_E \colon E \to \Sigma^2$ for the ***induced edge function***, which is defined as

$$f_E(xy) \coloneqq \big(f(x), f(y)\big) \text{ for each edge } xy \in E. \tag{6.2}$$

Let $F \colon E^k \to (\Sigma^2)^k$ be an edge k-tuple function; namely, for an edge k-tuple $\vec{e}$, a pair of symbols in Σ^2 are assigned to the endpoints of each edge in $\vec{e}$. Suppose that $F(\vec{e}) = ((\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$ for an edge k-tuple $\vec{e} = (e_1, \ldots, e_k) = (x_1 y_1, \ldots, x_k y_k) \in E^k$. Then, we use $F(\vec{e})[e_i] \in \Sigma^2$ to denote a pair of symbols assigned to the endpoints of e_i; namely, $F(\vec{e})[e_i] = (\alpha_i, \beta_i)$. By abuse of notation, for a vertex k-tuple $\vec{x} = (x_1, \ldots, x_k) \in V^k$, we use $F(\vec{e})[x_i] \in \Sigma$ to denote a symbol assigned to the endpoint x_i of e_i; namely, $F(\vec{e})[x_i] \coloneqq \alpha_i$, and use $F(\vec{e})[\vec{x}] \in \Sigma^k$ to denote a k-tuple of symbols assigned to the endpoints $x_1, \ldots, x_k$ of $e_1, \ldots, e_k$; namely, $F(\vec{e})[\vec{x}] \coloneqq (\alpha_1, \ldots, \alpha_k)$. For a k-tuple $\vec{x} = (x_1, \ldots, x_k)$ and a permutation $\pi \in \mathfrak{S}_k$, let $\vec{x} \circ \pi$ denote a k-tuple such that $(\vec{x} \circ \pi)_i \coloneqq x_{\pi(i)}$ for each $i \in [k]$; namely,

$$\vec{x} \circ \pi \coloneqq (x_{\pi(1)}, \ldots, x_{\pi(k)}). \tag{6.3}$$

Tolerant q-query Direct Product Tester for Edge k-tuple Functions. Before describing the gap-preserving reduction, we introduce the tolerant q-query direct product tester for an edge k-tuple function, denoted by $\mathcal{W}_q$. Let $G = (V, E)$ be a Δ-regular graph, $F \colon E^k \to (\Sigma^2)^k$ be an edge k-tuple function, and $\ell \coloneqq \Theta(\sqrt{k})$. Intuitively, F is supposed to be the k-wise direct product of the induced edge function of some assignment $f \colon V \to \Sigma$; namely, $f_E^k \colon E^k \to (\Sigma^2)^k$. Then, $\mathcal{W}_q$ tests if this is the case.

Tolerant q-query direct product tester $\mathcal{W}_q$ for an edge k-tuple function

Input: a Δ-regular graph $G = (V, E)$.
Oracle access: an edge k-tuple function $F \colon E^k \to (\Sigma^2)^k$.

let $\ell \coloneqq \Theta(\sqrt{k})$.
sample $\vec{\boldsymbol{x}}_1$ from V^k uniformly.
for each $i \in [q-1]$ **do**
  sample $\boldsymbol{I}_i$ from $\binom{[k]}{\ell}$ uniformly.
  sample $\vec{\boldsymbol{x}}_{i+1}$ from V^k uniformly conditioned on $\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}$.
for each $i \in [q]$ **do**
  sample $\vec{\boldsymbol{e}}_i$ from $E^k(\vec{\boldsymbol{x}}_i)$ uniformly.
  sample $\boldsymbol{\pi}_i$ from $\mathfrak{S}_k$ uniformly.
  read $F(\vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i)$.
for each $i \in [q-1]$ **do**
  if $|\{\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,k}, \boldsymbol{x}_{i+1,1}, \ldots, \boldsymbol{x}_{i+1,k}\}| < 2k - \ell$ **then** ▷ *duplicates found between $\vec{\boldsymbol{x}}_i$ and $\vec{\boldsymbol{x}}_{i+1}$.*
    continue.
  if $\left(F(\vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i) \circ \boldsymbol{\pi}_i^{-1}\right)\left[\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}\right] \underset{1/\ell}{\not\approx} \left(F(\vec{\boldsymbol{e}}_{i+1} \circ \boldsymbol{\pi}_{i+1}) \circ \boldsymbol{\pi}_{i+1}^{-1}\right)\left[\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right]$ **then**
    return 0.
return 1.

Our verifier $\mathcal{W}_q$ has the following completeness and soundness, whose proof is obtained by applying Theorem 5.1 and is deferred to Section 6.2.

Lemma 6.3. *Let $\varepsilon > 0$ be a real, $G = (V, E)$ be a Δ-regular n-vertex graph, and $F: E^k \to (\Sigma^2)^k$ be an edge k-tuple function. If k and n are sufficiently large, the following hold:*

(Completeness) *Suppose that $F \in \mathsf{mix}(f_E^k, g_E^k)$ for two functions $f, g: V \to \Sigma$ that differ in at most a single vertex. Then, $\mathcal{W}_q$ accepts F with probability* 1.

(Soundness) *Suppose that $\mathcal{W}_q$ accepts F with probability at least $\frac{1}{2^{q-1}} + \varepsilon$. Then, there exists a function $f: V \to \Sigma$ such that*

$$\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}} \in E^k}\left[F(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} f_E^k(\vec{\boldsymbol{e}})\right] \geqslant \frac{1}{2} + \frac{\varepsilon}{4}, \tag{6.4}$$

where $\eta \in (0,1)$ is a universal constant.

6.1 Proof of Lemma 6.2

Reduction. Our gap-preserving reduction from Maxmin 2-CSP Reconfiguration to Maxmin q-CSP Reconfiguration is described as follows. Let $s \in (0,1)$ and $\varepsilon > 0$ be reals, and $q \geqslant 2$, σ and Δ be positive integers. Let $(G, f_{\mathsf{start}}, f_{\mathsf{end}})$ be an instance of Gap$_{1,s}$ 2-CSP$_\sigma$ Reconfiguration, where $G = (V, E, \Sigma, \Psi = (\psi_e)_{e \in E})$ is a satisfiable Δ-regular n-vertex 2-CSP instance with $|\Sigma| = \sigma$, and $f_{\mathsf{start}}, f_{\mathsf{end}}: V \to \Sigma$ are a pair of its satisfying assignments. We construct a new instance of Maxmin q-CSP Reconfiguration in the form of q-query verifier. Let k be a positive integer. Without loss of generality, we can assume that k and n are sufficiently large so that Lemma 6.3 can be applied.

Our q-query verifier $\mathcal{V}_q$ for Maxmin 2-CSP Reconfiguration is described below. Given oracle access to an edge k-tuple function $F: E^k \to (\Sigma^2)^k$, $\mathcal{V}_q$ first tests if F is close to f_E^k for some assignment $f: V \to \Sigma$ by running $\mathcal{W}_q$ and then tests if the qk edges selected by $\mathcal{W}_q$ are satisfied by F.

q-query verifier $\mathcal{V}_q$ for Maxmin 2-CSP Reconfiguration

Input: a regular 2-CSP instance $G = (V, E, \Sigma, \Psi = (\psi_e)_{e \in E})$.
Oracle access: an edge k-tuple function $F: E^k \to (\Sigma^2)^k$.

run $\mathcal{W}_q(V, E)$ on F.
if run of $\mathcal{W}_q(V, E)$ returned 0 **then**
  return 0.
for each $i \in [q]$ **do**
  let $\vec{\boldsymbol{e}}_i \coloneqq (\boldsymbol{e}_{i,1}, \ldots, \boldsymbol{e}_{i,k})$ be the i^{th} edge k-tuple selected by $\mathcal{W}_q(V, E)$.
  for each $j \in [k]$ **do**
    if $\psi_{\boldsymbol{e}_{i,j}}\left(F(\vec{\boldsymbol{e}}_i)[\boldsymbol{e}_{i,j}]\right) = 0$ **then**
      return 0.
return 1.

The starting and ending functions $F_{\text{start}}, F_{\text{end}} \colon E^k \to (\Sigma^2)^k$ are defined as $F_{\text{start}} \coloneqq (f_{\text{start}})_E^k$ and $F_{\text{end}} \coloneqq (f_{\text{end}})_E^k$; namely, for each edge k-tuple $(x_1y_1, \ldots, x_ky_k) \in E^k$,

$$\begin{aligned} F_{\text{start}}(x_1y_1, \ldots, x_ky_k) &\coloneqq \Big(\big(f_{\text{start}}(x_1), f_{\text{start}}(y_1)\big), \ldots, \big(f_{\text{start}}(x_k), f_{\text{start}}(y_k)\big) \Big), \\ F_{\text{end}}(x_1y_1, \ldots, x_ky_k) &\coloneqq \Big(\big(f_{\text{end}}(x_1), f_{\text{end}}(y_1)\big), \ldots, \big(f_{\text{end}}(x_k), f_{\text{end}}(y_k)\big) \Big). \end{aligned} \tag{6.5}$$

Observe that $\mathcal{V}_q$ accepts both F_{start} and F_{end} with probability 1 by Lemma 6.3, which completes the description of the reduction.

For a function $F \colon E^k \to (\Sigma^2)^k$, the ***value*** $\mathsf{val}_{\mathcal{V}_q}(F)$ is defined as the probability that $\mathcal{V}_q$ accepts F. For a reconfiguration sequence $\vec{F} = (F^{(1)}, \ldots, F^{(T)})$, the ***value*** $\mathsf{val}_{\mathcal{V}_q}(\vec{F})$ is defined as the minimum value over all functions in $\vec{F}$; namely,

$$\mathsf{val}_{\mathcal{V}_q}(\vec{F}) \coloneqq \min_{1 \leqslant t \leqslant T} \big\{ \mathsf{val}_{\mathcal{V}_q}(F^{(t)}) \big\}. \tag{6.6}$$

The ***optimal value*** $\mathsf{opt}_{\mathcal{V}_q}(F_{\text{start}} \leftrightsquigarrow F_{\text{end}})$ is defined as the maximum value over all possible reconfiguration sequences $\vec{F}$ from F_{start} to F_{end}; namely,

$$\mathsf{opt}_{\mathcal{V}_q}(F_{\text{start}} \leftrightsquigarrow F_{\text{end}}) \coloneqq \max_{\vec{F} = (F_{\text{start}}, \ldots, F_{\text{end}})} \big\{ \mathsf{val}_{\mathcal{V}_q}(\vec{F}) \big\}. \tag{6.7}$$

Correctness. We first show the completeness.

Lemma 6.4. *If* $\mathsf{opt}_G(f_{\text{start}} \leftrightsquigarrow f_{\text{end}}) = 1$, *then* $\mathsf{opt}_{\mathcal{V}_q}(F_{\text{start}} \leftrightsquigarrow F_{\text{end}}) = 1$.

Proof. Suppose first that f_{start} and f_{end} differ in a single vertex. Consider a trivial reconfiguration sequence $\vec{F}$ from F_{start} to F_{end} obtained by the following procedure.

Reconfiguration sequence $\vec{F}$ from F_{start} to F_{end}

1: ▷ *start with F_{start}.* ◁
2: **for each** edge k-tuple $\vec{e} \in E^k$ such that $F_{\text{start}}(\vec{e}) \neq F_{\text{end}}(\vec{e})$ **do**
3: change the current assignment to $\vec{e}$ from $F_{\text{start}}(\vec{e})$ to $F_{\text{end}}(\vec{e})$.
4: ▷ *end with F_{end}.* ◁

We show that any intermediate assignment F° in $\vec{F}$ is accepted by $\mathcal{V}_q$ with probability 1. By construction, $F^\circ \in \mathsf{mix}(F_{\text{start}}, F_{\text{end}}) = \mathsf{mix}\big((f_{\text{start}})_E^k, (f_{\text{end}})_E^k\big)$. Since f_{start} and f_{end} differ in a single vertex, by Lemma 6.3, $\mathcal{W}_q$ accepts F° with probability 1. Since f_{start} and f_{end} satisfy G, we have $\psi_{e_i}\big(F^\circ(e_1, \ldots, e_k)[e_i]\big) = 1$ for every edge k-tuple $(e_1, \ldots, e_k) \in E^k$ and every $i \in [k]$. Therefore, $\mathcal{V}_q$ accepts F° with probability 1. In particular, $\mathsf{opt}_{\mathcal{V}_q}(F_{\text{start}} \leftrightsquigarrow F_{\text{end}}) = 1$, as desired.

Suppose now that f_{start} and f_{end} differ in multiple vertices. Since $\mathsf{opt}_G(f_{\text{start}} \leftrightsquigarrow f_{\text{end}}) = 1$, there exists a reconfiguration sequence $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$ from f_{start} to f_{end} consisting of satisfying assignments for G. For each $t \in [T]$, define $F^{(t)} \coloneqq (f^{(t)})_E^k$. By applying the above argument to each pair of $f^{(t)}$ and $f^{(t+1)}$, we have $\mathsf{opt}_{\mathcal{V}_q}(F^{(t)} \leftrightsquigarrow F^{(t+1)}) = 1$, implying that $\mathsf{opt}_{\mathcal{V}_q}(F_{\text{start}} \leftrightsquigarrow F_{\text{end}}) = 1$, as desired. □

We then show the soundness.

Lemma 6.5. *If* $\mathsf{opt}_G(f_{\mathrm{start}} \leftrightsquigarrow f_{\mathrm{end}}) < s$*, then* $\mathsf{opt}_{\mathcal{V}_q}(F_{\mathrm{start}} \leftrightsquigarrow F_{\mathrm{end}}) < \frac{1}{2^{q-1}} + \varepsilon$.

By applying Lemmas 6.4 and 6.5, we can prove Lemma 6.2.

Proof of Lemma 6.2. Construct a q-CSP instance H that represents $\mathcal{V}_q$, whose vertex set is E^k and alphabet is $(\Sigma^2)^k$. By Lemma 6.4, if $\mathsf{opt}_G(f_{\mathrm{start}} \leftrightsquigarrow f_{\mathrm{end}}) = 1$, then $\mathsf{opt}_H(F_{\mathrm{start}} \leftrightsquigarrow F_{\mathrm{end}}) = 1$. By Lemma 6.5, if $\mathsf{opt}_G(f_{\mathrm{start}} \leftrightsquigarrow f_{\mathrm{end}}) < s$, then $\mathsf{opt}_H(F_{\mathrm{start}} \leftrightsquigarrow F_{\mathrm{end}}) < \frac{1}{2^{q-1}} + \varepsilon$. In order to show that the underlying hypergraph of H is regular, we show that $\mathcal{W}_q$ is "regular" in a sense that any edge k-tuple $\vec{e} \in E^k$ appears in $\mathcal{W}_q$'s query with the same probability. Let $(\vec{\boldsymbol{x}}_1, \ldots, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{e}}_1, \ldots, \vec{\boldsymbol{e}}_q, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_q) \sim \mathcal{W}_q$ denote q vertex k-tuples, q edge k-tuples, and q permutations selected by $\mathcal{W}_q$. Observe that any vertex k-tuple $\vec{x} \in V^k$ appears in $\{\vec{\boldsymbol{x}}_1, \ldots, \vec{\boldsymbol{x}}_q\}$ with the same probability. Since each $\vec{\boldsymbol{e}}_i$ is uniformly distributed in E^k as the underlying graph of G is regular, any edge k-tuple $\vec{e} \in E^k$ appears in $\{\vec{\boldsymbol{e}}_1, \ldots, \vec{\boldsymbol{e}}_q\}$ with the same probability. Since each $\vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i$ is obtained by randomly ordering $\vec{\boldsymbol{e}}_i$, any edge k-tuple $\vec{e} \in E^k$ appears in $\{\vec{\boldsymbol{e}}_1 \circ \boldsymbol{\pi}_1, \ldots, \vec{\boldsymbol{e}}_q \circ \boldsymbol{\pi}_q\}$ with the same probability. Therefore, each "vertex" $\vec{e} \in E^k$ of H appears in the same number of hyperedges of H. Consequently, there exists a polynomial-time reduction from $\textsc{Gap}_{1,s}$ 2-CSP_σ $\textsc{Reconfiguration}$ whose underlying graph is Δ-regular to $\textsc{Gap}_{1,\frac{1}{2^{q-1}}+\varepsilon}$ q-$\mathrm{CSP}_{\sigma^{2k}}$ $\textsc{Reconfiguration}$ whose underlying hypergraph is regular, as desired. □

The remainder of this section is devoted to the proof of Lemma 6.5.

Proof of Lemma 6.5. Assume that $\mathsf{opt}_G(f_{\mathrm{start}} \leftrightsquigarrow f_{\mathrm{end}}) < s$. Let $\vec{F} = (F^{(1)}, \ldots, F^{(T)})$ be any reconfiguration sequence from F_{start} to F_{end}. We would like to show that $\mathsf{val}_{\mathcal{V}_q}(\vec{F}) < \frac{1}{2^{q-1}} + \varepsilon$. Suppose first that there exists some function $F^{(t)}$ in $\vec{F}$ such that

$$\mathbf{Pr}\left[\mathcal{W}_q^{F^{(t)}} = 1\right] < \frac{1}{2^{q-1}} + \varepsilon. \tag{6.8}$$

Since $\mathbf{Pr}[\mathcal{V}_q^{F^{(t)}} = 1] \leqslant \mathbf{Pr}[\mathcal{W}_q^{F^{(t)}} = 1]$ by construction, $\mathcal{V}_q$ accepts $F^{(t)}$ with probability below $\frac{1}{2^{q-1}} + \varepsilon$; i.e., $\mathsf{val}_{\mathcal{V}_q}(\vec{F}) < \frac{1}{2^{q-1}} + \varepsilon$, as desired. Hereafter, we assume that for every function $F^{(t)}$ in $\vec{F}$,

$$\mathbf{Pr}\left[\mathcal{W}_q^{F^{(t)}} = 1\right] \geqslant \frac{1}{2^{q-1}} + \varepsilon. \tag{6.9}$$

By applying Lemma 6.3 to each function $F^{(t)}$ in $\vec{F}$, there exists an assignment $f^{(t)} \colon V \to \Sigma$ such that

$$\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}} \in E^k}\left[F^{(t)}(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} (f^{(t)})_E^k(\vec{\boldsymbol{e}})\right] \geqslant \frac{1}{2} + \frac{\varepsilon}{4}. \tag{6.10}$$

Consider a sequence $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$ over Σ^V such that $f^{(1)} := f_{\mathrm{start}}$, $f^{(T)} := f_{\mathrm{end}}$, and $f^{(t)}$ for each $2 \leqslant t \leqslant T-1$ is any function satisfying Eq. (6.10). Although $\vec{f}$ is not necessarily a reconfiguration sequence, $f^{(t)}$ and $f^{(t+1)}$ can be made arbitrarily close by taking k sufficiently large, as described below.

Claim 6.6. *Let* $F_1, F_2 \colon E^k \to (\Sigma^2)^k$ *be any edge k-tuple functions, and* $f_1, f_2 \colon V \to \Sigma$ *be any vertex functions. Suppose that* F_1 *and* F_2 *differ in at most a single coordinate, and*

$$\begin{aligned}
&\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}} \in E^k}\left[F_1(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} f_{1,E}^k(\vec{\boldsymbol{e}})\right] \geqslant \frac{1}{2} + \frac{\varepsilon}{4}, \\
&\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}} \in E^k}\left[F_2(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} f_{2,E}^k(\vec{\boldsymbol{e}})\right] \geqslant \frac{1}{2} + \frac{\varepsilon}{4},
\end{aligned} \tag{6.11}$$

where $f_{1,E}, f_{2,E} \colon E \to \Sigma^2$ denote the induced edge functions of f_1, f_2, respectively, and $\eta \in (0,1)$ is a universal constant appearing in Lemma 6.3. Then,

$$\mathrm{dist}(f_1, f_2) \leqslant \delta(k,\varepsilon) := \sqrt{\frac{1}{2k}\ln\frac{2}{\varepsilon}} + 3k^{-\eta}. \tag{6.12}$$

Proof. By the triangle inequality, for any edge k-tuple $\vec{e} \in E^k$, we have

$$\begin{aligned}
\mathrm{dist}\big(f_{1,E}^k(\vec{e}), f_{2,E}^k(\vec{e})\big) &\leqslant \mathrm{dist}\big(f_{1,E}^k(\vec{e}), F_1(\vec{e})\big) + \mathrm{dist}\big(F_1(\vec{e}), F_2(\vec{e})\big) + \mathrm{dist}\big(F_2(\vec{e}), f_{2,E}^k(\vec{e})\big) \\
&\leqslant \mathrm{dist}\big(f_{1,E}^k(\vec{e}), F_1(\vec{e})\big) + \frac{1}{|E|^k} + \mathrm{dist}\big(F_2(\vec{e}), f_{2,E}^k(\vec{e})\big) \\
&\underbrace{\leqslant}_{|E|\geqslant 2} \mathrm{dist}\big(F_1(\vec{e}), f_{1,E}^k(\vec{e})\big) + \mathrm{dist}\big(F_2(\vec{e}), f_{2,E}^k(\vec{e})\big) + k^{-\eta}.
\end{aligned} \tag{6.13}$$

By assumption and Eq. (6.13), we have

$$\begin{aligned}
\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}}\in E^k}\left[f_{1,E}^k(\vec{\boldsymbol{e}}) \underset{3k^{-\eta}}{\approx} f_{2,E}^k(\vec{\boldsymbol{e}})\right] &\geqslant \mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}}\in E^k}\left[F_1(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} f_{1,E}^k(\vec{\boldsymbol{e}}) \text{ and } F_2(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} f_{2,E}^k(\vec{\boldsymbol{e}})\right] \\
&\geqslant \underbrace{\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}}\in E^k}\left[F_1(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} f_{1,E}^k(\vec{\boldsymbol{e}})\right]}_{\geqslant \frac{1}{2}+\frac{\varepsilon}{4}} + \underbrace{\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}}\in E^k}\left[F_2(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} f_{2,E}^k(\vec{\boldsymbol{e}})\right]}_{\geqslant \frac{1}{2}+\frac{\varepsilon}{4}} - 1 \geqslant \frac{\varepsilon}{2}.
\end{aligned} \tag{6.14}$$

On the other hand, $\mathrm{dist}\big(f_{1,E}^k(\vec{\boldsymbol{e}}), f_{2,E}^k(\vec{\boldsymbol{e}})\big)$ over a random edge k-tuple $\vec{\boldsymbol{e}} = (\boldsymbol{x}_1\boldsymbol{y}_1, \ldots, \boldsymbol{x}_k\boldsymbol{y}_k) \in E^k$ can be thought of as the sum of k independent random variables $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_k$ such that

$$\boldsymbol{Z}_i := \frac{1}{k}[\![f_{1,E}(\boldsymbol{x}_i\boldsymbol{y}_i) \neq f_{2,E}(\boldsymbol{x}_i\boldsymbol{y}_i)]\!] = \frac{1}{k}[\![f_1(\boldsymbol{x}_i) \neq f_2(\boldsymbol{x}_i) \text{ or } f_1(\boldsymbol{y}_i) \neq f_2(\boldsymbol{y}_i)]\!]. \tag{6.15}$$

