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Cube-Root Soundness Is Optimal for the Line–versus–Point Test

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Abstract

The line–versus–point test, studied by Rubinfeld and Sudan [RS96] and Arora and Sudan [AS03], takes as input a point table $f\colon \mathbb{F}_q^m \to \mathbb{F}_q$ and a polynomial P_L of degree at most d for each affine line L. It samples a uniformly random incident pair (x, L) such that $x \in L$ and checks if $f(x) = P_L(x)$. This test and its variants have played a central role in PCP constructions. A well-studied quantitative question about this test is the following. What is the largest exponent $\tau > 0$ for which there are constants $C, c > 0$ such that acceptance $\varepsilon \geq C(d/q)^\tau$ guarantees agreement at least $c\varepsilon$ with a polynomial of total degree at most d, over all finite fields, dimensions $m \geq 2$, and degrees $1 \leq d < q$?

We characterize the optimal soundness exponent of this test, showing that it is exactly $1/3$. More specifically, over every finite field $\mathbb{F}_q$, for every $m \geq 2$ and $d \geq 1$, whenever a point table and a degree-d line table pass the test with probability ε, there is a polynomial Q of total degree at most d such that

$$\Pr_{x \in \mathbb{F}_q^m}[Q(x) = f(x)] \geq \varepsilon - C\left(\frac{d}{q}\right)^{1/3},$$

where $C > 0$ is an absolute constant.

The previous best bounds, due to [HKSS24], are soundness thresholds $\Omega((d/q)^{1/48})$ for $m > 2$ and $\Omega((d/q)^{1/7})$ for $m = 2$.

To complement this upper bound, we give an explicit construction in every dimension $m \geq 3$ over cubic extension fields of size $q = s^3$, where $s \geq 3$ is a prime power. For every $1 \leq d \leq q$, our constructed tables have acceptance $d/q + (1 - d/q)q^{-1/3} + O(q^{-2/3})$, while every polynomial of total degree at most d agrees with the point table on at most a $d/q + O(q^{-2/3})$ fraction of points. Thus global agreement is a vanishing fraction of acceptance whenever $d = o(q^{2/3})$. In particular, no soundness exponent larger than $1/3$ is possible, even for a fixed degree d.

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1 Introduction

The Reed–Muller code is one of the basic algebraic codes in theoretical computer science. Its codewords are the evaluation tables of low-degree polynomials:

$$\mathrm{RM}_q(m,d) := \{(Q(x))_{x\in\mathbb{F}_q^m} : Q \in \mathbb{F}_q[X_1,\dots,X_m],\ \deg Q \le d\}.$$

Much of the usefulness of Reed–Muller codes comes from their strong locality. The restriction of a degree-d polynomial to any affine line is a univariate polynomial of degree at most d, and hence a Reed–Solomon codeword. Local testing and correction procedures use this to inspect only a small part of a word of length q^m.

One particularly important test arising from this structure is the line–versus–point test, a two-query consistency test. We are given a point table $f\colon \mathbb{F}_q^m \to \mathbb{F}_q$ and, for every affine line L, a purported degree-d restriction P_L. The test samples a uniformly random incident pair (x, L), queries $f(x)$ and $P_L(x)$, and accepts when $P_L(x) = f(x)$. If the point and line tables are induced by a single degree-d polynomial, the test always accepts. Its soundness asks for a converse: how small can the acceptance probability be while still forcing one global degree-d polynomial to agree with f on a noticeable fraction of points?

Low-degree tests, such as the line–versus–point test and its variants, have played a central role in PCP constructions and hardness of approximation since the earliest algebraic PCPs [PS94, RS96, AS98, ALM+98, RS97, AS03]. Their two-query form is particularly useful for constructing projection PCPs and Label Cover instances, which serve as starting points for many hardness-of-approximation reductions [MR10]. Higher-dimensional subspace agreement tests have also been central to the series of works leading to the 2-to-2 Games Theorem [KMS25, DKK+25, DKK+21, KMS23] and other PCP constructions with strong or near-optimal soundness [KS13, MZ26a, MZ26b]. More recently, the line–versus–point test has been used to construct query-efficient interactive oracle proofs of proximity for NP [MZ25].

Beyond these applications, the line–versus–point test is a natural local-to-global agreement problem in its own right, closely related to property testing of Reed–Muller codes [AKK+05, KR06, BKS+10, HSS13, RZS13, KM22, MZ23b]. It was introduced by Rubinfeld and Sudan [RS96]. Polishchuk and Spielman, and subsequently Friedl and Sudan, analyzed it in the low-error, or unique-decoding, regime when the field size is linear in the degree [PS94, FS95]. Arora and Sudan then obtained a high-error analysis when the field is polynomially larger than the degree [AS03]. Harsha, Kumar, Saptharishi, and Sudan gave the first high-error analysis when the field size is linear in the degree [HKSS24]. Their main theorems, which leave the exponent unspecified, show that acceptance $\varepsilon \ge (d/q)^{\Omega(1)}$ implies agreement $\varepsilon - (d/q)^{\Omega(1)}$ with a low-degree polynomial. The proofs give the thresholds $(d/q)^{1/7}$ in two dimensions [HKSS24, Theorem 4.2] and $(d/q)^{1/48}$ in general [HKSS24, Theorem 5.7].[1]

Quantitatively, we ask for the largest $\tau > 0$ for which there are constants $C, c > 0$ with the following property. For every finite field $\mathbb{F}_q$, every $m \ge 2$ and every $1 \le d < q$, if a point table and a degree-d line table pass the line–versus–point test with probability $\varepsilon \ge C(d/q)^\tau$, then some polynomial of total degree at most d agrees with the point table on at least a $c\varepsilon$ fraction of points. The constants C, c must be independent of q, d, m and the tables. Necessarily $\tau \le 1$, since every point table passes with probability at least $(d+1)/q$ against a line table interpolating it.

A benchmark for this question comes from tests that query higher