Let $\delta := \mathrm{dist}(f_1, f_2)$. Note that $0 \leqslant \boldsymbol{Z}_i \leqslant \frac{1}{k}$ for each $i \in [k]$, and that $\mathbf{E}\big[\mathrm{dist}(f_{1,E}^k(\vec{\boldsymbol{e}}), f_{2,E}^k(\vec{\boldsymbol{e}}))\big] = \mathbf{E}\big[\sum_{1\leqslant i\leqslant k}\boldsymbol{Z}_i\big] \geqslant \delta$ because

$$\begin{aligned}
\mathbf{E}\big[\boldsymbol{Z}_i\big] &= \mathop{\mathbf{E}}_{\boldsymbol{x}_i\boldsymbol{y}_i\in E}\left[\tfrac{1}{k}[\![f_1(\boldsymbol{x}_i) \neq f_2(\boldsymbol{x}_i) \text{ or } f_1(\boldsymbol{y}_i) \neq f_2(\boldsymbol{y}_i)]\!]\right] \\
&\geqslant \tfrac{1}{k}\mathop{\mathbf{Pr}}_{\boldsymbol{x}_i\boldsymbol{y}_i\in E}\big[f_1(\boldsymbol{x}_i) \neq f_2(\boldsymbol{x}_i)\big] = \tfrac{1}{k}\mathop{\mathbf{Pr}}_{\boldsymbol{x}_i\in V}\big[f_1(\boldsymbol{x}_i) \neq f_2(\boldsymbol{x}_i)\big] = \frac{\delta}{k},
\end{aligned} \tag{6.16}$$

where the second-to-the-last equality holds because the underlying graph of G is regular. By Hoeffding's inequality, for any real $t > 0$, we have

$$\mathbf{Pr}\left[\sum_{1\leqslant i\leqslant k}\boldsymbol{Z}_i \leqslant \mathbf{E}\left[\sum_{1\leqslant i\leqslant k}\boldsymbol{Z}_i\right] - t\right] \leqslant \exp\left(-\frac{2t^2}{\sum_{1\leqslant i\leqslant k}\left(\frac{1}{k}\right)^2}\right) = \exp(-2kt^2). \tag{6.17}$$

Setting $t := \delta - 3k^{-\eta}$, we obtain

$$\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}}\in E^k}\left[f_{1,E}^k(\vec{\boldsymbol{e}}) \underset{3k^{-\eta}}{\approx} f_{2,E}^k(\vec{\boldsymbol{e}})\right] \leqslant \exp\big(-2k(\delta - 3k^{-\eta})^2\big). \tag{6.18}$$

Consequently, we must have

$$\begin{aligned}
&\exp\left(-2k(\delta-3k^{-\eta})^2\right) \geqslant \Pr_{\vec{\boldsymbol{e}}\in E^k}\left[f_{1,E}^k(\vec{\boldsymbol{e}}) \underset{3k^{-\eta}}{\approx} f_{2,E}^k(\vec{\boldsymbol{e}})\right] \geqslant \frac{\varepsilon}{2}, \\
&\therefore \operatorname{dist}(f_1,f_2) = \delta \leqslant \sqrt{\frac{1}{2k}\ln\frac{2}{\varepsilon}} + 3k^{-\eta},
\end{aligned} \tag{6.19}$$

as desired. □

Since each quadruple of $F^{(t)}$, $F^{(t+1)}$, $f^{(t)}$, and $f^{(t+1)}$ satisfy the condition of Claim 6.6, we have $\operatorname{dist}(f^{(t)}, f^{(t+1)}) \leqslant \delta(k,\varepsilon)$. We then show that if every adjacent pair of assignments in $\vec{f}$ are sufficiently close, then some function in $\vec{f}$ has a value below $\frac{1+s}{2}$.

Claim 6.7. *Let $\vec{f} = (f^{(1)},\ldots,f^{(T)})$ be a sequence over Σ^V from f_{start} to f_{end}. If $\operatorname{dist}(f^{(t)}, f^{(t+1)}) \leqslant \gamma$ for every $t \in [T-1]$, where $\gamma := \frac{1-s}{4}$, then there exists a function $f^{(t)}$ in $\vec{f}$ with value less than $\frac{1+s}{2} < 1$.*

Proof. For each $t \in [T-1]$, let $\vec{g}_t$ be a trivial reconfiguration sequence from $f^{(t)}$ to $f^{(t+1)}$ obtained by merely changing the assignments of at most γn vertices. Concatenating $\vec{g}_t$ for every $t \in [T-1]$, we obtain a reconfiguration sequence $\vec{g}$ from f_{start} to f_{end}. By assumption, $\mathsf{val}_G(\vec{g}) < s$. In particular, there exists some assignment g° in $\vec{g}$ such that $\mathsf{val}_G(g^\circ) < s$. Suppose that g° appears in $\vec{g}_t$. Observe that for two assignments for G that differ in at most a single vertex, their values differ in at most $\frac{\Delta}{|E|}$. Since g° and $f^{(t)}$ differ in at most γn vertices, we derive

$$\mathsf{val}_G(f^{(t)}) \leqslant \mathsf{val}_G(g^\circ) + \frac{\Delta}{|E|}\gamma n \underbrace{=}_{|E|=\frac{|V|\Delta}{2}} \mathsf{val}_G(g^\circ) + \frac{2}{|V|}\frac{1-s}{4}n < s + \frac{1-s}{2} = \frac{1+s}{2}, \tag{6.20}$$

as desired. □

By taking k sufficiently large so that

$$\delta(k,\varepsilon) = \sqrt{\frac{1}{2k}\ln\frac{2}{\varepsilon}} + 3k^{-\eta} \leqslant \frac{1-s}{4}, \tag{6.21}$$

Claim 6.7 derives that there exists some assignment $f^{(t)}$ such that $\mathsf{val}_G(f^{(t)}) < \frac{1+s}{2}$. We are now ready to show that $\mathcal{V}_q$ accepts $F^{(t)}$ with probability at most $\frac{1}{2^{q-1}} + \varepsilon$.

Claim 6.8. *Let $F\colon E^k \to (\Sigma^2)^k$ be an edge k-tuple function and $f\colon V \to \Sigma$ be a vertex function such that*

$$\begin{aligned}
\Pr_{\vec{\boldsymbol{e}}\in E^k}\left[F(\vec{\boldsymbol{e}}) \underset{k^{-\eta}}{\approx} f_E^k(\vec{\boldsymbol{e}})\right] &\geqslant \frac{1}{2} + \frac{\varepsilon}{4}, \\
\mathsf{val}_G(f) &< \frac{1+s}{2}.
\end{aligned} \tag{6.22}$$

Then, $\mathcal{V}_q$ accepts F with probability less than $\frac{1}{2^{q-1}}$.

In the proof of Claim 6.8, we use the following claim, whose proof is obtained by applying Lemma 5.5 and deferred to Appendix B.

Claim 6.9 ($*$). *For any set $\mathcal{S} \subseteq E^k$,*

$$\Pr_{(\vec{\boldsymbol{e}}_1,\ldots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{1\leqslant i\leqslant q}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{S}\right]\leqslant\left(\frac{|\mathcal{S}|}{|E^k|}\right)^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{S}|}{|E^k|}\right)+\mathrm{O}_{k,q}\left(\frac{1}{n}\right), \tag{6.23}$$

where $(\vec{\boldsymbol{e}}_1,\ldots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q$ denote q edge k-tuples and q permutations selected by $\mathcal{W}_q$.

Proof of Claim 6.8. Define $\mathcal{D}$ as the set of edge k-tuples $\vec{e}\in E^k$ such that $F(\vec{e})$ does not approximately agree with $f_E^k(\vec{e})$, and S as the set of edges e satisfied by f; namely,

$$\begin{aligned}\mathcal{D} &:= \left\{\vec{e}\in E^k \;\middle|\; F(\vec{e}) \underset{k^{-\eta}}{\not\approx} f_E^k(\vec{e})\right\},\\ S &:= \left\{e\in E \;\middle|\; \psi_e\big(f_E(e)\big)=1\right\}.\end{aligned}\tag{6.24}$$

Let $(\vec{\boldsymbol{e}}_1,\ldots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q$. Observe that for $\mathcal{V}_q$ to accept F, (at least) either of the following two conditions must hold:

(C1) For every $i\in[q]$, it holds that $\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{D}$.

(C2) For some $i\in[q]$, at least $k-k^{1-\eta}$ edges of $\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i$ are in S.

To see why this is true, suppose that $\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\notin\mathcal{D}$ for some $i\in[q]$. Denote $(\boldsymbol{e}_{i,1},\ldots,\boldsymbol{e}_{i,k}):=\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i$. Since $F(\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i)\underset{k^{-\eta}}{\approx}f_E^k(\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i)$ by assumption, $F(\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i)[\boldsymbol{e}_{i,j}]=f_E(\boldsymbol{e}_{i,j})$ holds for at least $k-k^{1-\eta}$ edges of $\boldsymbol{e}_{i,1},\ldots,\boldsymbol{e}_{i,k}$. Therefore, $F(\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i)$ satisfies all of the k edges $\boldsymbol{e}_{i,1},\ldots,\boldsymbol{e}_{i,k}$ only if at least $k-k^{1-\eta}$ of them are in S.

Since $\frac{|\mathcal{D}|}{|E^k|}\leqslant\frac{1}{2}-\frac{\varepsilon}{4}$ by assumption, we apply Claim 6.9 to $\mathcal{D}$ and derive

$$\begin{aligned}\Pr_{(\vec{\boldsymbol{e}}_1,\ldots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\big[\text{(C1) holds}\big] &= \Pr_{(\vec{\boldsymbol{e}}_1,\ldots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{1\leqslant i\leqslant q}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{D}\right]\\ &\leqslant\left(\frac{|\mathcal{D}|}{|E^k|}\right)^q+\mathrm{O}\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{D}|}{|E^k|}\right)+\mathrm{O}_{q,k}\left(\frac{1}{n}\right)\\ &\leqslant\left(\frac{1}{2}-\frac{\varepsilon}{4}\right)^q+\mathrm{O}\left(\frac{\ell}{k}\right)+\mathrm{O}_{q,k}\left(\frac{1}{n}\right)\\ &\leqslant\frac{1}{2^q}+\mathrm{O}_q\left(\frac{1}{\sqrt{k}}\right)+\mathrm{O}_{q,k}\left(\frac{1}{n}\right).\end{aligned}\tag{6.25}$$

Since each $\vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i$ is uniformly distributed in E^k and $\frac{|S|}{|E|} \leqslant \frac{1+s}{2}$ by assumption, we derive for sufficiently large k,

$$\begin{aligned}
\Pr_{(\vec{\boldsymbol{e}}_1,\ldots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\Big[\text{(C2) holds}\Big] &= \Pr_{(\vec{\boldsymbol{e}}_1,\ldots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigvee_{1\leqslant i\leqslant q} \Big|\Big\{\boldsymbol{e}_{i,j} \in \vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i \;\Big|\; \boldsymbol{e}_{i,j} \in S\Big\}\Big| \geqslant k - k^{1-\eta}\right] \\
&\leqslant \sum_{1\leqslant i\leqslant q} \Pr_{(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k)\in E^k}\Big[\big|\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k\}\cap S\big| \geqslant k - k^{1-\eta}\Big] \\
&\leqslant q\binom{k}{k-k^{1-\eta}}\left(\frac{|S|}{|E|}\right)^{k-k^{1-\eta}} \\
&\leqslant q\left(\frac{\mathrm{e}k}{k^{1-\eta}}\right)^{k^{1-\eta}}\left(\frac{1+s}{2}\right)^{k-k^{1-\eta}} \\
&= q\cdot\exp\big(\Theta(k^{1-\eta}\log k^{1-\eta})\big)\cdot\exp\big(-\Theta(k)\big) \\
&= q\cdot\exp\big(-\Omega(k)\big).
\end{aligned} \tag{6.26}$$

Consequently, for sufficiently large k and n, we obtain

$$\begin{aligned}
\Pr\Big[\mathcal{V}_q^F = 1\Big] &\leqslant \Pr\big[\text{(C1) holds}\big] + \Pr\big[\text{(C2) holds}\big] \\
&\leqslant \frac{1}{2^q} + \underbrace{\mathrm{O}_q\left(\frac{1}{\sqrt{k}}\right) + \mathrm{O}_{q,k}\left(\frac{1}{n}\right) + q\cdot\exp\big(-\Omega(k)\big)}_{\leqslant \frac{1}{2^q}} \leqslant \frac{1}{2^{q-1}},
\end{aligned} \tag{6.27}$$

as desired. □

By Claim 6.8, $\mathcal{V}_q$ accepts $F^{(t)}$ with probability at most $\frac{1}{2^{q-1}}$, implying that $\mathsf{val}_{\mathcal{V}_q}(\vec{F}) \leqslant \mathsf{val}_{\mathcal{V}_q}(F^{(t)}) \leqslant \frac{1}{2^{q-1}} < \frac{1}{2^{q-1}} + \varepsilon$, which completes the proof of Lemma 6.5. □

6.2 Proof of Lemma 6.3

We prove Lemma 6.3, i.e., the completeness and soundness of $\mathcal{W}_q$. Let $\varepsilon > 0$ be a real. Let $G = (V, E)$ be a Δ-regular n-vertex graph, and $F\colon E^k \to (\Sigma^2)^k$ be an edge k-tuple function. Hereafter, we assume that k and n are sufficiently large so that Theorem 5.1 can be applied. We first prove the completeness part.

Proof of the completeness part of Lemma 6.3. Suppose that $F \in \mathsf{mix}(f_E^k, g_E^k)$ for two functions $f, g\colon V \to \Sigma$ that differ in at most a single vertex. We will show that $\mathcal{W}_q$ never rejects during the iteration in lines 10–14 for each $i \in [q-1]$ by the following case analysis:

(Case 1) Suppose that $f = g$. Then, F is the k-wise direct product of f_E; thus,

$$\big(F(\vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i) \circ \boldsymbol{\pi}_i^{-1}\big)\big[\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}\big] = \big(F(\vec{\boldsymbol{e}}_{i+1} \circ \boldsymbol{\pi}_{i+1}) \circ \boldsymbol{\pi}_{i+1}^{-1}\big)\big[\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\big], \tag{6.28}$$

implying that $\mathcal{W}_q$ does not reject.

(Case 2) Suppose that f and g differ in a single vertex, say $x^* \in V$. Note that for each edge k-tuple $\vec{e} = (x_1y_1, \ldots, x_ky_k) \in E^k$, we have $F(\vec{e})[x_i] = f(x_i) = g(x_i)$ whenever $x_i \neq x^*$.

(Case 2-1) Suppose that $|\{\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,k},\boldsymbol{x}_{i+1,1},\ldots,\boldsymbol{x}_{i+1,k}\}| < 2k-\ell$. Obviously, $\mathcal{W}_q$ does not reject.

(Case 2-2) Suppose that $|\{\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,k},\boldsymbol{x}_{i+1,1},\ldots,\boldsymbol{x}_{i+1,k}\}| \geqslant 2k-\ell$. Since $\boldsymbol{x}_{i,j} = \boldsymbol{x}_{i+1,j}$ for every $j \in \boldsymbol{I}_i$, we have $|\{\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,k},\boldsymbol{x}_{i+1,1},\ldots,\boldsymbol{x}_{i+1,k}\}| = 2k-\ell$. Thus, x^* appears in $\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}$ at most once.

(Case 2-2-1) If x^* does not appear in $\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}$, then we have

$$\left(F(\vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i) \circ \boldsymbol{\pi}_i^{-1}\right)\left[\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}\right] = \left(F(\vec{\boldsymbol{e}}_{i+1} \circ \boldsymbol{\pi}_{i+1}) \circ \boldsymbol{\pi}_{i+1}^{-1}\right)\left[\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right], \tag{6.29}$$

implying that $\mathcal{W}_q$ does not reject.

(Case 2-2-2) If x^* appears in $\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}$ once, then there exists a unique $j^* \in \boldsymbol{I}_i$ such that $x_{i,j^*} = x_{i+1,j^*} = x^*$. For each $j \in \boldsymbol{I}_i \setminus \{j^*\}$, we have

$$\left(F(\vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i) \circ \boldsymbol{\pi}_i^{-1}\right)\left[\boldsymbol{x}_{i,j}\right] = \left(F(\vec{\boldsymbol{e}}_{i+1} \circ \boldsymbol{\pi}_{i+1}) \circ \boldsymbol{\pi}_{i+1}^{-1}\right)\left[\boldsymbol{x}_{i+1,j}\right]. \tag{6.30}$$

Therefore, we have

$$\left(F(\vec{\boldsymbol{e}}_i \circ \boldsymbol{\pi}_i) \circ \boldsymbol{\pi}_i^{-1}\right)\left[\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}\right] \underset{1/\ell}{\approx} \left(F(\vec{\boldsymbol{e}}_{i+1} \circ \boldsymbol{\pi}_{i+1}) \circ \boldsymbol{\pi}_{i+1}^{-1}\right)\left[\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right], \tag{6.31}$$

implying that $\mathcal{W}_q$ does not reject. □

In the remainder of this subsection, we prove the soundness part of Lemma 6.3. Suppose that $\mathcal{W}_q$ accepts F with probability at least $\frac{1}{2^{q-1}} + \varepsilon$. Let $F_{\mathrm{tuple}} \colon V^k \times [\Delta]^k \to \Sigma^k$ be a k-tuple function whose value is determined based on F by the following procedure.

***k*-tuple function $F_{\mathrm{tuple}} \colon V^k \times [\Delta]^k \to \Sigma^k$**

Input: a vertex k-tuple $\vec{x} = (x_1,\ldots,x_k) \in V^k$, and an integer k-tuple $\vec{a} = (a_1,\ldots,a_k) \in [\Delta]^k$.

```
1: for each i ∈ [k] do
2:     let e_i ∈ E(x_i) be the a_i-th edge incident to x_i.
3: return F(e_1, ..., e_k)[x].
```

Let $\mathcal{T}_{\mathrm{tuple}}$ be a tolerant q-query direct product tester for a k-tuple function described as follows.

Tolerant q-query direct product tester $\mathcal{T}_{\text{tuple}}$ for a k-tuple function

Input: positive integers ℓ and m with $1 \leqslant m \leqslant \ell \leqslant k$.
Oracle access: a k-tuple function $F_{\text{tuple}} \colon V^k \times \mathcal{R} \to \Sigma^k$.

```
sample x⃗_1 = (x_{1,1}, ..., x_{1,k}) from V^k uniformly.
for each i ∈ [q − 1] do
    sample I_i from binom([k], ℓ) uniformly.
    sample x⃗_{i+1} = (x_{i+1,1}, ..., x_{i+1,k}) from V^k uniformly conditioned on x⃗_{i+1}|_{I_i} = x⃗_i|_{I_i}.
for each i ∈ [q] do
    sample π_i from 𝔖_k uniformly.
    sample r_i from ℛ uniformly.
    read F_tuple(x⃗_i ∘ π_i; r_i).
for each i ∈ [q − 1] do
    if (F_tuple(x⃗_i ∘ π_i; r_i) ∘ π_i^{−1})|_{I_i} ≉_{m/ℓ} (F_tuple(x⃗_{i+1} ∘ π_{i+1}; r_{i+1}) ∘ π_{i+1}^{−1})|_{I_i} then
        return 0.
return 1.
```

Consider running $\mathcal{T}_{\text{tuple}}(\ell, 1)$ on F_{tuple}, where $\mathcal{R} := [\Delta]^k$. We show that $\mathcal{W}_q$ and $\mathcal{T}_{\text{tuple}}(\ell, 1)$ have nearly the same acceptance probability, whose proof is deferred to Appendix B.

Claim 6.10 ($*$)**.** *If $\mathcal{W}_q$ accepts F with probability at least p, then $\mathcal{T}_{\text{tuple}}(\ell, 1)$ accepts F_{tuple} with probability at least $p - \frac{qk^2}{n}$.*

Let $F_{\text{set}} \colon \binom{V}{k} \times [\Delta]^k \times \mathfrak{S}_k \to \Sigma^k$ be a k-set function whose value is determined based on F_{tuple} by the following procedure.

k-set function $F_{\text{set}} \colon \binom{V}{k} \times [\Delta]^k \times \mathfrak{S}_k \to \Sigma^k$

Input: a k-set $X \in \binom{V}{k}$, an integer k-tuple $\vec{a} \in [\Delta]^k$, and a permutation $\pi \in \mathfrak{S}_k$.

```
let (x_1, ..., x_k) be a canonical ordering of k vertices in X.
let X ∘ π := (x_{π(1)}, ..., x_{π(k)}).
return F_tuple(X ∘ π; a⃗) ∘ π^{−1}.
```

Consider running $\mathcal{T}_q(\ell, 1)$ on F_{set}, where $\mathcal{R} := [\Delta]^k \times \mathfrak{S}_k$. (See Section 5 for the description of $\mathcal{T}_q$.) We show that $\mathcal{T}_{\text{tuple}}(\ell, 1)$ and $\mathcal{T}_q(\ell, 1)$ have nearly the same acceptance probability, whose proof is well-known (cf. [DS14, Section A] and [IKW12, Footnote 5]) and deferred to Appendix B.

Claim 6.11 ($*$)**.** *If $\mathcal{T}_{\text{tuple}}(\ell, 1)$ accepts F_{tuple} with probability at least p, then $\mathcal{T}_q(\ell, 1)$ accepts F_{set} with probability at least $p - \frac{qk^2}{n}$.*

By applying Theorem 5.1, we show that if $\mathcal{T}_q(\ell, 1)$ accepts F_{set} with probability at least p, then there exists a function $f \colon V \to \Sigma$ such that F_{set} approximately agrees with f^k with probability $\approx p^{\frac{1}{q-1}}$.

Claim 6.12. *Let $\eta := \frac{1}{128}$. Suppose that $\mathcal{T}_q(\ell,1)$ accepts F_{set} with probability at least $p \geqslant k^{-\eta}$. Then, there exists a function $f\colon V \to \Sigma$ such that*

$$\Pr_{\substack{\boldsymbol{X} \in \binom{V}{k} \\ (\vec{\boldsymbol{a}},\boldsymbol{\pi}) \in [\Delta]^k \times \mathfrak{S}_k}} \left[F_{\text{set}}(\boldsymbol{X};\vec{\boldsymbol{a}},\boldsymbol{\pi}) \underset{\delta}{\approx} f^k(\boldsymbol{X}) \right] \geqslant p^{\frac{1}{q-1}}(1-k^{-\Omega(1)}), \text{ where } \delta := k^{-\eta}. \tag{6.32}$$

Proof. Substituting $\Theta(\sqrt{k})$ for ℓ and 1 for m, we evaluate the value of $\varepsilon_{k,\ell,m}$ (see Eq. (5.1) for the definition) as

$$\varepsilon_{k,\ell,m} = \widetilde{\Theta}\left(\max\left\{\frac{\ell}{k}, \frac{m^{\frac{1}{8}}}{\ell^{\frac{1}{8}}}, \frac{1}{\ell^{\frac{1}{16}}}\right\}\right) = \widetilde{\Theta}\left(\max\left\{\frac{1}{k^{\frac{1}{2}}}, \frac{1}{k^{\frac{1}{16}}}, \frac{1}{k^{\frac{1}{32}}}\right\}\right) = \widetilde{\Theta}\left(\frac{1}{k^{\frac{1}{32}}}\right). \tag{6.33}$$

Suppose that $\mathcal{T}_q(\ell,1)$ accepts F_{set} with probability $p \geqslant k^{-\eta}$. Since $k^{-\eta} \geqslant \sqrt{\varepsilon_{k,\ell,m}} = \widetilde{\Theta}(k^{-\frac{1}{64}})$, by Theorem 5.1, there exists a function $f\colon V \to \Sigma$ such that

$$\Pr_{\substack{\boldsymbol{X} \in \binom{V}{k} \\ (\vec{\boldsymbol{a}},\boldsymbol{\pi}) \in [\Delta]^k \times \mathfrak{S}_k}} \left[F_{\text{set}}(\boldsymbol{X};\vec{\boldsymbol{a}},\boldsymbol{\pi}) \underset{k^{-\eta}}{\approx} f^k(\boldsymbol{X}) \right] \geqslant p^{\frac{1}{q-1}}\left(1-k^{-\Omega(1)}\right), \tag{6.34}$$

as desired. □

We show that if F_{set} approximately agrees with f^k with probability p, then so does F_{tuple} with probability $\approx p$, whose proof is well-known (cf. [DS14, Section A] and [IKW12, Footnote 5]) and deferred to Appendix B.

Claim 6.13 (∗)**.**

$$\Pr_{\substack{\boldsymbol{X} \in \binom{V}{k} \\ (\vec{\boldsymbol{a}},\boldsymbol{\pi}) \in [\Delta]^k \times \mathfrak{S}_k}} \left[F_{\text{set}}(\boldsymbol{X};\vec{\boldsymbol{a}},\boldsymbol{\pi}) \underset{\delta}{\approx} f^k(\boldsymbol{X}) \right] \geqslant p \implies \Pr_{\substack{\vec{\boldsymbol{x}} \in V^k \\ \vec{\boldsymbol{a}} \in [\Delta]^k}} \left[F_{\text{tuple}}(\vec{\boldsymbol{x}};\vec{\boldsymbol{a}}) \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}}) \right] \geqslant p\left(1-\frac{k^2}{n}\right). \tag{6.35}$$

We show that if F_{tuple} approximately agrees with f^k with probability p, then F approximately agrees with f_E^k with probability $\approx p$, whose proof is based on [IKW12, Proof of Theorem 6.1].

Claim 6.14.

$$\Pr_{\substack{\vec{\boldsymbol{x}} \in V^k \\ \vec{\boldsymbol{a}} \in [\Delta]^k}} \left[F_{\text{tuple}}(\vec{\boldsymbol{x}};\vec{\boldsymbol{a}}) \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}}) \right] \geqslant p \implies \Pr_{\vec{\boldsymbol{e}} \in E^k} \left[F(\vec{\boldsymbol{e}}) \underset{4\delta}{\approx} f_E^k(\vec{\boldsymbol{e}}) \right] \geqslant p - \exp\left(-\Omega(\delta k)\right). \tag{6.36}$$

Proof. For an edge k-tuple $\vec{e} = (e_1,\ldots,e_k) = (x_1y_1,\ldots,x_ky_k)$, let $\prod \vec{e}$ denote the set of vertex k-tuples obtained by selecting k endpoints from $e_1,\ldots,e_k$; namely,

$$\prod \vec{e} := \prod_{1 \leqslant i \leqslant k} \{x_i,y_i\} = \{x_1,y_1\} \times \cdots \times \{x_k,y_k\}. \tag{6.37}$$

By assumption, we have

$$\Pr_{\substack{\vec{\boldsymbol{x}}=(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)\in V^k\\ \vec{\boldsymbol{a}}=(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_k)\in[\Delta]^k}}\left[F_{\text{tuple}}(\vec{\boldsymbol{x}};\vec{\boldsymbol{a}}) \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}})\right] \geqslant p, \text{ which implies } \Pr_{\substack{\vec{\boldsymbol{e}}=(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_k)\in E^k\\ \vec{\boldsymbol{x}}=(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)\in\prod\vec{\boldsymbol{e}}}}\left[F(\vec{\boldsymbol{e}})[\vec{\boldsymbol{x}}] \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}})\right] \geqslant p, \tag{6.38}$$

where we used the fact that the $\boldsymbol{a}_i{}^{\text{th}}$ edge incident to $\boldsymbol{x}_i$ is uniformly distributed over E, where $\boldsymbol{x}_i \in V$ and $\boldsymbol{a}_i \in [\Delta]$, since the underlying graph of G is Δ-regular. We will claim that

$$\Pr_{\vec{\boldsymbol{e}}\in E^k}\left[F(\vec{\boldsymbol{e}}) \underset{4\delta}{\approx} f_E^k(\vec{\boldsymbol{e}})\right] \geqslant 1 - \frac{1-p}{1-\exp(-\delta k/6)}, \tag{6.39}$$

which is sufficient to conclude the proof because

$$\begin{aligned} 1 - \frac{1-p}{1-\exp(-\delta k/6)} &= \frac{p-\exp(-\delta k/6)}{1-\exp(-\delta k/6)} \\ &\geqslant \big(p-\exp(-\delta k/6)\big)\big(1+\exp(-\delta k/6)\big) \\ &\geqslant p - 2\exp(-\delta k/6) = p - \exp\big(-\Omega(\delta k)\big). \end{aligned} \tag{6.40}$$

For this purpose, we show the contrapositive; namely,

$$\Pr_{\vec{\boldsymbol{e}}\in E^k}\left[F(\vec{\boldsymbol{e}}) \underset{4\delta}{\not\approx} f_E^k(\vec{\boldsymbol{e}})\right] \geqslant \frac{1-p}{1-\exp(-\delta k/6)} \implies \Pr_{\substack{\vec{\boldsymbol{e}}\in E^k\\ \vec{\boldsymbol{x}}\in\prod\vec{\boldsymbol{e}}}}\left[F(\vec{\boldsymbol{e}})[\vec{\boldsymbol{x}}] \underset{\delta}{\not\approx} f^k(\vec{\boldsymbol{x}})\right] \geqslant 1-p. \tag{6.41}$$

Suppose that for a fixed edge k-tuple $\vec{e} = (e_1,\ldots,e_k) \in E^k$,

$$F(\vec{e}) \underset{4\delta}{\not\approx} f_E^k(\vec{e}). \tag{6.42}$$

Observe that $k \cdot \operatorname{dist}\big(F(\vec{e})[\vec{\boldsymbol{x}}], f^k(\vec{\boldsymbol{x}})\big)$ over a random vertex k-tuple $\vec{\boldsymbol{x}} = (\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k) \in \prod\vec{e}$ can be thought of as the sum of k independent Bernoulli random variables $\boldsymbol{Z}_1,\ldots,\boldsymbol{Z}_k$ such that

$$\boldsymbol{Z}_i := [\![F(\vec{e})[\boldsymbol{x}_i] \neq f(\boldsymbol{x}_i)]\!]. \tag{6.43}$$

Since $\mu := \mathbf{E}\left[\sum_{1\leqslant i\leqslant k}\boldsymbol{Z}_i\right] \geqslant 2\delta k$, by the Chernoff bound, we have

$$\Pr\left[\sum_{1\leqslant i\leqslant k}\boldsymbol{Z}_i < \frac{\mu}{2}\right] < 2\exp\left(-\frac{\left(\frac{1}{2}\right)^2}{3}\mu\right) < \exp(-\delta k/6), \tag{6.44}$$

implying that

$$\Pr_{\vec{\boldsymbol{x}}\in\prod\vec{e}}\left[F(\vec{e})[\vec{\boldsymbol{x}}] \underset{\delta}{\not\approx} f^k(\vec{\boldsymbol{x}})\right] > 1 - \exp(-\delta k/6). \tag{6.45}$$

Consequently, by the assumption of Eq. (6.41), we derive

$$\begin{aligned} &\Pr_{\vec{\boldsymbol{e}}\in E^k}\left[\Pr_{\vec{\boldsymbol{x}}\in\prod\vec{\boldsymbol{e}}}\left[F(\vec{\boldsymbol{e}})[\vec{\boldsymbol{x}}] \underset{\delta}{\not\approx} f^k(\vec{\boldsymbol{x}})\right] \geqslant 1-\exp(-\delta k/6)\right] \geqslant \frac{1-p}{1-\exp(-\delta k/6)}, \\ &\therefore \Pr_{\substack{\vec{\boldsymbol{e}}\in E^k\\ \vec{\boldsymbol{x}}\in\prod\vec{\boldsymbol{e}}}}\left[F(\vec{\boldsymbol{e}})[\vec{\boldsymbol{x}}] \underset{\delta}{\not\approx} f^k(\vec{\boldsymbol{x}})\right] \geqslant \big(1-\exp(-\delta k/6)\big)\frac{1-p}{1-\exp(-\delta k/6)} = 1-p, \end{aligned} \tag{6.46}$$

as desired. □

We are now ready to complete the proof of the soundness part of Lemma 6.3.

Proof of the soundness part of Lemma 6.3. Suppose that $\mathcal{W}_q$ accepts F with probability $\frac{1}{2^{q-1}} + \varepsilon$. By Claims 6.10 to 6.14, there exists a function $f : V \to \Sigma$ such that

$$\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}} \in E^k}\left[F(\vec{\boldsymbol{e}}) \underset{4\delta}{\approx} f_E^k(\vec{\boldsymbol{e}})\right] \geqslant \left(\frac{1}{2^{q-1}} + \varepsilon - \frac{qk^2}{n} - \frac{qk^2}{n}\right)^{\frac{1}{q-1}} \left(1 - k^{-\Omega(1)}\right)\left(1 - \frac{k^2}{n}\right) - \exp\left(-\Omega(\delta k)\right), \tag{6.47}$$

which implies that for sufficiently large k and n,

$$\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}} \in E^k}\left[F(\vec{\boldsymbol{e}}) \underset{k^{-\eta/2}}{\approx} f_E^k(\vec{\boldsymbol{e}})\right] \geqslant \frac{1}{2} + \frac{\varepsilon}{4}, \tag{6.48}$$

completing the proof. □

7 NP-membership of $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor Approximation for MAXMIN q-CSP RECONFIGURATION

In this section, we prove Theorem 1.2; i.e., NP-membership of a $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation for MAXMIN q-CSP RECONFIGURATION in the perfect completeness case.

Theorem 7.1. *For any positive integers $q \geqslant 2$ and σ, and any real $\varepsilon > 0$, let $G = (V,E,\Sigma,\Psi)$ be a satisfiable q-CSP instance on n vertices with $|\Sigma| = \sigma$, and $f_{\mathsf{start}}, f_{\mathsf{end}} : V \to \Sigma$ be a pair of its satisfying assignments. If there exists a reconfiguration sequence from f_{start} to f_{end} consisting only of satisfying assignments for G, then there exists a reconfiguration sequence $\vec{f}$ from f_{start} to f_{end} such that*

$$\mathsf{val}_G(\vec{f}) \geqslant \frac{1}{2^{q-1}} - \varepsilon, \tag{7.1}$$

and the length of $\vec{f}$ is at most $n^{O_{q,\sigma,\varepsilon}(1)}$. In particular, for any positive integers $q \geqslant 2$ and σ, and any real $\varepsilon > 0$, GAP$_{1,\frac{1}{2^{q-1}}-\varepsilon}$ q-CSP$_\sigma$ RECONFIGURATION is in NP.

Hereafter, we fix positive integers $q \geqslant 2$ and σ, and a real $\varepsilon > 0$. Let $G = (V,E,\Sigma,\Psi)$ be a satisfiable q-CSP instance with $|\Sigma| = \sigma$. Let $n := |V|$, $m := |E|$, and

$$\Delta := \frac{\varepsilon^2 m}{100q\log n}. \tag{7.2}$$

We say that a vertex of G is ***low degree*** if its degree is at most Δ and ***high degree*** otherwise. The proof of Theorem 7.1 is based on the following interpolation lemma for low-degree variables.

Lemma 7.2. *Let $f_{\mathsf{start}}, f_{\mathsf{end}} : V \to \Sigma$ be a pair of satisfying assignments for G. Define*

$$D := \left\{x \in V \;\middle|\; f_{\mathsf{start}}(x) \neq f_{\mathsf{end}}(x)\right\}. \tag{7.3}$$

Suppose that every vertex $x \in D$ has degree at most Δ. Then, for any sufficiently large n, there exists a reconfiguration sequence $\vec{f}$ from f_{start} to f_{end} of length $|D|+1$ such that

$$\mathsf{val}_G(\vec{f}) \geqslant \frac{1}{2^{q-1}} - \varepsilon. \tag{7.4}$$

Proof. Choose independent random labels $(\boldsymbol{\lambda}_x)_{x\in D}$, each uniformly distributed on $(0,1)$. For each real $\theta \in [0,1]$, define an assignment $\boldsymbol{h}_\theta : V \to \Sigma$ as

$$\boldsymbol{h}_\theta(x) := \begin{cases} f_{\text{end}}(x) & \text{if } \boldsymbol{\lambda}_x \leqslant \theta, \\ f_{\text{start}}(x) & \text{otherwise.} \end{cases} \tag{7.5}$$

Note that $\boldsymbol{h}_0 = f_{\text{start}}$ and $\boldsymbol{h}_1 = f_{\text{end}}$. For each real $\theta \in [0,1]$, let

$$\boldsymbol{X}_\theta := \left|\left\{e \in E \;\middle|\; \boldsymbol{h}_\theta \text{ satisfies } e\right\}\right| = m \cdot \mathsf{val}_G(\boldsymbol{h}_\theta). \tag{7.6}$$

We show that there exists a realization of the random variables for which $\boldsymbol{X}_\theta \geqslant \left(\frac{1}{2^{q-1}} - \varepsilon\right)m$ holds for every $\theta \in [0,1]$.

(Step 1) Lower bound on $\mathbf{E}[\boldsymbol{X}_\theta]$ for fixed θ. Fix a real $\theta \in [0,1]$ and a hyperedge $e \in E$. Let $c_e := |e \cap D|$. If $c_e = 0$, then $\boldsymbol{h}_\theta|_e = f_{\text{start}}|_e = f_{\text{end}}|_e$; thus, e is always satisfied. Assume now that $c_e \geqslant 1$. Since f_{start} and f_{end} are satisfying assignments, $\boldsymbol{h}_\theta$ certainly satisfies e if each of the following two cases holds:

- No vertex in $e \cap D$ has switched to its value in f_{end}, which happens with probability $(1-\theta)^{c_e}$.
- Every vertex in $e \cap D$ has switched to its value in f_{end}, which happens with probability θ^{c_e}.

Since these two events are disjoint, we have

$$\mathbf{Pr}\left[\boldsymbol{h}_\theta \text{ satisfies } e\right] \geqslant (1-\theta)^{c_e} + \theta^{c_e} \underbrace{\geqslant}_{1 \leqslant c_e \leqslant q} \theta^q + (1-\theta)^q \geqslant \left(\frac{1}{2}\right)^q + \left(\frac{1}{2}\right)^q = \frac{1}{2^{q-1}}, \tag{7.7}$$

where the second-to-last inequality holds because the function $\varphi_q(\theta) := \theta^q + (1-\theta)^q$ attains its minimum at $\theta = \frac{1}{2}$. Combining the cases $c_e = 0$ and $c_e \geqslant 1$, we obtain

$$\mathbf{Pr}\left[\boldsymbol{h}_\theta \text{ satisfies } e\right] \geqslant \frac{1}{2^{q-1}}, \qquad \therefore \mathbf{E}\left[\boldsymbol{X}_\theta\right] \geqslant \frac{m}{2^{q-1}}. \tag{7.8}$$

(Step 2) Concentration on fixed θ. For a fixed real $\theta \in [0,1]$, regard $\boldsymbol{X}_\theta$ as a function $g_\theta : (0,1)^D \to \mathbb{N}$ of the random labels defined as

$$g_\theta\left((\boldsymbol{\lambda}_x)_{x\in D}\right) := \boldsymbol{X}_\theta. \tag{7.9}$$

If only the coordinate $\boldsymbol{\lambda}_x$ is changed, then only the assignment to x may change. Therefore, g_θ satisfies the bounded difference property with bounds $(d(x))_{x\in D}$. By applying McDiarmid's inequality to Eq. (7.8), we obtain

$$\mathbf{Pr}\left[\boldsymbol{X}_\theta < \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right)m\right] \leqslant \mathbf{Pr}\left[\boldsymbol{X}_\theta < \mathbf{E}[\boldsymbol{X}_\theta] - \frac{\varepsilon m}{2}\right] \leqslant \exp\left(-\frac{2\left(\frac{\varepsilon m}{2}\right)^2}{\sum_{x\in D} d(x)^2}\right). \tag{7.10}$$

Since $d(x) \leqslant \Delta$ for every vertex $x \in D$,

$$\sum_{x\in D} d(x)^2 \leqslant \Delta \sum_{x\in D} d(x) \leqslant \Delta q m. \tag{7.11}$$

Hence, we derive

$$\mathbf{Pr}\left[\boldsymbol{X}_\theta < \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right)m\right] \leqslant \exp\left(-\frac{\varepsilon^2 m}{2\Delta q}\right) \underbrace{\leqslant}_{\text{Eq. (7.2)}} n^{-50}. \tag{7.12}$$

(Step 3) One realization works for every $\theta \in [0,1]$. Let $M := n^4$ and

$$\Gamma := \left\{ \frac{j}{M} \;\middle|\; 0 \leqslant j \leqslant M \right\}. \tag{7.13}$$

Let $\mathcal{E}_1$ be the event that the value of $\boldsymbol{h}_\theta$ is at least $\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}$ for every $\theta \in \Gamma$; namely,

$$\mathcal{E}_1 := \left\{ \boldsymbol{X}_\theta \geqslant \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2} \right) m \text{ for every } \theta \in \Gamma \right\}. \tag{7.14}$$

By applying the union bound to Eq. (7.12), we derive

$$\mathbf{Pr}\left[\neg \mathcal{E}_1\right] \leqslant (M+1) n^{-50} = n^{-\Omega(1)}. \tag{7.15}$$

For each $j \in [M]$, define the interval I_j as

$$I_j := \left(\frac{j-1}{M}, \frac{j}{M} \right]. \tag{7.16}$$

Set

$$B := \left\lfloor \frac{\varepsilon m}{4\Delta} \right\rfloor \geqslant \frac{25 q \log n}{\varepsilon} - 1. \tag{7.17}$$

Let $\mathcal{E}_2$ be the event that every interval I_j contains at most B labels; namely,

$$\mathcal{E}_2 := \left\{ \left| \left\{ x \in D \mid \boldsymbol{\lambda}_x \in I_j \right\} \right| \leqslant B \text{ for every } j \in [M] \right\}. \tag{7.18}$$

For each $j \in [M]$, we have

$$\begin{aligned} \mathbf{Pr}\left[\left| \left\{ x \in D \mid \boldsymbol{\lambda}_x \in I_j \right\} \right| \geqslant B+1 \right] &= \mathbf{Pr}\left[\exists S \in \tbinom{S}{B+1} \text{ s.t. } \forall x \in S,\ \boldsymbol{\lambda}_x \in I_j \right] \\ &\leqslant \binom{|D|}{B+1} \left(\frac{1}{M} \right)^{B+1} \leqslant n^{B+1} \cdot n^{-4(B+1)} = n^{-3(B+1)}, \end{aligned} \tag{7.19}$$

which implies that

$$\mathbf{Pr}\left[\neg \mathcal{E}_2\right] \leqslant \sum_{j \in [M]} \mathbf{Pr}\left[\left| \left\{ x \in D \mid \boldsymbol{\lambda}_x \in I_j \right\} \right| \geqslant B+1 \right] \leqslant M n^{-3(B+1)} = n^{-\Omega_{q,\varepsilon}(\log n)}. \tag{7.20}$$

By Eqs. (7.15) and (7.20), for any sufficiently large n, we have

$$\mathbf{Pr}\left[\mathcal{E}_1 \wedge \mathcal{E}_2 \right] > 0. \tag{7.21}$$

Fix a realization of the random labels $(\boldsymbol{\lambda}_x)_{x \in D}$ in this event. We can safely assume that all labels are distinct.

We claim that

$$\boldsymbol{X}_\theta \geqslant \left(\frac{1}{2^{q-1}} - \varepsilon \right) m \qquad \text{for every } \theta \in [0,1]. \tag{7.22}$$

Fix a real $\theta \in [0,1]$, and let $\theta' := \frac{\lfloor M\theta \rfloor}{M} \in \Gamma$ be the "left endpoint" of the interval containing θ. Since $\mathcal{E}_1$ holds,

$$\boldsymbol{X}_{\theta'} \geqslant \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right) m. \tag{7.23}$$

The assignments $\boldsymbol{h}_{\theta'}$ and $\boldsymbol{h}_\theta$ differ only on vertices $x \in D$ with $\boldsymbol{\lambda}_x \in (\theta', \theta]$. Since $\mathcal{E}_2$ holds, there are at most B such vertices, denoted by $D_\theta \subseteq D$. Therefore, we have

$$\begin{aligned} &|\boldsymbol{X}_\theta - \boldsymbol{X}_{\theta'}| \leqslant \sum_{x \in D_\theta} d(x) \leqslant |D_\theta| \Delta \leqslant B\Delta \leqslant \frac{\varepsilon m}{4}, \\ &\Longrightarrow \boldsymbol{X}_\theta \geqslant \boldsymbol{X}_{\theta'} - \frac{\varepsilon m}{4} \geqslant \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right) m - \frac{\varepsilon m}{4} \geqslant \left(\frac{1}{2^{q-1}} - \varepsilon\right) m, \end{aligned} \tag{7.24}$$

as desired.

(Step 4) Extract a discrete reconfiguration sequence. Let $N := |D|$. Order the vertices of D as $v_1, \ldots, v_N$ so that

$$\boldsymbol{\lambda}_{v_1} < \cdots < \boldsymbol{\lambda}_{v_N}. \tag{7.25}$$

For each $0 \leqslant t \leqslant N$, we define an assignment $\boldsymbol{f}^{(t)} \colon V \to \Sigma$ as

$$\boldsymbol{f}^{(t)}(x) := \begin{cases} f_{\mathrm{end}}(x) & \text{if } x \in \{v_1, \ldots, v_t\}, \\ f_{\mathrm{start}}(x) & \text{otherwise.} \end{cases} \tag{7.26}$$

Then, $\vec{\boldsymbol{f}} := (\boldsymbol{f}^{(0)}, \boldsymbol{f}^{(1)}, \ldots, \boldsymbol{f}^{(N)})$ is a reconfiguration sequence from f_{start} to f_{end}, whose length is $N+1$. By applying Eq. (7.22) to each assignment $\boldsymbol{f}^{(t)}$, we obtain

$$\mathsf{val}_G(\vec{\boldsymbol{f}}) \geqslant \frac{1}{2^{q-1}} - \varepsilon, \tag{7.27}$$

as desired. $\square$

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. We can safely assume that n is sufficiently large so that Lemma 7.2 holds. Without loss of generality, we can assume that each vertex of G appears in some hyperedge of G, implying that $qm \geqslant n$. Therefore, for sufficiently large n,

$$m \geqslant \frac{n}{q} > \frac{100 q \log n}{\varepsilon^2}. \tag{7.28}$$

Let $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$ be a reconfiguration sequence from f_{start} to f_{end} consisting only of satisfying assignments. Partition the vertices of G into

$$\begin{aligned} H &:= \left\{ v \in V \;\middle|\; d(v) > \Delta \right\}, \\ L &:= \left\{ v \in V \;\middle|\; d(v) \leqslant \Delta \right\}. \end{aligned} \tag{7.29}$$

Since the sum of all vertex degrees is qm, the number of vertices in H is

$$|H| \leqslant \frac{qm}{\Delta} = \frac{100q^2 \log n}{\varepsilon^2} = \mathrm{O}_{q,\varepsilon}(\log n). \tag{7.30}$$

Consider the following sparsification procedure, which extracts a subsequence of $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$.

Sparsifying $\vec{f} = (f^{(1)}, \ldots, f^{(T)})$

let $I \coloneqq [T]$.
for each index $t_\ell \in I$ **do**
  find the largest index $t_r \in I$ such that $f^{(t_\ell)}|_H = f^{(t_r)}|_H$.
  if $t_\ell + 1 \leqslant t_r - 1$ **then**
    delete $t_\ell + 1, \ldots, t_r - 1$ from I.
return the subsequence $(f^{(t)})_{t \in I}$.

Let $\vec{g} = (g^{(1)}, \ldots, g^{(T')})$ denote the subsequence of $\vec{f}$ obtained by the above procedure. Note that $g^{(1)} = f^{(1)}$ and $g^{(T')} = f^{(T)}$. We first bound the length T' of $\vec{g}$. Let $\alpha \colon H \to \Sigma$ be a partial assignment to H. By construction, α appears in $g^{(1)}|_H, \ldots, g^{(T')}|_H$ at most twice. Otherwise, there exist $t_1, t_2, t_3 \in I$ such that $t_1 < t_2 < t_3$ and $f^{(t_1)}|_H = f^{(t_2)}|_H = f^{(t_3)}|_H$; thus, t_2 would have been removed from I by the sparsification procedure. Moreover, the number of such partial assignments $\alpha \colon H \to \Sigma$ is $|\Sigma^H| = n^{\mathrm{O}_{q,\sigma,\varepsilon}(1)}$ by Eq. (7.30). Therefore, $T' \leqslant 2|\Sigma^H| = n^{\mathrm{O}_{q,\sigma,\varepsilon}(1)}$.

We now construct a polynomial-length reconfiguration sequence. For each $t \in [T'-1]$, either $g^{(t)}|_H = g^{(t+1)}|_H$, or $g^{(t)}$ and $g^{(t+1)}$ differ in at most a single vertex. In the former case, by applying Lemma 7.2, we obtain a reconfiguration sequence $\vec{h}_t$ from $g^{(t)}$ to $g^{(t+1)}$ with value at least $\frac{1}{2^{q-1}} - \varepsilon$, whose length is at most $n+1$. In the latter case, $\vec{h}_t \coloneqq (g^{(t)}, g^{(t+1)})$ is already a valid reconfiguration sequence. Concatenating $\vec{h}_t$ for every $t \in [T'-1]$, we obtain a reconfiguration sequence $\vec{h}$ from $g^{(1)}$ to $g^{(T')}$ whose value is at least $\frac{1}{2^{q-1}} - \varepsilon$. Moreover, the length of $\vec{h}$ is at most $T'(n+1) = n^{\mathrm{O}_{q,\sigma,\varepsilon}(1)}$, as desired. □

8 $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor Approximation for Maxmin q-CSP Reconfiguration on Regular Instances

In this section, we prove Theorem 1.4; i.e., we develop a deterministic $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation algorithm for Maxmin q-CSP Reconfiguration on regular instances. The proof is based on a bucket-label construction together with an explicit exact $2q$-wise independent family.

Theorem 8.1. *For an integer $q \geqslant 2$ and a real $\varepsilon > 0$, let G be a satisfiable n-vertex regular q-CSP instance and f_{start} and f_{end} be a pair of its satisfying assignments. If n is sufficiently large, then there exists a polynomial-length reconfiguration sequence $\vec{f}$ from f_{start} to f_{end} such that*

$$\mathsf{val}_G(\vec{f}) \geqslant \frac{1}{2^{q-1}} - \varepsilon. \tag{8.1}$$

Moreover, such $\vec{f}$ can be found in deterministic polynomial time. In particular, there exists a deterministic $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation algorithm for Maxmin q-CSP Reconfiguration *on regular q-CSP instances.*

The idea is as follows. If we choose a uniformly random ordering of the vertices and change them one by one from f_{start} to f_{end}, then with positive probability every intermediate assignment satisfies at least about a 2^{1-q} fraction of the constraints. For the deterministic algorithm, instead of using a fully random ordering, we partition the vertices into $L = \Theta(q/\varepsilon)$ random groups and then change the groups in order. Thus we only need to analyze the assignments at the $L+1$ group boundaries and the sizes of the L groups. These quantities are controlled by first and second moments of random variables depending on at most $2q$ labels, so an explicit exact $2q$-wise independent label family is enough for the analysis, and we can deterministically search over that family.

The next lemma gives the exact k-wise independent label family that we use. It follows immediately from [Vad12, Corollary 3.34]; for the classical low-degree polynomial construction underlying this result, see also Joffe [Jof74].

Lemma 8.2 (Exact k-wise independent labels [Vad12, Corollary 3.34])**.** *Let $k \geqslant 1$, and let L be a power of two. Let M be the smallest power of two such that $M \geqslant \max\{n, L\}$. Then there exists an explicit family $\mathcal{H}$ of functions $\ell\colon [n] \to [L]$ such that*

1. *$|\mathcal{H}| = M^k$, and*
2. *if ℓ is chosen uniformly from $\mathcal{H}$, then for every $t \leqslant k$ and every distinct vertices $v_1, \ldots, v_t \in [n]$, the tuple*

$$\big(\ell(v_1), \ldots, \ell(v_t)\big) \tag{8.2}$$

is uniformly distributed over $[L]^t$.

Proof Sketch. This follows immediately from [Vad12, Corollary 3.34]. Take the explicit family of k-wise independent functions from $[M]$ to $[M]$ given there; it has size M^k. Restrict the domain to $[n]$ and compose the output with any fixed balanced map from $[M]$ to $[L]$. Since L divides M, every label in $[L]$ has exactly M/L preimages, so the resulting labels are uniform on $[L]$, and every t-tuple of distinct labels remains uniform over $[L]^t$ for $t \leqslant k$. □

Proof of Theorem 8.1. If $\varepsilon \geqslant 2^{1-q}$, then the inequality

$$\mathsf{val}_G(\vec{f}) \geqslant \frac{1}{2^{q-1}} - \varepsilon \tag{8.3}$$

is trivial for every reconfiguration sequence because $\mathsf{val}_G(\vec{f}) \geqslant 0$. Thus we may assume that $0 < \varepsilon < 2^{1-q}$. If $\Delta = 0$, then every assignment satisfies all constraints and the theorem is immediate. Hence we may further assume that $\Delta \geqslant 1$.

Let $(G, f_{\text{start}}, f_{\text{end}})$ be an instance of MAXMIN q-CSP RECONFIGURATION, where G is a satisfiable Δ-regular q-CSP instance on n vertices and m constraints, and f_{start} and f_{end} are satisfying assignments for G. We identify $V(G)$ with $[n]$. Since G is q-ary and Δ-regular,

$$m = \frac{n\Delta}{q}. \tag{8.4}$$

Set L to be the smallest power of two such that $L \geqslant 4q/\varepsilon$. Let M be the smallest power of two such that $M \geqslant \max\{n, L\}$, and apply Lemma 8.2 with $k = 2q$. Thus we obtain an explicit family $\mathcal{H}$ of functions $\ell\colon [n] \to [L]$ such that

$$|\mathcal{H}| = M^{2q} \leqslant (2\max\{n, L\})^{2q} = n^{\mathrm{O}_{q,\varepsilon}(1)}, \tag{8.5}$$

and a uniformly random $\ell \in \mathcal{H}$ is exact $2q$-wise independent and uniform on $[L]$.

Fix $\ell \in \mathcal{H}$. For each $i \in [L]$, define the ith bucket by

$$B_i(\ell) \coloneqq \{v \in [n] \mid \ell(v) = i\}, \tag{8.6}$$

and for each $0 \leqslant j \leqslant L$, define

$$A_j(\ell) \coloneqq \bigcup_{1 \leqslant i \leqslant j} B_i(\ell), \tag{8.7}$$

where $A_0(\ell) \coloneqq \emptyset$. Let $\prec_\ell$ be the total order on $[n]$ obtained by sorting vertices first by increasing label $\ell(v)$ and then by their natural order in $[n]$. Let

$$\vec{f}_\ell = \left(f_\ell^{(0)}, \ldots, f_\ell^{(n)}\right) \tag{8.8}$$

be the reconfiguration sequence that changes vertices from f_{start} to f_{end} in the order $\prec_\ell$. For each $0 \leqslant j \leqslant L$, let

$$t_j(\ell) \coloneqq |A_j(\ell)|. \tag{8.9}$$

Then $f_\ell^{(t_j(\ell))}$ agrees with f_{end} on $A_j(\ell)$ and with f_{start} on $[n] \setminus A_j(\ell)$.

For $0 \leqslant j \leqslant L$, define

$$Y_j(\ell) \coloneqq \sum_{e \in E(G)} \left[\!\!\left[e \subseteq A_j(\ell) \text{ or } e \cap A_j(\ell) = \emptyset \right]\!\!\right], \tag{8.10}$$

and for $i \in [L]$, define

$$Z_i(\ell) \coloneqq |B_i(\ell)|. \tag{8.11}$$

For convenience, let $Z_0(\ell) \coloneqq 0$. Thus $Y_j(\ell)$ counts the constraints all of whose vertices lie on one side of the jth checkpoint, and $Z_i(\ell)$ is the size of the ith bucket. We call ℓ ***good*** if, for every $0 \leqslant i \leqslant L$,

$$Y_i(\ell) \geqslant \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right) m \qquad \text{and} \qquad Z_i(\ell) \leqslant \frac{2n}{L}. \tag{8.12}$$

We show that a uniformly random $\ell \in \mathcal{H}$ is good with positive probability.

Checkpoint bounds. Fix $0 \leqslant j \leqslant L$. For each constraint $e \in E(G)$, define

$$I_{e,j}(\ell) := \llbracket e \subseteq A_j(\ell) \text{ or } e \cap A_j(\ell) = \emptyset \rrbracket. \tag{8.13}$$

Then $Y_j = \sum_{e \in E(G)} I_{e,j}$. Because ℓ is exact $2q$-wise independent and uniform on $[L]$, for every fixed constraint e the labels on the q vertices of e are fully independent and uniform. Hence

$$\mathbf{E}[I_{e,j}] = \left(\frac{j}{L}\right)^q + \left(1 - \frac{j}{L}\right)^q, \tag{8.14}$$

and therefore

$$\mathbf{E}[Y_j] = m\left[\left(\frac{j}{L}\right)^q + \left(1 - \frac{j}{L}\right)^q\right] \geqslant \frac{m}{2^{q-1}}, \tag{8.15}$$

because the function $p \mapsto p^q + (1-p)^q$ is convex on $[0,1]$ and is minimized at $p = 1/2$.

We next bound the variance of Y_j. If two distinct constraints $e, e' \in E(G)$ are disjoint, then $I_{e,j}$ and $I_{e',j}$ are independent, because they depend on the labels on at most $2q$ vertices and ℓ is exact $2q$-wise independent. Thus

$$\begin{aligned}
\operatorname{Var}(Y_j) &= \sum_{e \in E(G)} \operatorname{Var}(I_{e,j}) + \sum_{\substack{e,e' \in E(G) \\ e \neq e'}} \operatorname{Cov}(I_{e,j}, I_{e',j}) \\
&\leqslant \sum_{e \in E(G)} \operatorname{Var}(I_{e,j}) + \sum_{\substack{e,e' \in E(G) \\ e \neq e',\ e \cap e' \neq \emptyset}} \left|\operatorname{Cov}(I_{e,j}, I_{e',j})\right| \\
&\leqslant m + mq(\Delta - 1).
\end{aligned} \tag{8.16}$$

Indeed, $\operatorname{Var}(I_{e,j}) \leqslant 1$ for every e, and the number of ordered pairs (e,e') of distinct intersecting constraints is at most $mq(\Delta - 1)$: for each constraint e and each of its q vertices, there are at most $\Delta - 1$ other constraints containing that vertex. Using $m = n\Delta/q$ and $\Delta \geqslant 1$, we obtain

$$mq(\Delta - 1) \leqslant mq\Delta = \frac{q^2 m^2}{n} \qquad \text{and} \qquad m \geqslant \frac{n}{q}. \tag{8.17}$$

Hence Chebyshev's inequality gives

$$\begin{aligned}
\Pr_{\ell \in \mathcal{H}}\left[Y_j(\ell) < \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right)m\right] &\leqslant \Pr_{\ell \in \mathcal{H}}\left[Y_j(\ell) < \mathbf{E}[Y_j] - \frac{\varepsilon}{2}m\right] \\
&\leqslant \frac{4\operatorname{Var}(Y_j)}{\varepsilon^2 m^2} \\
&\leqslant \frac{4}{\varepsilon^2 m} + \frac{4q^2}{\varepsilon^2 n} \\
&\leqslant \frac{4q(1+q)}{\varepsilon^2 n}.
\end{aligned} \tag{8.18}$$

Bucket-size bounds. Fix $i \in [L]$. For each vertex $v \in [n]$, let

$$J_{v,i}(\ell) := \llbracket \ell(v) = i \rrbracket. \tag{8.19}$$

Then $Z_i = \sum_{v \in [n]} J_{v,i}$. Since each label is uniform on $[L]$ and any two vertex labels are independent,

$$\mathbf{E}[Z_i] = \frac{n}{L} \qquad \text{and} \qquad \mathrm{Var}(Z_i) = \sum_{v \in [n]} \mathrm{Var}(J_{v,i}) \leqslant \frac{n}{L}. \tag{8.20}$$

Another application of Chebyshev's inequality yields

$$\Pr_{\ell \in \mathcal{H}}\left[Z_i(\ell) > \frac{2n}{L}\right] \leqslant \Pr_{\ell \in \mathcal{H}}\left[\left|Z_i(\ell) - \frac{n}{L}\right| \geqslant \frac{n}{L}\right] \leqslant \frac{L}{n}. \tag{8.21}$$

By the union bound,

$$\Pr_{\ell \in \mathcal{H}}\left[\exists i \in \{0,1,\ldots,L\} : \left(Y_i(\ell) < \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right)m \text{ or } Z_i(\ell) > \frac{2n}{L}\right)\right] \leqslant \frac{4(L+1)q(1+q)}{\varepsilon^2 n} + \frac{L^2}{n}. \tag{8.22}$$

Since $L = \mathrm{O}_{q,\varepsilon}(1)$, this upper bound is smaller than 1 for all sufficiently large n. Therefore, for all sufficiently large n, there exists a good labeling $\ell^\star \in \mathcal{H}$. Because $\mathcal{H}$ is explicit and has polynomial size, we can find such an $\ell^\star$ deterministically by enumerating all labelings in $\mathcal{H}$ and checking the above conditions. For a fixed labeling ℓ, all values $Z_i(\ell)$ and $Y_j(\ell)$ can be computed in $\mathrm{O}\big(L(m+n)\big)$ time, so this search runs in polynomial time.

It remains to show that the reconfiguration sequence $\vec{f}_{\ell^\star}$ has value at least $\frac{1}{2^{q-1}} - \varepsilon$. Write $t_j := t_j(\ell^\star)$ for $0 \leqslant j \leqslant L$. Fix $j \in \{0,1,\ldots,L-1\}$. At time t_j, the assignment $f_{\ell^\star}^{(t_j)}$ agrees with f_{end} on $A_j(\ell^\star)$ and with f_{start} on $[n] \setminus A_j(\ell^\star)$. Hence every constraint counted by $Y_j(\ell^\star)$ is certainly satisfied by $f_{\ell^\star}^{(t_j)}$, because all its vertices lie entirely on one side of the cut and both f_{start} and f_{end} satisfy that constraint. Now let t be any time with $t_j \leqslant t \leqslant t_{j+1}$. The assignments $f_{\ell^\star}^{(t)}$ and $f_{\ell^\star}^{(t_j)}$ differ only on vertices of the bucket $B_{j+1}(\ell^\star)$. Thus only constraints incident to $B_{j+1}(\ell^\star)$ can change their satisfaction status, and there are at most $Z_{j+1}(\ell^\star)\Delta$ such constraints. Consequently,

$$\begin{aligned}
\#\{\text{constraints satisfied by } f_{\ell^\star}^{(t)}\} &\geqslant Y_j(\ell^\star) - Z_{j+1}(\ell^\star)\Delta \\
&\geqslant \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right)m - \frac{2n\Delta}{L} \\
&= \left(\frac{1}{2^{q-1}} - \frac{\varepsilon}{2}\right)m - \frac{2q}{L}m \\
&\geqslant \left(\frac{1}{2^{q-1}} - \varepsilon\right)m,
\end{aligned} \tag{8.23}$$

where the last inequality uses $L \geqslant 4q/\varepsilon$. Since this holds for every block and every intermediate time,

$$\mathsf{val}_G(\vec{f}_{\ell^\star}) \geqslant \frac{1}{2^{q-1}} - \varepsilon. \tag{8.24}$$

This proves the theorem for all sufficiently large n.

Finally, fix q and ε, and let $n_0 = n_0(q, \varepsilon)$ be large enough so that the argument above works for every $n \geqslant n_0$. For instances with $n < n_0$, we solve the problem exactly. Indeed, the reconfiguration graph on all assignments in Σ^V has $|\Sigma|^n = |\Sigma|^{O_{q,\varepsilon}(1)}$ vertices, which is polynomial in the input size because n is bounded by the constant n_0. We can therefore construct this graph explicitly and compute, by a standard bottleneck-path algorithm, a reconfiguration sequence from f_{start} to f_{end} of maximum value. Combining this exact algorithm for $n < n_0$ with the algorithm above for $n \geqslant n_0$, we obtain a deterministic polynomial-time $\left(\frac{1}{2^{q-1}} - \varepsilon\right)$-factor approximation algorithm for MAXMIN q-CSP RECONFIGURATION on regular q-CSP instances. $\square$

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The authors used ChatGPT 5.5 Pro in the course of developing proofs of Theorems 7.1 and 8.1. Before using ChatGPT, the authors proved Lemma 7.2 separately. The authors then used ChatGPT to generate initial proof drafts for Theorems 7.1 and 8.1. The drafts were subsequently rewritten and refined by the authors. The authors verified the correctness and originality of all content.

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A Proof of Theorem 5.6

In this section, we prove Theorem 5.6. It suffices to consider the case of $q = 2$.

Theorem A.1. *Let $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ be a k-set function and $n := |V|$. Let ℓ and m be positive integers with $1 \leqslant m \leqslant \ell \leqslant k$. If k and n are sufficiently large, the following holds. Suppose that $\mathcal{T}_2(\ell,m)$ accepts F with probability $p = \omega\left(\frac{\ell}{k}\right)$. Then, there exists a function $f\colon V \to \Sigma$ such that*

$$\Pr_{(\boldsymbol{X},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}}\left[F(\boldsymbol{X};\boldsymbol{r}) \underset{\delta_{k,\ell,m}}{\approx} f^k(\boldsymbol{X})\right] \geqslant \Omega\left(p^6\right), \tag{A.1}$$

where $\delta_{k,\ell,m}$ is defined as

$$\delta_{k,\ell,m} = \widetilde{\Theta}\left(\max\left\{\frac{\ell}{k}, \frac{m}{\ell}\right\}\right). \tag{A.2}$$

Hereafter, fix a function $F\colon \binom{V}{k} \times \mathscr{R} \to \Sigma^k$ and positive integers ℓ and m with $1 \leqslant m \leqslant \ell \leqslant k$. Define $n := |V|$ and $\sigma := |\Sigma|$. For $v \in V$ and $S \subseteq V$, define

$$\binom{V}{k}_v := \left\{X \in \binom{V}{k} \;\middle|\; v \in X\right\}, \quad \binom{V}{k}_S := \left\{X \in \binom{V}{k} \;\middle|\; S \subseteq X\right\}. \tag{A.3}$$

A.1 Goodness and Excellence

We introduce goodness and excellence from [IKW12].

Definition A.2. Let $I_0 \in \binom{V}{\ell}$, $A_0, A \in \binom{V\setminus I_0}{k-\ell}$, and $r_0, r \in \mathscr{R}$. We say that (A,r) is ***consistent*** with (I_0,A_0,r_0) if $F(I_0\cup A;r)|_{I_0} \underset{m/\ell}{\approx} F(I_0\cup A_0;r_0)|_{I_0}$. We define $\mathsf{Cons}(I_0,A_0,r_0)$ as

$$\begin{aligned}\mathsf{Cons}(I_0,A_0,r_0) &:= \left\{(A,r) \in \binom{V\setminus I_0}{k-\ell} \times \mathscr{R} \;\middle|\; (A,r) \text{ is consistent with } (I_0,A_0,r_0)\right\} \\ &= \left\{(A,r) \in \binom{V\setminus I_0}{k-\ell} \times \mathscr{R} \;\middle|\; F(I_0\cup A;r)|_{I_0} \underset{m/\ell}{\approx} F(I_0\cup A_0;r_0)|_{I_0}\right\}.\end{aligned} \tag{A.4}$$

Definition A.3. Let $I_0 \in \binom{V}{\ell}$, $A_0 \in \binom{V\setminus I_0}{k-\ell}$, and $r_0 \in \mathscr{R}$. We say that (I_0, A_0, r_0) is ***ε-good*** if

$$\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\binom{V\setminus I_0}{k-\ell}\times\mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r}) \in \mathsf{Cons}(I_0, A_0, r_0)\right] \geqslant \varepsilon. \tag{A.5}$$

Lemma A.4 (Lemma 3.5 of [IKW12])**.** *Suppose that $\mathcal{T}_2(\ell, m)$ accepts F with probability ε. Then,*

$$\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\(\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[(\boldsymbol{I}_0, \boldsymbol{A}_0, \boldsymbol{r}_0) \textit{ is } \tfrac{\varepsilon}{2}\textit{-good}\right] \geqslant \tfrac{\varepsilon}{2}. \tag{A.6}$$

Proof. By an averaging argument, we derive

$$\begin{aligned}
&\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\(\boldsymbol{A}_0,\boldsymbol{r}_0),(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[F(\boldsymbol{I}_0\cup\boldsymbol{A}_0;\boldsymbol{r}_0)|_{\boldsymbol{I}_0} \underset{m/\ell}{\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0}\right] \geqslant \varepsilon, \\
\Longrightarrow \quad &\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\(\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[\Pr_{(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}\left[F(\boldsymbol{I}_0\cup\boldsymbol{A}_0;\boldsymbol{r}_0)|_{\boldsymbol{I}_0} \underset{m/\ell}{\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0}\right] \geqslant \tfrac{\varepsilon}{2}\right] \geqslant \tfrac{\varepsilon}{2}, \qquad (A.7)\\
\Longrightarrow \quad &\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\(\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[(\boldsymbol{I}_0, \boldsymbol{A}_0, \boldsymbol{r}_0) \text{ is } \tfrac{\varepsilon}{2}\text{-good}\right] \geqslant \tfrac{\varepsilon}{2},
\end{aligned}$$

as desired. ∎

Definition A.5. Let $I_0 \in \binom{V}{\ell}$, $A_0 \in \binom{V\setminus I_0}{k-\ell}$, and $r_0 \in \mathscr{R}$. We say that (I_0, A_0, r_0) is $(\alpha,\gamma,\varepsilon)$***-excellent*** if (I_0, A_0, r_0) is ε-good and

$$\Pr_{\substack{\boldsymbol{B}\in\binom{V\setminus I_0}{\ell}\\(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V\setminus I_0}{k-\ell}_{\boldsymbol{B}}\times\mathscr{R}\\(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V\setminus I_0}{k-\ell}_{\boldsymbol{B}}\times\mathscr{R}}}\left[\begin{matrix}(\boldsymbol{A}_1,\boldsymbol{r}_1) \in \mathsf{Cons}(I_0, A_0, r_0)\\(\boldsymbol{A}_2,\boldsymbol{r}_2) \in \mathsf{Cons}(I_0, A_0, r_0)\end{matrix} \text{ and } F(I_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\alpha}{\not\approx} F(I_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}}\right] \leqslant \gamma. \tag{A.8}$$

Lemma A.6 (Lemma 3.6 of [IKW12])**.** *Let $\alpha \geqslant \omega\left(\frac{m}{\ell}\right)$. Then,*

$$\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\(\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[(\boldsymbol{I}_0, \boldsymbol{A}_0, \boldsymbol{r}_0) \textit{ is } \tfrac{\varepsilon}{2}\textit{-good but not } \left(\alpha,\gamma,\tfrac{\varepsilon}{2}\right)\textit{-excellent}\right] < \frac{\exp\left(-\Omega(\alpha\ell)\right)}{\gamma}. \tag{A.9}$$

Proof. We will bound the following probability:

$$\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\(\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}\\\boldsymbol{B}\in\binom{V\setminus \boldsymbol{I}_0}{\ell}\\(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}_{\boldsymbol{B}}\times\mathscr{R}\\(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}_{\boldsymbol{B}}\times\mathscr{R}}}\left[\begin{matrix}(\boldsymbol{A}_1,\boldsymbol{r}_1) \in \mathsf{Cons}(\boldsymbol{I}_0, \boldsymbol{A}_0, \boldsymbol{r}_0)\\(\boldsymbol{A}_2,\boldsymbol{r}_2) \in \mathsf{Cons}(\boldsymbol{I}_0, \boldsymbol{A}_0, \boldsymbol{r}_0)\end{matrix} \text{ and } F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\alpha}{\not\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}}\right]. \tag{A.10}$$

Observing that

$$
\begin{aligned}
&\big((\boldsymbol{A}_1,\boldsymbol{r}_1)\in \mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0) \text{ and } (\boldsymbol{A}_2,\boldsymbol{r}_2)\in \mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\big)\\
&\quad\implies F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0} \underset{2m/\ell}{\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0},
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
&F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\alpha}{\not\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}}\\
&\quad\implies F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0\cup\boldsymbol{B}} \underset{\alpha/2}{\not\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0\cup\boldsymbol{B}},
\end{aligned}
\tag{A.12}
$$

we derive that Eq. (A.10) is at least

$$
\Pr_{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0,\boldsymbol{B},\boldsymbol{A}_1,\boldsymbol{r}_1,\boldsymbol{A}_2,\boldsymbol{r}_2}\left[\begin{matrix} F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0} \underset{2m/\ell}{\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0} \\ F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0\cup\boldsymbol{B}} \underset{\alpha/2}{\not\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0\cup\boldsymbol{B}}\end{matrix}\right].
\tag{A.13}
$$

Conditioning $\boldsymbol{I}_0\cup\boldsymbol{B}$ on the event that $F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0\cup\boldsymbol{B}} \underset{\alpha/2}{\not\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0\cup\boldsymbol{B}}$, let

$$
\boldsymbol{Z} := \ell\cdot \mathrm{dist}\big(F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0}, F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0}\big) = \sum_{i\in\boldsymbol{I}_0}\big[\!\big[F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_i \neq F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_i\big]\!\big].
\tag{A.14}
$$

Since $\boldsymbol{I}_0$ is uniformly distributed over $\binom{\boldsymbol{I}_0\cup\boldsymbol{B}}{\ell}$, $\boldsymbol{Z}$ follows a hypergeometric distribution where the population size is 2ℓ, the number of successes is $2\ell\cdot\mathrm{dist}\big(F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0\cup\boldsymbol{B}}, F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0\cup\boldsymbol{B}}\big)$, and the sample size is ℓ. Since $\alpha \geqslant \omega\big(\frac{m}{\ell}\big)$ by assumption and $\mathbf{E}[\boldsymbol{Z}] \geqslant \frac{\alpha}{2}\ell$, we have

$$
\frac{2m}{\ell}\ell \leqslant \frac{\alpha}{200}\ell \leqslant \frac{\mathbf{E}[\boldsymbol{Z}]}{100}.
\tag{A.15}
$$

By the Chernoff bound, we have

$$
\begin{aligned}
&\Pr_{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0,\boldsymbol{B},\boldsymbol{A}_1,\boldsymbol{r}_1,\boldsymbol{A}_2,\boldsymbol{r}_2}\left[\begin{matrix} F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0} \underset{2m/\ell}{\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0} \\ F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0\cup\boldsymbol{B}} \underset{\alpha/2}{\not\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0\cup\boldsymbol{B}}\end{matrix}\right]\\
&\leqslant \Pr_{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0,\boldsymbol{B},\boldsymbol{A}_1,\boldsymbol{r}_1,\boldsymbol{A}_2,\boldsymbol{r}_2}\left[F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0} \underset{2m/\ell}{\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0} \;\middle|\; F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{I}_0\cup\boldsymbol{B}} \underset{\alpha/2}{\not\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{I}_0\cup\boldsymbol{B}}\right]\\
&\leqslant \Pr\left[\boldsymbol{Z}\leqslant \tfrac{2m}{\ell}\ell\right]\\
&\leqslant \Pr\left[\boldsymbol{Z}\leqslant \tfrac{1}{100}\mathbf{E}[\boldsymbol{Z}]\right]\\
&< \exp\big(-\Omega(\alpha\ell)\big).
\end{aligned}
\tag{A.16}
$$

By an averaging argument, we have

$$
\begin{aligned}
&\Pr_{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0}\left[\Pr_{\boldsymbol{B},\boldsymbol{A}_1,\boldsymbol{r}_1,\boldsymbol{A}_2,\boldsymbol{r}_2}\left[\begin{matrix}(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\end{matrix} \text{ and } F(\boldsymbol{I}_0\cup\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\alpha}{\not\approx} F(\boldsymbol{I}_0\cup\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}}\right]\geqslant\gamma\right] < \frac{\exp\big(-\Omega(\alpha\ell)\big)}{\gamma},\\
&\implies \Pr_{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0}\left[(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\text{ is }\tfrac{\varepsilon}{2}\text{-good but not }\big(\alpha,\gamma,\tfrac{\varepsilon}{2}\big)\text{-excellent}\right] < \frac{\exp\big(-\Omega(\alpha\ell)\big)}{\gamma}.
\end{aligned}
\tag{A.17}
$$

as desired. □

As an immediate corollary of Lemmas A.4 and A.6, we obtain the following.

Corollary A.7 (Corollary 3.7 of [IKW12]). *Suppose that $\mathcal{T}_2(\ell,m)$ accepts f with probability ε. Then,*

$$\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\(\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0) \text{ is } \left(\alpha,\gamma,\tfrac{\varepsilon}{2}\right)\text{-excellent} \;\middle|\; (\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0) \text{ is } \tfrac{\varepsilon}{2}\text{-good}\right] \geqslant 1-\frac{\exp\big(-\Omega(\alpha\ell)\big)}{\gamma\varepsilon}. \tag{A.18}$$

Proof. By Lemmas A.4 and A.6, we have

$$\begin{aligned}
&\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\(\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus \boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0) \text{ is not } \left(\alpha,\gamma,\tfrac{\varepsilon}{2}\right)\text{-excellent} \;\middle|\; (\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0) \text{ is } \tfrac{\varepsilon}{2}\text{-good}\right]\\
&= \frac{\Pr_{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0}\left[(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0) \text{ is } \tfrac{\varepsilon}{2}\text{-good but not } \left(\alpha,\gamma,\tfrac{\varepsilon}{2}\right)\text{-excellent}\right]}{\Pr_{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0}\left[(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0) \text{ is } \tfrac{\varepsilon}{2}\text{-good}\right]} \qquad (A.19)\\
&< \frac{\exp\big(-\Omega(\alpha\ell)\big)}{\gamma}\cdot\frac{2}{\varepsilon} = \frac{\exp\big(-\Omega(\alpha\ell)\big)}{\gamma\varepsilon},
\end{aligned}$$

as desired. □

A.2 Excellence Implies Local Agreement

We show that we can perform unique decoding on $\mathsf{Cons}(I_0,A_0,r_0)$. We first define the plurality function.

Definition A.8. Let $I_0\in\binom{V}{\ell}$, $A_0\in\binom{V\setminus I_0}{k-\ell}$, and $r_0\in\mathscr{R}$. The ***plurality function*** of (I_0,A_0,r_0) is defined as a function $f_{I_0,A_0,r_0}\colon V\to\Sigma$ such that for each $v\in V$,

$$f_{I_0,A_0,r_0}(v) := \operatorname*{plurality}_{(A,r)\in\mathsf{Cons}(I_0,A_0,r_0):A\ni v}\left\{F(I_0\cup A;r)|_v\right\}. \tag{A.20}$$

Lemma A.9 (Lemma 3.8 of [IKW12]). *Suppose that $\mathcal{T}_2(\ell,m)$ accepts F with probability $\varepsilon\geqslant\omega\left(\frac{\ell}{k}\right)$ and that (I_0,A_0,r_0) is $\left(\alpha,\gamma,\frac{\varepsilon}{2}\right)$-excellent, where*

$$\alpha=\omega\left(\max\left\{\frac{\ell}{k},\frac{m}{\ell}\right\}\log\varepsilon^{-1}\right) \text{ and } \gamma=o\big(\varepsilon^3\big). \tag{A.21}$$

Then,

$$\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\mathsf{Cons}(I_0,A_0,r_0)}\left[F(I_0\cup\boldsymbol{A};\boldsymbol{r})\not\approx_{2\beta} f^k_{I_0,A_0,r_0}(I_0\cup\boldsymbol{A})\right]<\delta, \text{ where } \beta:=16\sigma\alpha \text{ and } \delta:=\frac{256\sigma\gamma}{\varepsilon^2}. \tag{A.22}$$

Definition A.10. Let $F\colon \mathcal{D} \to \Sigma^k$ be a function, where $\mathcal{D}$ is a subset of $\binom{V}{k} \times \mathcal{R}$. We say that f is (α,γ)-***excellent*** if

$$\Pr_{\substack{\boldsymbol{B}\in\binom{V}{\ell} \\ (\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V}{k}_{\boldsymbol{B}}\times\mathcal{R} \\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V}{k}_{\boldsymbol{B}}\times\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\mathcal{D} \\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D}\end{matrix} \text{ and } F(\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\alpha}{\not\approx} F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}}\right] \leqslant \gamma. \tag{A.23}$$

Proposition A.11 (Lemma 3.10 of [IKW12])**.** *Let $F\colon \mathcal{D} \to \Sigma^k$ be a function, where $\mathcal{D}$ is a subset of $\binom{V}{k} \times \mathcal{R}$. Suppose that $\frac{|\mathcal{D}|}{\left|\binom{V}{k}\times\mathcal{R}\right|} = \varepsilon \geqslant \omega\left(\frac{\ell}{k}\right)$ and F is (α,γ)-excellent, where*

$$\alpha = \omega\left(\max\left\{\frac{\ell}{k}, \frac{m}{\ell}\right\}\log\varepsilon^{-1}\right) \text{ and } \gamma = o\left(\varepsilon^3\right). \tag{A.24}$$

Let $f\colon V \to \Sigma$ be a function such that

$$f(v) \coloneqq \underset{(A,r)\in\mathcal{D}:A\ni v}{\mathsf{plurality}} \left\{F(A;r)|_v\right\}. \tag{A.25}$$

Then,

$$\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}}\left[F(\boldsymbol{A};\boldsymbol{r}) \underset{\beta}{\not\approx} f^k(\boldsymbol{A})\right] \leqslant \delta, \text{ where } \beta \coloneqq 16\sigma\alpha \text{ and } \delta \coloneqq \frac{64\sigma\gamma}{\varepsilon^2}. \tag{A.26}$$

By Proposition A.11, we can prove Lemma A.9.

Proof of Lemma A.9. By applying Proposition A.11 to $f|_{\mathcal{D}}$ with $\mathcal{D} = \mathsf{Cons}(I_0, A_0, r_0)$, $\frac{|\mathcal{D}|}{\left|\binom{V\setminus I_0}{k-\ell}\times\mathcal{R}\right|} \geqslant \frac{\varepsilon}{2}$, and $k \leftarrow k-\ell$. we obtain

$$F(I_0\cup\boldsymbol{A};\boldsymbol{r})|_{\boldsymbol{A}} \underset{\beta}{\approx} f^{k-\ell}(\boldsymbol{A}) \implies \operatorname{dist}\left(F(I_0\cup\boldsymbol{A};\boldsymbol{r}), f^k(I_0\cup\boldsymbol{A})\right) \leqslant \tfrac{k-\ell}{k}\cdot\beta + \tfrac{\ell}{k}\cdot 1 \leqslant \left(1+o(1)\right)\beta \leqslant 2\beta, \tag{A.27}$$

as desired. ☐

A.3 Proof of Proposition A.11

In this subsection, we prove Proposition A.11.

For $\mathcal{D} \subset \binom{V}{k} \times \mathcal{R}$, $v \in V$, and $S \subseteq V$, define

$$\mathcal{D}_v \coloneqq \left\{(A,r)\in\mathcal{D} \;\middle|\; v\in A\right\}, \quad \mathcal{D}_S \coloneqq \left\{(A,r)\in\mathcal{D} \;\middle|\; S\subseteq A\right\}. \tag{A.28}$$

Definition A.12. For $(A,r) \in \binom{V}{k} \times \mathcal{R}$, define the event $\mathcal{E}_{\text{A.29}}(A,r)$ as

$$(A,r)\in\mathcal{D} \text{ and } F(A;r) \underset{\beta}{\not\approx} f^k(A). \tag{A.29}$$

For $B \in \binom{V}{\ell}$, define the event $\mathcal{E}_{\text{A.30}}(B)$ as

$$\begin{aligned} &\frac{|\mathcal{D}_B|}{\left|\binom{V}{k}_B \times \mathscr{R}\right|} \geqslant \frac{\varepsilon}{2} \left(\text{i.e., } \Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \binom{V}{k}_B \times \mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r}) \in \mathcal{D}\right] \geqslant \frac{\varepsilon}{2}\right), \text{ and} \\ &\Pr_{\boldsymbol{v} \in B}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \mathcal{D}_B}\left[F(\boldsymbol{A};\boldsymbol{r})|_{\boldsymbol{v}} = f(\boldsymbol{v})\right] \geqslant \tfrac{1}{2\sigma}\right] \geqslant 1 - \mathrm{O}\left(\tfrac{\log \varepsilon^{-1}}{k/\ell}\right). \end{aligned} \tag{A.30}$$

Remark A.13. *The event $\mathcal{E}_{\text{A.30}}(B)$ can be thought of as a "B-restricted" version of the following assumptions:*

$$\frac{|\mathcal{D}|}{\left|\binom{V}{k} \times \mathscr{R}\right|} \geqslant \varepsilon \textit{ and } \Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \mathcal{D}_v}\left[F(\boldsymbol{A})|_v = f(v)\right] \geqslant \tfrac{1}{\sigma}. \tag{A.31}$$

(Step 1) We first show that $\mathcal{E}_{\text{A.30}}(\boldsymbol{B})$ holds with probability $\frac{1}{2}$.

Lemma A.14 (Proof of Proposition A.11: Part 1)**.**

$$\Pr_{\boldsymbol{B} \in \binom{V}{\ell}}\left[\mathcal{E}_{\text{A.30}}(\boldsymbol{B})\right] \geqslant \frac{1}{2}. \tag{A.32}$$

Claim A.15 (Claim 3.11 of [IKW12])**.**

$$\Pr_{\boldsymbol{v} \in V}\left[\frac{|\mathcal{D}_{\boldsymbol{v}}|}{\left|\binom{V}{k}_{\boldsymbol{v}}\right|} \geqslant \frac{\varepsilon}{2}\right] = \Pr_{\boldsymbol{v} \in V}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \binom{V}{k}_{\boldsymbol{v}} \times \mathscr{R}}\left[\boldsymbol{A} \in \mathcal{D}\right] \geqslant \tfrac{\varepsilon}{2}\right] \geqslant 1 - \mathrm{O}\left(\tfrac{\log \varepsilon^{-1}}{k}\right). \tag{A.33}$$

Proof. By [IKW12, Lemma 2.6], we have

$$\Pr_{\boldsymbol{v} \in V}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \binom{V}{k}_{\boldsymbol{v}} \times \mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r}) \in \mathcal{D}\right] \geqslant \tfrac{\varepsilon}{2}\right] \geqslant 1 - \mathrm{O}\left(\tfrac{\log \varepsilon^{-1}}{k}\right), \tag{A.34}$$

as desired. □

Claim A.16 (Claim 3.12 of [IKW12])**.** *Suppose that $v \in V$ satisfies $\frac{|\mathcal{D}_v|}{\left|\binom{V}{k}_v \times \mathscr{R}\right|} \geqslant \frac{\varepsilon}{2}$. Then,*

$$\Pr_{\boldsymbol{B} \in \binom{V}{\ell}_v}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \mathcal{D}_{\boldsymbol{B}}}\left[F(\boldsymbol{A};\boldsymbol{r})|_v = f(v)\right] \geqslant \tfrac{1}{2\sigma}\right] \geqslant 1 - \mathrm{O}\left(\tfrac{\log \varepsilon^{-1}}{k/\ell}\right). \tag{A.35}$$

Proof. Since $f(v)$ is determined based on the plurality, we have

$$\Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \mathcal{D}_v}\left[F(\boldsymbol{A})|_v = f(v)\right] \geqslant \tfrac{1}{\sigma}. \tag{A.36}$$

Define

$$\mu_1 := \Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \binom{V}{k}_v \times \mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r}) \in \mathcal{D}\right], \tag{A.37}$$

$$\mu_2 := \Pr_{(\boldsymbol{A},\boldsymbol{r}) \in \binom{V}{k}_v \times \mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r}) \in \mathcal{D} \text{ and } F(\boldsymbol{A};\boldsymbol{r})|_v = f(v)\right]. \tag{A.38}$$

Note that $\frac{\mu_2}{\mu_1} \geqslant \frac{1}{\sigma}$. By applying [IKW12, Corollary 2.7], we have

$$\Pr_{\boldsymbol{B}\in\binom{V}{\ell}_v}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}\right]\in\left(0.8\mu_1, 1.2\mu_1\right)\right] \geqslant 1-\mathrm{O}\left(\frac{\log\mu_1^{-1}}{k/\ell}\right), \tag{A.39}$$

$$\Pr_{\boldsymbol{B}\in\binom{V}{\ell}_v}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}\text{ and }F(\boldsymbol{A};\boldsymbol{r})|_v=f(v)\right]\in\left(0.8\mu_2, 1.2\mu_2\right)\right] \geqslant 1-\mathrm{O}\left(\frac{\log\mu_2^{-1}}{k/\ell}\right). \tag{A.40}$$

Note that $\boldsymbol{A}$ and $\boldsymbol{B}$ always contain v. By assumption, $\mu_1 \geqslant \frac{\varepsilon}{2}$ and thus $\mu_2 \geqslant \frac{\varepsilon}{2\sigma}$; i.e., $\mu_1 = \Omega(\varepsilon)$ and $\mu_2 = \Omega(\varepsilon)$. By taking the union bound, with probability at least $1-\mathrm{O}\left(\frac{\log\varepsilon^{-1}}{k/\ell}\right)$ over $\boldsymbol{B}\in\binom{V}{\ell}_v$, we have

$$\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}_{\boldsymbol{B}}}\left[F(\boldsymbol{A};\boldsymbol{r})|_v=f(v)\right] = \frac{\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}\text{ and }F(\boldsymbol{A};\boldsymbol{r})|_v=f(v)\right]}{\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}\right]} \geqslant \frac{0.8\mu_2}{1.2\mu_1} \geqslant \frac{1}{2\sigma}, \tag{A.41}$$

as desired. $\square$

Claim A.17 (Claim 3.13 of [IKW12])**.**

$$\Pr_{\substack{\boldsymbol{v}\in V\\ \boldsymbol{B}\in\binom{V}{\ell}_{\boldsymbol{v}}}}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}_{\boldsymbol{B}}}\left[F(\boldsymbol{A};\boldsymbol{r})|_{\boldsymbol{v}}=f(\boldsymbol{v})\right]\geqslant\frac{1}{2\sigma}\right] \geqslant 1-\mathrm{O}\left(\frac{\log\varepsilon^{-1}}{k/\ell}\right). \tag{A.42}$$

Proof. By applying the union bound to Claims A.15 and A.16, we obtain the desired result. $\square$

Claim A.18 (Claim 3.14 of [IKW12])**.**

$$\Pr_{\boldsymbol{B}\in\binom{V}{\ell}}\left[\Pr_{\boldsymbol{v}\in\boldsymbol{B}}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}_{\boldsymbol{B}}}\left[F(\boldsymbol{A};\boldsymbol{r})|_{\boldsymbol{v}}=f(\boldsymbol{v})\right]\geqslant\frac{1}{2\sigma}\right]\geqslant 1-\mathrm{O}\left(\frac{\log\varepsilon^{-1}}{k/\ell}\right)\right]\geqslant 0.99. \tag{A.43}$$

Proof. By swapping the order of choosing $\boldsymbol{v}$ and $\boldsymbol{B}$ and applying an averaging argument to Claim A.17, we obtain the desired result. $\square$

Claim A.19 (Claim 3.15 of [IKW12])**.**

$$\Pr_{\boldsymbol{B}\in\binom{V}{\ell}}\left[\frac{|\mathcal{D}_{\boldsymbol{B}}|}{\left|\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}\right|}\geqslant\frac{\varepsilon}{2}\right] = \Pr_{\boldsymbol{B}\in\binom{V}{\ell}}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}\right]\geqslant\frac{\varepsilon}{2}\right]\geqslant 1-\mathrm{O}\left(\frac{\log\varepsilon^{-1}}{k/\ell}\right). \tag{A.44}$$

Proof. By applying [IKW12, Corollary 2.7], we have

$$\Pr_{\boldsymbol{B}\in\binom{V}{\ell}}\left[\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}\left[(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}\right]\geqslant\frac{\varepsilon}{2}\right]\geqslant 1-\mathrm{O}\left(\frac{\log\varepsilon^{-1}}{k/\ell}\right), \tag{A.45}$$

as desired. $\square$

We now prove Lemma A.14.

Proof of Lemma A.14. By applying the union bound to Claims A.18 and A.19, we have

$$\Pr_{\boldsymbol{B}\in\binom{V}{\ell}}\left[\mathcal{E}_{\text{A.30}}(\boldsymbol{B})\right] \geqslant 0.99 - \mathrm{O}\left(\frac{\log \varepsilon^{-1}}{k/\ell}\right) \geqslant 0.99 - o(1) \geqslant \frac{1}{2}, \tag{A.46}$$

as desired. □

(Step 2) By applying Lemma A.14, we next show the following.

Lemma A.20 (Proof of Proposition A.11: Part 2)**.**

$$\Pr_{\substack{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}\\ \boldsymbol{B}\in\binom{\boldsymbol{A}}{\ell}}}\left[\mathcal{E}_{\text{A.30}}(\boldsymbol{B}) \text{ and } F(\boldsymbol{A};\boldsymbol{r})|_{\boldsymbol{B}} \underset{\beta/2}{\not\approx} f^{\ell}(\boldsymbol{B}) \;\middle|\; \mathcal{E}_{\text{A.29}}(\boldsymbol{A},\boldsymbol{r})\right] \geqslant \frac{1}{8}. \tag{A.47}$$

Proof. Condition $(\boldsymbol{A},\boldsymbol{r}) \in \binom{V}{k} \times \mathscr{R}$ on the event that $\mathcal{E}_{\text{A.29}}(\boldsymbol{A},\boldsymbol{r})$, which implies $F(\boldsymbol{A};\boldsymbol{r}) \underset{\beta}{\not\approx} f^{k}(\boldsymbol{A})$. By the Chernoff bound, we have

$$\Pr_{\substack{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}\\ \boldsymbol{B}\in\binom{\boldsymbol{A}}{\ell}}}\left[F(\boldsymbol{A};\boldsymbol{r})|_{\boldsymbol{B}} \underset{\beta/2}{\not\approx} f^{\ell}(\boldsymbol{B}) \;\middle|\; \mathcal{E}_{\text{A.29}}(\boldsymbol{A},\boldsymbol{r})\right] \geqslant 1 - \exp\big(-\Omega(\beta\ell)\big). \tag{A.48}$$

By applying [IKW12, Lemmas 2.3 and 2.4] to Lemma A.14, we have

$$\Pr_{\substack{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}\\ \boldsymbol{B}\in\binom{\boldsymbol{A}}{\ell}}}\left[\mathcal{E}_{\text{A.30}}(\boldsymbol{B}) \;\middle|\; \mathcal{E}_{\text{A.29}}(\boldsymbol{A},\boldsymbol{r})\right] \geqslant \frac{1}{6}. \tag{A.49}$$

By the union bound, we have

$$\Pr_{\substack{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}\times\mathscr{R}\\ \boldsymbol{B}\in\binom{\boldsymbol{A}}{\ell}}}\left[\mathcal{E}_{\text{A.30}}(\boldsymbol{B}) \text{ and } F(\boldsymbol{A};\boldsymbol{r})|_{\boldsymbol{B}} \underset{\beta/2}{\not\approx} f^{\ell}(\boldsymbol{B}) \;\middle|\; \mathcal{E}_{\text{A.29}}(\boldsymbol{A},\boldsymbol{r})\right] \geqslant \frac{1}{6} - \exp\big(-\Omega(\beta\ell)\big) \geqslant \frac{1}{6} - o(1) \geqslant \frac{1}{8}, \tag{A.50}$$

where we used the assumption that $\beta = \omega\left(\frac{\log \varepsilon^{-1}}{\ell}\right)$, as desired. □

(Step 3) We then show the following.

Lemma A.21 (Proof of Proposition A.11: Part 3)**.** *Suppose that* $(A_1, r_1) \in \binom{V}{k} \times \mathscr{R}$ *satisfies* $\mathcal{E}_{\text{A.29}}(A_1, r_1)$, $B \in \binom{A_1}{\ell}$ *satisfies* $\mathcal{E}_{\text{A.30}}(B)$, *and* $F(A_1; r_1)|_B \underset{\beta/2}{\not\approx} f^{\ell}(B)$. *Then,*

$$\Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V}{k}_B\times\mathscr{R}}\left[(\boldsymbol{A}_2,\boldsymbol{r}_2) \in \mathcal{D} \text{ and } F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_B \underset{\beta/(16\sigma)}{\not\approx} F(A_1;r_1)|_B\right] > \frac{\varepsilon}{8\sigma}. \tag{A.51}$$

Proof. Define

$$B^\circ := \left\{ v \in B \;\middle|\; F(A_1;r_1)|_v \neq f(v) \text{ and } \Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D}_B}\left[F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_v = f(v)\right] \geqslant \tfrac{1}{2\sigma} \right\}. \tag{A.52}$$

Since $F(A_1;r_1)|_B \underset{\beta/2}{\not\approx} f^\ell(B)$ and $\mathcal{E}_{\text{A.30}}(B)$ holds by assumption, we have

$$\begin{aligned} |B^\circ| &\geqslant |B|\left(\Pr_{\boldsymbol{v}\in B}\left[F(A_1;r_1)|_{\boldsymbol{v}} \neq f(\boldsymbol{v})\right] + \Pr_{\boldsymbol{v}\in B}\left[\Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D}_B}\left[F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{v}} = f(\boldsymbol{v})\right] \geqslant \tfrac{1}{2\sigma}\right] - 1\right) \\ &\geqslant |B|\left(\tfrac{\beta}{2} + 1 - \mathrm{O}\left(\tfrac{\log\varepsilon^{-1}}{k/\ell}\right) - 1\right) \geqslant |B|\tfrac{\beta}{2}\left(1-o(1)\right) > \tfrac{\beta}{4}|B|, \end{aligned} \tag{A.53}$$

where we used the assumption that $\beta = \omega\left(\frac{\log\varepsilon^{-1}}{k/\ell}\right)$. Observe that for every $v \in B^\circ$,

$$\Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D}_B}\left[F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_v = f(v) \neq F(A_1;r_1)|_v\right] \geqslant \tfrac{1}{2\sigma}, \tag{A.54}$$

implying that by an averaging argument,

$$\begin{aligned} &\Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D}_B}\Bigg[\underbrace{\Pr_{\boldsymbol{v}\in B^\circ}\left[F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{v}} \neq F(A_1;r_1)|_{\boldsymbol{v}}\right]}_{=\mathrm{dist}(F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{B^\circ},F(A_1;r_1)|_{B^\circ})} \geqslant \tfrac{1}{4\sigma}\Bigg] \geqslant \tfrac{1}{4\sigma}, \\ \Longrightarrow\; &\Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D}_B}\left[\mathrm{dist}\left(F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_B, F(A_1;r_1)|_B\right) > \tfrac{\beta}{16\sigma}\right] \geqslant \tfrac{1}{4\sigma}, \end{aligned} \tag{A.55}$$

where we used the inequality that

$$\begin{aligned} \mathrm{dist}\left(F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_B, F(A_1;r_1)|_B\right) &\geqslant \tfrac{|B^\circ|}{|B|}\,\mathrm{dist}\left(F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{B^\circ}, F(A_1;r_1)|_{B^\circ}\right) \\ &\underbrace{>}_{\text{Eq. (A.53)}} \tfrac{\beta}{4}\,\mathrm{dist}\left(F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{B^\circ}, F(A_1;r_1)|_{B^\circ}\right). \end{aligned} \tag{A.56}$$

Since $\frac{|\mathcal{D}_B|}{\left|\binom{V}{k}_B \times \mathcal{R}\right|} \geqslant \frac{\varepsilon}{2}$ due to $\mathcal{E}_{\text{A.30}}(B)$, we have

$$\begin{aligned} &\Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V}{k}_B\times\mathcal{R}}\left[(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D} \text{ and } F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_B \underset{\beta/(16\sigma)}{\not\approx} F(A_1,r_1)|_B\right] \\ &= \Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D}_B}\left[F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_B \underset{\beta/(16\sigma)}{\not\approx} F(A_1;r_1)|_B\right] \cdot \Pr_{(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V}{k}_B\times\mathcal{R}}\left[(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D}\right] \geqslant \frac{1}{4\sigma}\cdot\frac{\varepsilon}{2} = \frac{\varepsilon}{8\sigma}, \end{aligned} \tag{A.57}$$

as desired. □

Proof of Proposition A.11. We are now ready to prove Proposition A.11.

Proof of Proposition A.11. Suppose for contradiction that

$$\Pr_{(\boldsymbol{A},\boldsymbol{r})\in\mathcal{D}}\left[F(\boldsymbol{A};\boldsymbol{r}) \underset{\beta}{\not\approx} f^k(\boldsymbol{A})\right] > \delta, \qquad \therefore \Pr_{(\boldsymbol{A},\boldsymbol{r})\in\binom{V}{k}\times\mathcal{R}}\left[\mathcal{E}_{\text{A.29}}(\boldsymbol{A},\boldsymbol{r})\right] > \delta\varepsilon. \tag{A.58}$$

By Lemma A.21, we have

$$\Pr_{\substack{(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V}{k}\times\mathscr{R}\\ \boldsymbol{B}\in\binom{\boldsymbol{A}_1}{\ell}\\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}}\left[\begin{array}{l|l}(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D} & \mathcal{E}_{\text{A.29}}(\boldsymbol{A}_1,\boldsymbol{r}_1)\\ F(\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\beta/(16\sigma)}{\not\approx} F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}} & \mathcal{E}_{\text{A.30}}(\boldsymbol{B}) \text{ and } F(\boldsymbol{A};\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\beta/2}{\not\approx} f^{\ell}(\boldsymbol{B})\end{array}\right] > \frac{\varepsilon}{8\sigma}. \tag{A.59}$$

By Lemma A.20, we have

$$\Pr_{\substack{(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V}{k}\times\mathscr{R}\\ \boldsymbol{B}\in\binom{\boldsymbol{A}_1}{\ell}\\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}}\left[(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D} \text{ and } F(\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\beta/(16\sigma)}{\not\approx} F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}} \;\middle|\; \mathcal{E}_{\text{A.29}}(\boldsymbol{A}_1,\boldsymbol{r}_1)\right] > \frac{\varepsilon}{8\sigma}\cdot\frac{1}{8} = \frac{\varepsilon}{64\sigma}. \tag{A.60}$$

By assumption, we have

$$\begin{aligned}&\Pr_{\substack{(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V}{k}\times\mathscr{R}\\ \boldsymbol{B}\in\binom{\boldsymbol{A}_1}{\ell}\\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}}\left[(\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D} \text{ and } F(\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\beta/(16\sigma)}{\not\approx} F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}} \text{ and } \mathcal{E}_{\text{A.29}}(\boldsymbol{A}_1,\boldsymbol{r}_1)\right] > \frac{\varepsilon}{64\sigma}\cdot\delta\varepsilon,\\ \implies &\Pr_{\substack{\boldsymbol{B}\in\binom{V}{\ell}\\ (\boldsymbol{A}_1,\boldsymbol{r}_1)\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}\\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\binom{V}{k}_{\boldsymbol{B}}\times\mathscr{R}}}\left[(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\mathcal{D} \text{ and } (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathcal{D} \text{ and } F(\boldsymbol{A}_1;\boldsymbol{r}_1)|_{\boldsymbol{B}} \underset{\alpha}{\not\approx} F(\boldsymbol{A}_2;\boldsymbol{r}_2)|_{\boldsymbol{B}}\right] > \gamma,\end{aligned} \tag{A.61}$$

which contradicts to the assumption that F is (α,γ)-excellent, as desired. □

A.4 Proof of Theorem 5.6

Suppose hereafter that

$$\begin{aligned}&\varepsilon = \omega\left(\frac{\ell}{k}\right),\\ &\alpha = \omega\left(\max\left\{\frac{\ell}{k},\frac{m}{\ell}\right\}\log\varepsilon^{-1}\right), \quad && \beta := 16\sigma\alpha,\\ &\gamma = o(\varepsilon^3), && \delta := \frac{256\sigma\gamma}{\varepsilon^2} = o(\varepsilon).\end{aligned} \tag{A.62}$$

Consider the following procedure Sample that generates $(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X})$:

Sample

- sample $\boldsymbol{I}_1\in\binom{V}{\ell}$ and $\boldsymbol{I}_2\in\binom{V}{\ell}$ such that $\boldsymbol{I}_1\cap\boldsymbol{I}_2=\emptyset$.
- sample $\boldsymbol{A}_1\in\binom{V\setminus\boldsymbol{I}_1}{k-\ell}$ and $\boldsymbol{A}_2\in\binom{V\setminus\boldsymbol{I}_2}{k-\ell}$.
- sample $\boldsymbol{X}\in\binom{V}{k}$ such that $\boldsymbol{X}\supset\boldsymbol{I}_1\cup\boldsymbol{I}_2$.

For each $i \in [2]$, define $\boldsymbol{f}_i\colon V \to \Sigma$, which depends on $\boldsymbol{I}_i$ and $\boldsymbol{A}_i$, as follows:

$$\boldsymbol{f}_i(v) := \operatorname*{plurality}_{(A,r)\in\mathsf{Cons}(\boldsymbol{I}_i,\boldsymbol{A}_i)\times\mathcal{R}:A\ni v} \left\{F(\boldsymbol{I}_i \cup A; r)|_v\right\}. \tag{A.63}$$

We prove the following claims about SAMPLE.

Claim A.22 (Claim 3.18 of [IKW12])**.**

$$\Pr_{\substack{(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X})\sim\textsc{Sample}\\ \boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}'_1,\boldsymbol{r}'_2\in\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1) \textit{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\textit{-excellent}\\ (\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2) \textit{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\textit{-excellent}\end{matrix} \textit{ and } \begin{matrix}(\boldsymbol{X}\setminus\boldsymbol{I}_1,\boldsymbol{r}'_1)\in\mathsf{Cons}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1)\\ (\boldsymbol{X}\setminus\boldsymbol{I}_2,\boldsymbol{r}'_2)\in\mathsf{Cons}(\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2)\end{matrix}\right] \geqslant \Omega(\varepsilon^5). \tag{A.64}$$

Proof. Consider the following procedure that generates SAMPLE$'$: $(\boldsymbol{X},\mathcal{P},\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2)$:

SAMPLE$'$

- sample $\boldsymbol{X} \in \binom{V}{k}$ and its k/ℓ-partition $\mathcal{P} = (\boldsymbol{P}_1,\ldots,\boldsymbol{P}_{k/\ell})$ such that $|\boldsymbol{P}_1| = \cdots = |\boldsymbol{P}_{k/\ell}| = \ell$.
- sample $\{\boldsymbol{I}_1,\boldsymbol{I}_2\} \in \binom{\mathcal{P}}{2}$.
- sample $\boldsymbol{A}_1 \in \binom{V\setminus\boldsymbol{I}_1}{k-\ell}$ and $\boldsymbol{A}_2 \in \binom{V\setminus\boldsymbol{I}_2}{k-\ell}$.

Note that SAMPLE and SAMPLE$'$ have the same distribution of $(\boldsymbol{X},\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2)$. By Lemma A.4 and Corollary A.7, we have

$$\Pr_{\substack{(\boldsymbol{X},\mathcal{P})\sim\textsc{Sample}'\\ \boldsymbol{I}\in\mathcal{P}\\ \boldsymbol{r}\in\mathcal{R}}}\left[(\boldsymbol{I},\boldsymbol{X}\setminus\boldsymbol{I},\boldsymbol{r}) \text{ is } \left(\alpha,\gamma,\tfrac{\varepsilon}{2}\right)\text{-excellent}\right] \geqslant \frac{\varepsilon}{2}\left(1-\frac{\exp(-\Omega(\alpha\ell))}{\gamma\varepsilon}\right) \geqslant \frac{\varepsilon}{2}(1-o(1)) \geqslant \frac{\varepsilon}{4}, \tag{A.65}$$

implying that by an averaging argument,

$$\Pr_{(\boldsymbol{X},\mathcal{P})\sim\textsc{Sample}'}\left[\Pr_{\substack{\boldsymbol{I}\in\mathcal{P}\\ \boldsymbol{r}\in\mathcal{R}}}\left[(\boldsymbol{I},\boldsymbol{X}\setminus\boldsymbol{I},\boldsymbol{r}) \text{ is } \left(\alpha,\gamma,\tfrac{\varepsilon}{2}\right)\text{-excellent}\right] \geqslant \tfrac{\varepsilon}{8}\right] \geqslant \tfrac{\varepsilon}{8}. \tag{A.66}$$

Conditioning $\boldsymbol{X},\mathcal{P}$ on the above event, we have

$$\begin{aligned}&\Pr_{\substack{(\boldsymbol{X},\mathcal{P},\boldsymbol{I}_1,\boldsymbol{I}_2)\sim\textsc{Sample}'\\ \boldsymbol{r}'_1,\boldsymbol{r}'_2\in\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{I}_1,\boldsymbol{X}\setminus\boldsymbol{I}_1,\boldsymbol{r}'_1) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\\ (\boldsymbol{I}_2,\boldsymbol{X}\setminus\boldsymbol{I}_2,\boldsymbol{r}'_2) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\end{matrix} \;\middle|\; \Pr_{\substack{\boldsymbol{I}\in\mathcal{P}\\ \boldsymbol{r}\in\mathcal{R}}}\left[(\boldsymbol{I},\boldsymbol{X}\setminus\boldsymbol{I},\boldsymbol{r}) \text{ is } \left(\alpha,\gamma,\tfrac{\varepsilon}{2}\right)\text{-excellent}\right] \geqslant \tfrac{\varepsilon}{8}\right]\\ &\geqslant \tfrac{\varepsilon}{8}\left(\tfrac{\varepsilon}{8}-\tfrac{\ell}{k}\right) \geqslant \Omega(\varepsilon^2),\end{aligned} \tag{A.67}$$

where we used the assumption that $\varepsilon = \omega\left(\frac{\ell}{k}\right)$. Conditioning $\boldsymbol{X},\boldsymbol{I}_1,\boldsymbol{I}_2$ on the above event, we have

$$\begin{aligned}&\Pr_{\substack{(\boldsymbol{X},\mathcal{P},\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2)\sim\textsc{Sample}'\\ \boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}'_1,\boldsymbol{r}'_2\in\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\mathsf{Cons}(\boldsymbol{I}_1,\boldsymbol{X}\setminus\boldsymbol{I}_1,\boldsymbol{r}'_1)\\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathsf{Cons}(\boldsymbol{I}_2,\boldsymbol{X}\setminus\boldsymbol{I}_2,\boldsymbol{r}'_2)\end{matrix} \;\middle|\; \begin{matrix}(\boldsymbol{I}_1,\boldsymbol{X}\setminus\boldsymbol{I}_1,\boldsymbol{r}'_1) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\\ (\boldsymbol{I}_2,\boldsymbol{X}\setminus\boldsymbol{I}_2,\boldsymbol{r}'_2) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\end{matrix}\right]\\ &\geqslant \frac{\varepsilon}{2}\cdot\frac{\varepsilon}{2} = \Omega(\varepsilon^2).\end{aligned} \tag{A.68}$$

Putting the above three inequalities together, we have

$$\Pr_{\substack{(\boldsymbol{X},\mathcal{P},\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2)\sim\textsc{Sample}' \\ \boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_1',\boldsymbol{r}_2'\in\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{I}_1,\boldsymbol{X}\setminus\boldsymbol{I}_1,\boldsymbol{r}_1') \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent} \\ (\boldsymbol{I}_2,\boldsymbol{X}\setminus\boldsymbol{I}_2,\boldsymbol{r}_2') \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\end{matrix} \text{ and } \begin{matrix}(\boldsymbol{A}_1,\boldsymbol{r}_1)\in\mathsf{Cons}(\boldsymbol{I}_1,\boldsymbol{X}\setminus\boldsymbol{I}_1,\boldsymbol{r}_1') \\ (\boldsymbol{A}_2,\boldsymbol{r}_2)\in\mathsf{Cons}(\boldsymbol{I}_2,\boldsymbol{X}\setminus\boldsymbol{I}_2,\boldsymbol{r}_2')\end{matrix}\right] \geqslant \Omega(\varepsilon^5), \tag{A.69}$$

$$\Longrightarrow \Pr_{\substack{(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X})\sim\textsc{Sample} \\ \boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_1',\boldsymbol{r}_2'\in\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent} \\ (\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\end{matrix} \text{ and } \begin{matrix}(\boldsymbol{X}\setminus\boldsymbol{I}_1,\boldsymbol{r}_1')\in\mathsf{Cons}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1) \\ (\boldsymbol{X}\setminus\boldsymbol{I}_2,\boldsymbol{r}_2')\in\mathsf{Cons}(\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2)\end{matrix}\right] \geqslant \Omega(\varepsilon^5), \tag{A.70}$$

where we note that $\boldsymbol{A}_i$ and $\boldsymbol{X}\setminus\boldsymbol{I}_i$ can be swapped, as desired. □

Claim A.23 (Claim 3.19 of [IKW12])**.**

$$\Pr_{\substack{(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X})\sim\textsc{Sample} \\ \boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1) \textit{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\textit{-excellent} \\ (\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2) \textit{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\textit{-excellent}\end{matrix} \textit{ and } \boldsymbol{f}_1^k(\boldsymbol{X}) \underset{4\beta}{\approx} \boldsymbol{f}_2^k(\boldsymbol{X})\right] \geqslant \Omega(\varepsilon^5). \tag{A.71}$$

Proof. By Lemma A.9, we have

$$\Pr_{\substack{(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X})\sim\textsc{Sample} \\ \boldsymbol{r}_1,\boldsymbol{r}_1'\in\mathcal{R}}}\left[F(\boldsymbol{X};\boldsymbol{r}_1') \underset{2\beta}{\not\approx} \boldsymbol{f}_1^k(\boldsymbol{X}) \;\middle|\; \begin{matrix}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent} \\ (\boldsymbol{X}\setminus\boldsymbol{I}_1,\boldsymbol{r}_1')\in\mathsf{Cons}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1)\end{matrix}\right] < \delta, \tag{A.72}$$

$$\Pr_{\substack{(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X})\sim\textsc{Sample} \\ \boldsymbol{r}_2,\boldsymbol{r}_2'\in\mathcal{R}}}\left[F(\boldsymbol{X};\boldsymbol{r}_2') \underset{2\beta}{\not\approx} \boldsymbol{f}_2^k(\boldsymbol{X}) \;\middle|\; \begin{matrix}(\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent} \\ (\boldsymbol{X}\setminus\boldsymbol{I}_2,\boldsymbol{r}_2')\in\mathsf{Cons}(\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_1)\end{matrix}\right] < \delta. \tag{A.73}$$

By Claim A.22, the conditions in the above formulas happen with probability $\Omega(\varepsilon^5)$, implying that

$$\begin{aligned}&\Pr_{\substack{(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{X})\sim\textsc{Sample} \\ \boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_1',\boldsymbol{r}_2'\in\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent} \\ (\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2) \text{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\end{matrix} \text{ and } \begin{matrix}F(\boldsymbol{X};\boldsymbol{r}_1') \underset{2\beta}{\approx} \boldsymbol{f}_1^k(\boldsymbol{X}) \\ F(\boldsymbol{X};\boldsymbol{r}_2') \underset{2\beta}{\approx} \boldsymbol{f}_2^k(\boldsymbol{X})\end{matrix}\right] \\ &\geqslant \Omega(\varepsilon^5)(1-2\delta) \geqslant \Omega(\varepsilon^5)\big(1-o(\varepsilon)\big) = \Omega(\varepsilon^5).\end{aligned} \tag{A.74}$$

Observe that

$$\left(F(\boldsymbol{X};\boldsymbol{r}_1') \underset{2\beta}{\approx} \boldsymbol{f}_1^k(\boldsymbol{X}) \text{ and } F(\boldsymbol{X};\boldsymbol{r}_2') \underset{2\beta}{\approx} \boldsymbol{f}_2^k(\boldsymbol{X})\right) \Longrightarrow \boldsymbol{f}_1^k(\boldsymbol{X}) \underset{4\beta}{\approx} \boldsymbol{f}_2^k(\boldsymbol{X}), \tag{A.75}$$

as desired. □

Claim A.24 (Claim 3.20 of [IKW12])**.**

$$\Pr_{\substack{(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2)\sim\textsc{Sample} \\ \boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{R}}}\left[\begin{matrix}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1) \textit{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\textit{-excellent} \\ (\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2) \textit{ is } \left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\textit{-excellent}\end{matrix} \textit{ and } \operatorname{dist}\big(\boldsymbol{f}_1,\boldsymbol{f}_2\big) \leqslant 8\beta\right] \geqslant \Omega(\varepsilon^5). \tag{A.76}$$

Proof. By Claim A.23 and an averaging argument, we have

$$\Pr_{\substack{(\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{A}_1,\boldsymbol{A}_2)\sim\textsc{Sample}\\ \boldsymbol{r}_1,\boldsymbol{r}_2\in\mathscr{R}}}\left[\begin{matrix}(\boldsymbol{I}_1,\boldsymbol{A}_1,\boldsymbol{r}_1)\text{ is }\left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\\ (\boldsymbol{I}_2,\boldsymbol{A}_2,\boldsymbol{r}_2)\text{ is }\left(\alpha,\gamma,\frac{\varepsilon}{2}\right)\text{-excellent}\end{matrix}\text{ and }\Pr_{\boldsymbol{X}\in\binom{V}{k}:\boldsymbol{X}\supset\boldsymbol{I}_1\cup\boldsymbol{I}_2}\left[\boldsymbol{f}_1^k(\boldsymbol{X})\underset{4\beta}{\approx}\boldsymbol{f}_2^k(\boldsymbol{X})\right]\geqslant\Omega(\varepsilon^5)\right]\geqslant\Omega(\varepsilon^5), \tag{A.77}$$

which implies Claim A.24 immediately for the following reason. Suppose for contradiction that for fixed $I_1,I_2\in\binom{V}{\ell}$ with $I_1\cap I_2=\emptyset$,

$$\begin{aligned}&\operatorname{dist}(\boldsymbol{f}_1,\boldsymbol{f}_2)>8\beta,\\ \Longrightarrow\ &\operatorname{dist}\left(\boldsymbol{f}_1|_{V\setminus(I_1\cup I_2)},\boldsymbol{f}_2|_{V\setminus(I_1\cup I_2)}\right)=\Pr_{\boldsymbol{v}\in V\setminus(I_1\cup I_2)}\left[\boldsymbol{f}_1(\boldsymbol{v})\neq\boldsymbol{f}_2(\boldsymbol{v})\right]>8\beta-\frac{2\ell}{n}>7\beta.\end{aligned} \tag{A.78}$$

By the Chernoff bound, we have

$$\begin{aligned}\Pr_{\boldsymbol{X}\in\binom{V}{k}:\boldsymbol{X}\supset I_1\cup I_2}\left[\boldsymbol{f}_1^k(\boldsymbol{X})\underset{4\beta}{\approx}\boldsymbol{f}_2^k(\boldsymbol{X})\right]&\leqslant\Pr_{\boldsymbol{X}\in\binom{V\setminus(I_1\cup I_2)}{k-2\ell}}\left[\boldsymbol{f}_1^{k-2\ell}(\boldsymbol{X})\underset{5\beta}{\approx}\boldsymbol{f}_2^{k-2\ell}(\boldsymbol{X})\right]\\ &\leqslant\exp\Big(-\Omega\big(\beta(k-2\ell)\big)\Big)\leqslant\exp\big(-\omega(\ell)\big)=o(\varepsilon^5),\end{aligned} \tag{A.79}$$

which is a contradiction. $\square$

We are now ready to conclude the proof of Theorem A.1.

Proof of Theorem A.1. By Claim A.24 and an averaging argument, there exists a function $f\colon V\to\Sigma$ such that

$$\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\ (\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus\boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\Big[\underbrace{(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\text{ is }\left(\alpha,\gamma,\tfrac{\varepsilon}{2}\right)\text{-excellent and }\operatorname{dist}(\boldsymbol{f}_0,f)\leqslant 8\beta}_{\mathcal{E}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0):=}\Big]\geqslant\Omega(\varepsilon^5). \tag{A.80}$$

On the one hand, by Lemma A.9, we have

$$\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\ (\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus\boldsymbol{I}_0}{k-\ell}\times\mathscr{R}\\ (\boldsymbol{A},\boldsymbol{r})\in\mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0)\times\mathscr{R}}}\left[F(\boldsymbol{I}_0\cup\boldsymbol{A};\boldsymbol{r})\underset{2\beta}{\not\approx}\boldsymbol{f}_0^k(\boldsymbol{I}_0\cup\boldsymbol{A})\,\middle|\,\mathcal{E}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\right]<\delta=o(1). \tag{A.81}$$

On the other hand, since $\operatorname{dist}(\boldsymbol{f}_0,f)\leqslant 8\beta$, by applying the Chernoff bound, we have

$$\begin{aligned}&\Pr_{\substack{\boldsymbol{I}_0\in\binom{V}{\ell}\\ (\boldsymbol{A}_0,\boldsymbol{r}_0)\in\binom{V\setminus\boldsymbol{I}_0}{k}\times\mathscr{R}\\ (\boldsymbol{A},\boldsymbol{r})\in\mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0)\times\mathscr{R}}}\left[\boldsymbol{f}_0^k(\boldsymbol{I}_0\cup\boldsymbol{A})\underset{16\beta}{\not\approx}f^k(\boldsymbol{I}_0\cup\boldsymbol{A})\,\middle|\,\mathcal{E}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\right]\\ =&\frac{\Pr_{\substack{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0\\ (\boldsymbol{A},\boldsymbol{r})\in\binom{V\setminus\boldsymbol{I}_0}{k}\times\mathscr{R}}}\left[\boldsymbol{f}_0^k(\boldsymbol{I}_0\cup\boldsymbol{A})\underset{16\beta}{\not\approx}f^k(\boldsymbol{I}_0\cup\boldsymbol{A})\text{ and }\underbrace{\boldsymbol{A}\in\mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)}_{\text{unnecessary}}\,\middle|\,\mathcal{E}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\right]}{\Pr_{\substack{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0\\ (\boldsymbol{A},\boldsymbol{r})\in\binom{V\setminus\boldsymbol{I}_0}{k}\times\mathscr{R}}}\left[(\boldsymbol{A},\boldsymbol{r})\in\mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\times\mathscr{R}\,\middle|\,\mathcal{E}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\right]}\\ <&\frac{\exp\big(-\Omega(\beta k)\big)}{\frac{\varepsilon}{2}}=o(1).\end{aligned} \tag{A.82}$$

By applying the union bound, we have

$$\Pr_{\substack{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0 \\ (\boldsymbol{A},\boldsymbol{r})\in\mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0)\times\mathscr{R}}}\left[\begin{array}{c|c} F(\boldsymbol{I}_0\cup\boldsymbol{A};\boldsymbol{r}) \underset{2\beta}{\approx} \boldsymbol{f}_0^k(\boldsymbol{I}_0\cup\boldsymbol{A}) & \\ & \mathcal{E}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0) \\ \boldsymbol{f}_0^k(\boldsymbol{I}_0\cup\boldsymbol{A}) \underset{16\beta}{\approx} f^k(\boldsymbol{I}_0\cup\boldsymbol{A}) & \end{array}\right] \geqslant 1-o(1) \geqslant \Omega(1). \tag{A.83}$$

Since $\frac{|\mathsf{Cons}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)|}{\left|\binom{V}{k-\ell}_{\boldsymbol{I}_0}\times\mathscr{R}\right|} \geqslant \frac{\varepsilon}{2}$ whenever $\mathcal{E}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)$ holds, we have

$$\begin{aligned}
&\Pr_{\substack{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0 \\ (\boldsymbol{A},\boldsymbol{r})\in\binom{V\setminus\boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[F(\boldsymbol{I}_0\cup\boldsymbol{A};\boldsymbol{r}) \underset{18\beta}{\approx} f^k(\boldsymbol{I}_0\cup\boldsymbol{A}) \,\middle|\, \mathcal{E}(\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0)\right] \geqslant \Omega(1)\cdot\frac{\varepsilon}{2} \geqslant \Omega(\varepsilon), \\
\Longrightarrow\quad &\Pr_{\substack{\boldsymbol{I}_0,\boldsymbol{A}_0,\boldsymbol{r}_0 \\ (\boldsymbol{A},\boldsymbol{r})\in\binom{V\setminus\boldsymbol{I}_0}{k-\ell}\times\mathscr{R}}}\left[F(\boldsymbol{I}_0\cup\boldsymbol{A};\boldsymbol{r}) \underset{18\beta}{\approx} f^k(\boldsymbol{I}_0\cup\boldsymbol{A})\right] \geqslant \Omega(\varepsilon^6), \qquad\qquad \text{(A.84)} \\
\Longrightarrow\quad &\Pr_{\boldsymbol{X}\in\binom{V}{k}}\left[F(\boldsymbol{X}) \underset{18\beta}{\approx} f^k(\boldsymbol{X})\right] \geqslant \Omega(\varepsilon^6),
\end{aligned}$$

as desired. □

B Omitted Proofs in Section 6

Proof of Claim 6.9. Let $J := J(n,k,\ell)$. For a vertex k-tuple $\vec{x} = (x_1,\ldots,x_k) \in V^k$ and an integer k-tuple $\vec{a} = (a_1,\ldots,a_k) \in [\Delta]^k$, we use $E^k(\vec{x})[\vec{a}]$ to denote the edge k-tuple $(e_1,\ldots,e_k) \in E^k$ such that each e_i is the ${a_i}^{\text{th}}$ edge incident to x_i.

Define $\mathcal{M}$ as the set of edge k-tuples in $\mathcal{S}$ that form a size-k matching; namely,

$$\mathcal{M} := \left\{\vec{e} = (e_1,\ldots,e_k) \in \mathcal{S} \,\middle|\, e_i \cap e_j = \emptyset \text{ for every } \{i,j\} \in \binom{[k]}{2}\right\}. \tag{B.1}$$

Note that $|\mathcal{M}| \leqslant |\mathcal{S}|$. We write

$$\left(\vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q,\vec{\boldsymbol{e}}_1,\ldots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q\right) \sim \mathcal{W}_q \tag{B.2}$$

for the random variables selected by $\mathcal{W}_q$. Observe first that

$$\begin{aligned}
&\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{M}\right]\\
&=\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{S}\text{ and }\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\text{ is a matching}\right]\\
&\geqslant\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{S}\right]-\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigvee_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\text{ is not a matching}\right]\\
&\geqslant\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{S}\right]-\sum_{i\in[q]}\underbrace{\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\text{ is not a matching}\right]}_{\leqslant\frac{2k^2}{n}}\\
&\geqslant\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{S}\right]-\frac{2qk^2}{n},
\end{aligned}\tag{B.3}$$

where we used the inequality that

$$\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}}=(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k)\in E^k}\left[\vec{\boldsymbol{e}}\text{ is not a matching}\right]\leqslant\sum_{\{i,j\}\in\binom{[k]}{2}}\mathop{\mathbf{Pr}}_{\vec{\boldsymbol{e}}=(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k)\in E^k}\left[\boldsymbol{e}_i\cap\boldsymbol{e}_j\neq\emptyset\right]\leqslant\binom{k}{2}\frac{4}{n}\leqslant\frac{2k^2}{n}.\tag{B.4}$$

We write $(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim\mathcal{T}_q(\ell,1)$ for q vertex sets selected by $\mathcal{T}_q(\ell,1)$. Observe next that

$$\begin{aligned}
&\mathop{\mathbf{Pr}}_{\substack{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim\mathcal{T}_q(\ell,1)\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \vec{\boldsymbol{a}}_1,\dots,\vec{\boldsymbol{a}}_q\in[\Delta]^k}}\left[\bigwedge_{i\in[q]}E^k(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i)[\vec{\boldsymbol{a}}_i]\in\mathcal{M}\right]\\
&=\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{x}}_1,\dots,\vec{\boldsymbol{x}}_q,\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{M}\;\middle|\;\bigwedge_{i\in[q]}\vec{\boldsymbol{x}}_i\in V^{\underline{k}}\right]\\
&\geqslant\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{x}}_1,\dots,\vec{\boldsymbol{x}}_q,\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{M}\text{ and }\bigwedge_{i\in[q]}\vec{\boldsymbol{x}}_i\in V^{\underline{k}}\right]\\
&\geqslant\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{M}\right]-\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{x}}_1,\dots,\vec{\boldsymbol{x}}_q)\sim\mathcal{W}_q}\left[\bigvee_{i\in[q]}\vec{\boldsymbol{x}}_i\notin V^{\underline{k}}\right]\\
&\geqslant\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{M}\right]-\sum_{i\in[q]}\underbrace{\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{x}}_1,\dots,\vec{\boldsymbol{x}}_q)\sim\mathcal{W}_q}\left[\vec{\boldsymbol{x}}_i\notin V^{\underline{k}}\right]}_{\leqslant\frac{k^2}{n}}\\
&\geqslant\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{M}\right]-\frac{qk^2}{n}.
\end{aligned}\tag{B.5}$$

Observe further that

$$
\begin{aligned}
&\Pr_{\substack{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathrm{RW}_q(J)\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \vec{\boldsymbol{a}}_1,\dots,\vec{\boldsymbol{a}}_q\in[\Delta]^k}}\left[\bigwedge_{i\in[q]} E^k(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i)[\vec{\boldsymbol{a}}_i]\in\mathcal{M}\right]\\
&=\Pr_{\substack{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathcal{T}_q(\ell,1)\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \vec{\boldsymbol{a}}_1,\dots,\vec{\boldsymbol{a}}_q\in[\Delta]^k}}\left[\bigwedge_{i\in[q]} E^k(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i)[\vec{\boldsymbol{a}}_i]\in\mathcal{M}\;\middle|\;\bigwedge_{i\in[q-1]}|\boldsymbol{X}_i\cap\boldsymbol{X}_{i+1}|=\ell\right]\\
&\geqslant\Pr_{\substack{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathcal{T}_q(\ell,1)\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \vec{\boldsymbol{a}}_1,\dots,\vec{\boldsymbol{a}}_q\in[\Delta]^k}}\left[\bigwedge_{i\in[q]} E^k(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i)[\vec{\boldsymbol{a}}_i]\in\mathcal{M}\text{ and }\bigwedge_{i\in[q-1]}|\boldsymbol{X}_i\cap\boldsymbol{X}_{i+1}|=\ell\right]\\
&\geqslant\Pr_{\substack{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathcal{T}_q(\ell,1)\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \vec{\boldsymbol{a}}_1,\dots,\vec{\boldsymbol{a}}_q\in[\Delta]^k}}\left[\bigwedge_{i\in[q]} E^k(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i)[\vec{\boldsymbol{a}}_i]\in\mathcal{M}\right]-\Pr_{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathcal{T}_q(\ell,1)}\left[\bigvee_{i\in[q-1]}|\boldsymbol{X}_i\cap\boldsymbol{X}_{i+1}|\neq\ell\right]\\
&\geqslant\Pr_{\substack{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathcal{T}_q(\ell,1)\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \vec{\boldsymbol{a}}_1,\dots,\vec{\boldsymbol{a}}_q\in[\Delta]^k}}\left[\bigwedge_{i\in[q]} E^k(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i)[\vec{\boldsymbol{a}}_i]\in\mathcal{M}\right]-\sum_{i\in[q-1]}\underbrace{\Pr_{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathcal{T}_q(\ell,1)}\left[|\boldsymbol{X}_i\cap\boldsymbol{X}_{i+1}|\neq\ell\right]}_{\leqslant\frac{k^2}{n}}\\
&\geqslant\Pr_{\substack{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathcal{T}_q(\ell,1)\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \vec{\boldsymbol{a}}_1,\dots,\vec{\boldsymbol{a}}_q\in[\Delta]^k}}\left[\bigwedge_{i\in[q]} E^k(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i)[\vec{\boldsymbol{a}}_i]\in\mathcal{M}\right]-\frac{qk^2}{n}.
\end{aligned}
\tag{B.6}
$$

Define

$$\widetilde{\mathcal{M}}\coloneqq\left\{(X,\pi,\vec{a})\in V(J\times\mathfrak{S}_k\times[\Delta]^k)\;\middle|\;E^k(X\circ\pi)[\vec{a}]\in\mathcal{M}\right\}. \tag{B.7}$$

Then, we obtain

$$\Pr_{\substack{(\boldsymbol{X}_1,\dots,\boldsymbol{X}_q)\sim \mathrm{RW}_q(J)\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \vec{\boldsymbol{a}}_1,\dots,\vec{\boldsymbol{a}}_q\in[\Delta]^k}}\left[\bigwedge_{i\in[q]} E^k(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i)[\vec{\boldsymbol{a}}_i]\in\mathcal{M}\right]=\Pr_{\left((\boldsymbol{X}_1,\boldsymbol{\pi}_1,\vec{\boldsymbol{a}}_1),\dots(\boldsymbol{X}_q,\boldsymbol{\pi}_q,\vec{\boldsymbol{a}}_q)\right)\sim\mathrm{RW}_q(J\times\mathfrak{S}_k\times[\Delta]^k)}\left[\bigwedge_{i\in[q]}(\boldsymbol{X}_i,\boldsymbol{\pi}_i,\vec{\boldsymbol{a}}_i)\in\widetilde{\mathcal{M}}\right]. \tag{B.8}$$

Let $\vec{e}=(e_1,\dots,e_k)\in E^k$ be an edge k-tuple that forms a size-k matching. Then, there are exactly 2^k triples $(X,\pi,\vec{a})\in V(J\times\mathfrak{S}_k\times[\Delta]^k)$ such that $E^k(X\circ\pi)[\vec{a}]=\vec{e}$. Since $\mathcal{M}$ consists only of size-k matchings, $|\widetilde{\mathcal{M}}|=2^k|\mathcal{M}|$. Since

$$|V(J\times\mathfrak{S}_k\times[\Delta]^k)|=\binom{n}{k}k!\Delta^k\text{ and }|E^k|=\left(\frac{n\Delta}{2}\right)^k, \tag{B.9}$$

we have

$$\frac{|\widetilde{\mathcal{M}}|}{|V(J\times\mathfrak{S}_k\times[\Delta]^k)|}=\frac{|E^k|}{|V(J\times\mathfrak{S}_k\times[\Delta]^k)|}\frac{|\widetilde{\mathcal{M}}|}{|E^k|}$$
$$=\frac{\left(\frac{n\Delta}{2}\right)^k}{\binom{n}{k}k!\Delta^k}\frac{2^k|\mathcal{M}|}{|E^k|}=\frac{n^k}{n^{\underline{k}}}\cdot\frac{|\mathcal{M}|}{|E^k|}=\frac{|\mathcal{M}|}{|E^k|}\left(1+\mathrm{O}_k\left(\frac{1}{n}\right)\right)\leqslant\frac{|\mathcal{S}|}{|E^k|}\left(1+\mathrm{O}_k\left(\frac{1}{n}\right)\right), \tag{B.10}$$

where we used the inequality that

$$\frac{n^k}{n^{\underline{k}}}\leqslant\frac{n^k}{(n-k)^k}=\left(1+\frac{k}{n-k}\right)^k=1+\mathrm{O}\left(\frac{k^2}{n}\right)\qquad(\text{as } n=\omega(k^2)). \tag{B.11}$$

By applying Lemma 5.5 to $\widetilde{\mathcal{M}}$, we have

$$\begin{aligned}
&\mathop{\mathbf{Pr}}_{\left((\boldsymbol{X}_1,\boldsymbol{\pi}_1,\vec{\boldsymbol{a}}_1),\dots(\boldsymbol{X}_q,\boldsymbol{\pi}_q,\vec{\boldsymbol{a}}_q)\right)\sim\mathrm{RW}_q(J\times\mathfrak{S}_k\times[\Delta]^k)}\left[\bigwedge_{i\in[q]}(\boldsymbol{X}_i,\boldsymbol{\pi}_i,\vec{\boldsymbol{a}}_i)\in\widetilde{\mathcal{M}}\right]\\
&\leqslant\left(\frac{|\widetilde{\mathcal{M}}|}{|V(J\times\mathfrak{S}_k\times[\Delta]^k)|}\right)^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\widetilde{\mathcal{M}}|}{|V(J\times\mathfrak{S}_k\times[\Delta]^k)|}\right)\\
&\underbrace{\leqslant}_{\text{Eq. (B.10)}}\left(\frac{|\mathcal{S}|}{|E^k|}\right)^q\left(1+\mathrm{O}_k\left(\frac{1}{n}\right)\right)^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{S}|}{|E^k|}\right)\left(1+\mathrm{O}_k\left(\frac{1}{n}\right)\right)\\
&=\left(\frac{|\mathcal{S}|}{|E^k|}\right)^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{S}|}{|E^k|}\right)+\mathrm{O}_{q,k}\left(\frac{1}{n}\right).
\end{aligned} \tag{B.12}$$

Combining Eqs. (B.3), (B.5), (B.6), (B.8) and (B.12), we obtain

$$\begin{aligned}
&\mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{e}}_1,\dots,\vec{\boldsymbol{e}}_q,\boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q)\sim\mathcal{W}_q}\left[\bigwedge_{i\in[q]}\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i\in\mathcal{S}\right]\\
&\leqslant\frac{2qk^2}{n}+\frac{qk^2}{n}+\frac{qk^2}{n}+\left(\frac{|\mathcal{S}|}{|E^k|}\right)^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{S}|}{|E^k|}\right)+\mathrm{O}_{k,q}\left(\frac{1}{n}\right)\\
&\leqslant\left(\frac{|\mathcal{S}|}{|E^k|}\right)^q+\mathrm{O}_q\left(\frac{\ell}{k}\right)\left(\frac{|\mathcal{S}|}{|E^k|}\right)+\mathrm{O}_{k,q}\left(\frac{1}{n}\right),
\end{aligned} \tag{B.13}$$

as desired. □

Proof of Claim 6.10. For $\boldsymbol{I}_i\in\binom{[k]}{\ell}$, $\vec{\boldsymbol{x}}_i,\vec{\boldsymbol{x}}_{i+1}\in V^k$ with $\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}$, $\vec{\boldsymbol{e}}_i\in E^k(\vec{\boldsymbol{x}}_i)$, $\vec{\boldsymbol{e}}_{i+1}\in E^k(\vec{\boldsymbol{x}}_{i+1})$, and $\boldsymbol{\pi}_i,\boldsymbol{\pi}_{i+1}\in\mathfrak{S}_k$, define the event $\mathcal{E}(\boldsymbol{I}_i,\vec{\boldsymbol{x}}_i,\vec{\boldsymbol{e}}_i,\boldsymbol{\pi}_i,\vec{\boldsymbol{x}}_{i+1},\vec{\boldsymbol{e}}_{i+1},\boldsymbol{\pi}_{i+1})$ as

$$\left(F(\vec{\boldsymbol{e}}_i\circ\boldsymbol{\pi}_i)\circ\boldsymbol{\pi}_i^{-1}\right)\left[\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}\right]\underset{1/\ell}{\approx}\left(F(\vec{\boldsymbol{e}}_{i+1}\circ\boldsymbol{\pi}_{i+1})\circ\boldsymbol{\pi}_{i+1}^{-1}\right)\left[\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right]. \tag{B.14}$$

By definition, $\mathcal{T}_{\text{tuple}}(\ell,1)$ accepts F_{tuple} with the following probability:

$$\begin{aligned}
&\mathbf{Pr}\Big[\mathcal{T}_{\text{tuple}}^{F_{\text{tuple}}}(\ell,1)=1\Big] \\
&= \mathop{\mathbf{Pr}}_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1}\in\binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q\in V^k \\ \vec{\boldsymbol{a}}_1,\ldots,\vec{\boldsymbol{a}}_q\in[\Delta]^k \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q\in\mathfrak{S}_k}}\left[\bigwedge_{i\in[q-1]}\big(F_{\text{tuple}}(\vec{\boldsymbol{x}}_i\circ\boldsymbol{\pi}_i,\vec{\boldsymbol{a}}_i)\circ\boldsymbol{\pi}_i^{-1}\big)|_{\boldsymbol{I}_i}\underset{1/\ell}{\approx}\big(F_{\text{tuple}}(\vec{\boldsymbol{x}}_{i+1}\circ\boldsymbol{\pi}_{i+1},\vec{\boldsymbol{a}}_{i+1})\circ\boldsymbol{\pi}_{i+1}^{-1}\big)|_{\boldsymbol{I}_i}\;\middle|\;\bigwedge_{i\in[q-1]}\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right] \\
&= \mathop{\mathbf{Pr}}_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1}\in\binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q\in V^k \\ \vec{\boldsymbol{e}}_i\in E^k(\vec{\boldsymbol{x}}_i)\ \forall i\in[q] \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q\in\mathfrak{S}_k}}\left[\bigwedge_{i\in[q-1]}\mathcal{E}(\boldsymbol{I}_i,\vec{\boldsymbol{x}}_i,\vec{\boldsymbol{e}}_i,\boldsymbol{\pi}_i,\vec{\boldsymbol{x}}_{i+1},\vec{\boldsymbol{e}}_{i+1},\boldsymbol{\pi}_{i+1})\;\middle|\;\bigwedge_{i\in[q-1]}\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right].
\end{aligned} \tag{B.15}$$

By definition, $\mathcal{W}_q$ accepts F with the following probability:

$$\begin{aligned}
&\mathbf{Pr}\Big[\mathcal{W}_q^F=1\Big] \\
&= \mathop{\mathbf{Pr}}_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1}\in\binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q\in V^k \\ \vec{\boldsymbol{e}}_i\in E^k(\vec{\boldsymbol{x}}_i)\ \forall i\in[q] \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q\in\mathfrak{S}_k}}\left[\bigwedge_{i\in[q-1]}\begin{array}{l}\mathcal{E}(\boldsymbol{I}_i,\vec{\boldsymbol{x}}_i,\vec{\boldsymbol{e}}_i,\boldsymbol{\pi}_i,\vec{\boldsymbol{x}}_{i+1},\vec{\boldsymbol{e}}_{i+1},\boldsymbol{\pi}_{i+1})\text{ or} \\ |\{\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,k},\boldsymbol{x}_{i+1,1},\ldots,\boldsymbol{x}_{i+1,k}\}|<2k-\ell\end{array}\;\middle|\;\bigwedge_{i\in[q-1]}\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right] \\
&\leqslant \mathop{\mathbf{Pr}}_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1}\in\binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q\in V^k \\ \vec{\boldsymbol{e}}_i\in E^k(\vec{\boldsymbol{x}}_i)\ \forall i\in[q] \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q\in\mathfrak{S}_k}}\left[\bigwedge_{i\in[q-1]}\mathcal{E}(\boldsymbol{I}_i,\vec{\boldsymbol{x}}_i,\vec{\boldsymbol{e}}_i,\boldsymbol{\pi}_i,\vec{\boldsymbol{x}}_{i+1},\vec{\boldsymbol{e}}_{i+1},\boldsymbol{\pi}_{i+1})\;\middle|\;\bigwedge_{i\in[q-1]}\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right] \\
&\qquad+\mathop{\mathbf{Pr}}_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1}\in\binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q\in V^k \\ \vec{\boldsymbol{e}}_i\in E^k(\vec{\boldsymbol{x}}_i)\ \forall i\in[q] \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q\in\mathfrak{S}_k}}\left[\bigvee_{i\in[q-1]}|\{\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,k},\boldsymbol{x}_{i+1,1},\ldots,\boldsymbol{x}_{i+1,k}\}|<2k-\ell\;\middle|\;\bigwedge_{i\in[q-1]}\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right] \qquad \text{(B.16)} \\
&\leqslant \mathbf{Pr}\Big[\mathcal{T}_{\text{tuple}}^{F_{\text{tuple}}}(\ell,1)=1\Big] \\
&\qquad+\sum_{i\in[q-1]}\underbrace{\mathop{\mathbf{Pr}}_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1}\in\binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q\in V^k \\ \vec{\boldsymbol{e}}_i\in E^k(\vec{\boldsymbol{x}}_i)\ \forall i\in[q] \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q\in\mathfrak{S}_k}}\Big[|\{\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,k},\boldsymbol{x}_{i+1,1},\ldots,\boldsymbol{x}_{i+1,k}\}|<2k-\ell\;\Big|\;\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\Big]}_{\leqslant\frac{k^2}{n}} \\
&\leqslant \mathbf{Pr}\Big[\mathcal{T}_{\text{tuple}}^{F_{\text{tuple}}}(\ell,1)=1\Big]+\frac{qk^2}{n},
\end{aligned}$$

Consequently, we have

$$\mathbf{Pr}\left[\mathcal{T}_{\mathrm{tuple}}^{F_{\mathrm{tuple}}}(\ell,1)=1\right] \geqslant \mathbf{Pr}\left[\mathcal{W}_q^F=1\right]-\frac{qk^2}{n} \geqslant p-\frac{qk^2}{n}, \tag{B.17}$$

as desired. $\square$

Proof of Claim 6.11. Instead of directly proving Claim 6.11, we prove the following slight generalization.

Lemma B.1. *Let $F_{\mathrm{tuple}}\colon V^k \times \mathscr{R} \to \Sigma^k$ be a k-tuple function, and let $F_{\mathrm{set}}\colon \binom{V}{k} \times \mathscr{R} \times \mathfrak{S}_k \to \Sigma^k$ be a k-set function such that $F_{\mathrm{set}}(X;r,\pi) := F_{\mathrm{tuple}}(X\circ\pi;r)\circ\pi^{-1}$. If $\mathcal{T}_{\mathrm{tuple}}(\ell,m)$ accepts F_{tuple} with probability at least p, then $\mathcal{T}_q(\ell,m)$ accepts F_{set} with probability at least $p-\frac{qk^2}{|V|}$.*

Proof. For $\boldsymbol{I}_i \in \binom{[k]}{\ell}$, $\vec{\boldsymbol{x}}_i,\vec{\boldsymbol{x}}_{i+1} \in V^k$, $\boldsymbol{\pi}_i,\boldsymbol{\pi}_{i+1} \in \mathfrak{S}_k$, $\boldsymbol{r}_i,\boldsymbol{r}_{i+1} \in \mathscr{R}$, define the event $\mathcal{E}_{\mathrm{tuple}}(\boldsymbol{I}_i,\vec{\boldsymbol{x}}_i,\boldsymbol{\pi}_i,\boldsymbol{r}_i,\vec{\boldsymbol{x}}_{i+1},\boldsymbol{\pi}_{i+1},\boldsymbol{r}_{i+1})$ as

$$\left(F_{\mathrm{tuple}}(\vec{\boldsymbol{x}}_i\circ\boldsymbol{\pi}_i;\boldsymbol{r}_i)\circ\boldsymbol{\pi}_i^{-1}\right)|_{\boldsymbol{I}_i} \underset{m/\ell}{\approx} \left(F_{\mathrm{tuple}}(\vec{\boldsymbol{x}}_{i+1}\circ\boldsymbol{\pi}_{i+1};\boldsymbol{r}_{i+1})\circ\boldsymbol{\pi}_{i+1}^{-1}\right)|_{\boldsymbol{I}_i}. \tag{B.18}$$

By assumption, we have

$$\begin{aligned} p &\leqslant \mathbf{Pr}\left[\mathcal{T}_{\mathrm{tuple}}^{F_{\mathrm{tuple}}}=1\right] \\ &= \mathop{\mathbf{Pr}}_{\substack{\boldsymbol{I}_1,\dots,\boldsymbol{I}_{q-1}\in\binom{[k]}{\ell}\\ \vec{\boldsymbol{x}}_1,\dots,\vec{\boldsymbol{x}}_q\in V^k\\ \boldsymbol{\pi}_1,\dots,\boldsymbol{\pi}_q\in\mathfrak{S}_k\\ \boldsymbol{r}_1,\dots,\boldsymbol{r}_q\in\mathscr{R}}}\left[\bigwedge_{i\in[q-1]}\mathcal{E}_{\mathrm{tuple}}(\boldsymbol{I}_i,\vec{\boldsymbol{x}}_i,\boldsymbol{\pi}_i,\boldsymbol{r}_i,\vec{\boldsymbol{x}}_{i+1},\boldsymbol{\pi}_{i+1},\boldsymbol{r}_{i+1}) \;\middle|\; \bigwedge_{i\in[q-1]}\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}\right]. \end{aligned} \tag{B.19}$$

We write

$$\left(\vec{\boldsymbol{I}}=(\boldsymbol{I}_i)_{i\in[q-1]},\vec{\boldsymbol{X}}=(\boldsymbol{X}_i)_{i\in[q]},\vec{\boldsymbol{r}}=(\boldsymbol{r}_i)_{i\in[q]},\vec{\boldsymbol{\pi}}=(\boldsymbol{\pi}_i)_{i\in[q]}\right)\sim\mathcal{T}_q(\ell,m) \tag{B.20}$$

for the random variables selected by $\mathcal{T}_q(\ell,m)$. By definition, $\mathcal{T}_q(\ell,m)$ accepts F with the following probability:

$$\begin{aligned} &\mathbf{Pr}\left[\mathcal{T}_q^{F_{\mathrm{set}}}(\ell,m)=1\right] \\ &= \mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}},\vec{\boldsymbol{\pi}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{i\in[q-1]} F_{\mathrm{set}}(\boldsymbol{X}_i;\boldsymbol{r}_i,\boldsymbol{\pi}_i)|_{\boldsymbol{I}_i} \underset{m/\ell}{\approx} F_{\mathrm{set}}(\boldsymbol{X}_{i+1};\boldsymbol{r}_{i+1},\boldsymbol{\pi}_{i+1})|_{\boldsymbol{I}_i}\right] \\ &= \mathop{\mathbf{Pr}}_{(\vec{\boldsymbol{I}},\vec{\boldsymbol{X}},\vec{\boldsymbol{r}},\vec{\boldsymbol{\pi}})\sim\mathcal{T}_q(\ell,m)}\left[\bigwedge_{i\in[q-1]} \left(F_{\mathrm{tuple}}\left(\boldsymbol{X}_i\circ\boldsymbol{\pi}_i;\boldsymbol{r}_i\right)\circ\boldsymbol{\pi}_i^{-1}\right)|_{\boldsymbol{I}_i} \underset{m/\ell}{\approx} \left(F_{\mathrm{tuple}}\left(\boldsymbol{X}_{i+1}\circ\boldsymbol{\pi}_{i+1};\boldsymbol{r}_{i+1}\right)\circ\boldsymbol{\pi}_{i+1}^{-1}\right)|_{\boldsymbol{I}_i}\right]. \end{aligned} \tag{B.21}$$

Observe that the following two distributions are equivalent:

- $(\vec{\boldsymbol{x}}_1\circ\boldsymbol{\pi}_1,\dots,\vec{\boldsymbol{x}}_q\circ\boldsymbol{\pi}_q)$ conditioned on the event that $\vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i}=\vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i}$ for every $i\in[q-1]$ and $\vec{\boldsymbol{x}}_i\in V^{\underline{k}}$ for every $i\in[q]$, where $\boldsymbol{I}_i\in\binom{[k]}{\ell}$ for every $i\in[q-1]$ and $\vec{\boldsymbol{x}}_i\in V^k$ and $\boldsymbol{\pi}_i\in\mathfrak{S}_k$ for every $i\in[q]$.

- $(\boldsymbol{X}_1 \circ \boldsymbol{\pi}_1, \ldots, \boldsymbol{X}_q \circ \boldsymbol{\pi}_q)$, where $(\vec{\boldsymbol{I}}, \vec{\boldsymbol{X}}, \vec{\boldsymbol{r}}, \vec{\boldsymbol{\pi}}) \sim \mathcal{T}_q(\ell, m)$.

Therefore, we derive

$$
\begin{aligned}
&\text{Eq. (B.21)} \\
&= \Pr_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1} \in \binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q \in V^k \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q \in \mathfrak{S}_k \\ \boldsymbol{r}_1,\ldots,\boldsymbol{r}_q \in \mathscr{R}}} \left[\bigwedge_{i \in [q-1]} \mathcal{E}_{\text{tuple}}(\boldsymbol{I}_i, \vec{\boldsymbol{x}}_i, \boldsymbol{\pi}_i, \boldsymbol{r}_i, \vec{\boldsymbol{x}}_{i+1}, \boldsymbol{\pi}_{i+1}, \boldsymbol{r}_{i+1}) \;\middle|\; \bigwedge_{i \in [q]} \vec{\boldsymbol{x}}_i \in V^{\underline{k}} \text{ and } \bigwedge_{i \in [q-1]} \vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i} \right] \\
&\geqslant \Pr_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1} \in \binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q \in V^k \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q \in \mathfrak{S}_k \\ \boldsymbol{r}_1,\ldots,\boldsymbol{r}_q \in \mathscr{R}}} \left[\bigwedge_{i \in [q-1]} \mathcal{E}_{\text{tuple}}(\boldsymbol{I}_i, \vec{\boldsymbol{x}}_i, \boldsymbol{\pi}_i, \boldsymbol{r}_i, \vec{\boldsymbol{x}}_{i+1}, \boldsymbol{\pi}_{i+1}, \boldsymbol{r}_{i+1}) \text{ and } \bigwedge_{i \in [q]} \vec{\boldsymbol{x}}_i \in V^{\underline{k}} \;\middle|\; \bigwedge_{i \in [q-1]} \vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i} \right] \\
&\geqslant \underbrace{\Pr_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1} \in \binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q \in V^k \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q \in \mathfrak{S}_k \\ \boldsymbol{r}_1,\ldots,\boldsymbol{r}_q \in \mathscr{R}}} \left[\bigwedge_{i \in [q-1]} \mathcal{E}_{\text{tuple}}(\boldsymbol{I}_i, \vec{\boldsymbol{x}}_i, \boldsymbol{\pi}_i, \boldsymbol{r}_i, \vec{\boldsymbol{x}}_{i+1}, \boldsymbol{\pi}_{i+1}, \boldsymbol{r}_{i+1}) \;\middle|\; \bigwedge_{i \in [q-1]} \vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i} \right]}_{\geqslant p} \\
&\quad - \underbrace{\Pr_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1} \in \binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q \in V^k \\ \boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_q \in \mathfrak{S}_k \\ \boldsymbol{r}_1,\ldots,\boldsymbol{r}_q \in \mathscr{R}}} \left[\bigvee_{i \in [q]} \vec{\boldsymbol{x}}_i \notin V^{\underline{k}} \;\middle|\; \bigwedge_{i \in [q-1]} \vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i} \right]}_{\leqslant \frac{qk^2}{|V|}} \\
&\geqslant p - \frac{qk^2}{|V|},
\end{aligned}
\tag{B.22}
$$

where we used the inequality that

$$
\begin{aligned}
\Pr_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1} \in \binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q \in V^k}} \left[\bigvee_{i \in [q]} \vec{\boldsymbol{x}}_i \notin V^{\underline{k}} \;\middle|\; \bigwedge_{i \in [q-1]} \vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i} \right] &\leqslant \sum_{i \in [q]} \Pr_{\substack{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_{q-1} \in \binom{[k]}{\ell} \\ \vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_q \in V^k}} \left[\vec{\boldsymbol{x}}_i \notin V^{\underline{k}} \;\middle|\; \bigwedge_{i \in [q-1]} \vec{\boldsymbol{x}}_i|_{\boldsymbol{I}_i} = \vec{\boldsymbol{x}}_{i+1}|_{\boldsymbol{I}_i} \right] \\
&= q \Pr_{\vec{\boldsymbol{x}} \in V^k} \left[\vec{\boldsymbol{x}} \notin V^{\underline{k}} \right] \leqslant \frac{qk^2}{|V|},
\end{aligned}
\tag{B.23}
$$

as desired. □

Proof of Claim 6.13. Instead of directly proving Claim 6.13, we prove the following slight generalization.

Lemma B.2. *Let $F_{\mathrm{tuple}}\colon V^k \times \mathscr{R} \to \Sigma^k$ be a k-tuple function, and let $F_{\mathrm{set}}\colon \binom{V}{k} \times \mathscr{R} \times \mathfrak{S}_k \to \Sigma^k$ be a k-set function such that $F_{\mathrm{set}}(X;r,\pi) \coloneqq F_{\mathrm{tuple}}(X\circ\pi;r)\circ\pi^{-1}$. Suppose that there exists a function $f\colon V\to\Sigma$ such that*

$$\Pr_{\substack{\boldsymbol{X}\in\binom{V}{k}\\ (\boldsymbol{r},\boldsymbol{\pi})\in\mathscr{R}\times\mathfrak{S}_k}}\left[F_{\mathrm{set}}(\boldsymbol{X};\boldsymbol{r},\boldsymbol{\pi}) \underset{\delta}{\approx} f^k(\boldsymbol{X})\right] \geqslant p \tag{B.24}$$

for some reals $p\in(0,1)$ and $\delta\in(0,1)$. Then,

$$\Pr_{\substack{\vec{\boldsymbol{x}}\in V^k\\ \boldsymbol{r}\in\mathscr{R}}}\left[F_{\mathrm{tuple}}(\vec{\boldsymbol{x}};\boldsymbol{r}) \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}})\right] \geqslant p\left(1-\frac{k^2}{|V|}\right). \tag{B.25}$$

Proof. By assumption, we have

$$\begin{aligned} p &\leqslant \Pr_{\substack{\boldsymbol{X}\in\binom{V}{k}\\ (\boldsymbol{r},\boldsymbol{\pi})\in\mathscr{R}\times\mathfrak{S}_k}}\left[F_{\mathrm{set}}(\boldsymbol{X};\boldsymbol{r},\boldsymbol{\pi}) \underset{\delta}{\approx} f^k(\boldsymbol{X})\right] \\ &= \Pr_{\substack{\boldsymbol{X}\in\binom{V}{k}\\ (\boldsymbol{r},\boldsymbol{\pi})\in\mathscr{R}\times\mathfrak{S}_k}}\left[F_{\mathrm{tuple}}(\boldsymbol{X}\circ\boldsymbol{\pi};\boldsymbol{r})\circ\boldsymbol{\pi}^{-1} \underset{\delta}{\approx} f^k(\boldsymbol{X})\right] \\ &= \Pr_{\substack{\vec{\boldsymbol{x}}\in V^k\\ \boldsymbol{r}\in\mathscr{R}}}\left[F_{\mathrm{tuple}}(\vec{\boldsymbol{x}};\boldsymbol{r}) \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}}) \;\middle|\; \vec{\boldsymbol{x}}\in V^{\underline{k}}\right], \end{aligned} \tag{B.26}$$

where the last equality holds because $\boldsymbol{X}\circ\boldsymbol{\pi}$ is uniformity distributed over $V^{\underline{k}}$. Therefore, we derive

$$\begin{aligned} \Pr_{\substack{\vec{\boldsymbol{x}}\in V^k\\ \boldsymbol{r}\in\mathscr{R}}}\left[F_{\mathrm{tuple}}(\vec{\boldsymbol{x}};\boldsymbol{r}) \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}})\right] &\geqslant \Pr_{\substack{\vec{\boldsymbol{x}}\in V^k\\ \boldsymbol{r}\in\mathscr{R}}}\left[F_{\mathrm{tuple}}(\vec{\boldsymbol{x}};\boldsymbol{r}) \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}}) \text{ and } \vec{\boldsymbol{x}}\in V^{\underline{k}}\right] \\ &= \underbrace{\Pr_{\substack{\vec{\boldsymbol{x}}\in V^k\\ \boldsymbol{r}\in\mathscr{R}}}\left[F_{\mathrm{tuple}}(\vec{\boldsymbol{x}};\boldsymbol{r}) \underset{\delta}{\approx} f^k(\vec{\boldsymbol{x}}) \;\middle|\; \vec{\boldsymbol{x}}\in V^{\underline{k}}\right]}_{\geqslant p} \cdot \underbrace{\Pr_{\vec{\boldsymbol{x}}\in V^k}\left[\vec{\boldsymbol{x}}\in V^{\underline{k}}\right]}_{\geqslant 1-\frac{k^2}{|V|}} \\ &\geqslant p\left(1-\frac{k^2}{|V|}\right), \end{aligned} \tag{B.27}$$

as desired. ☐

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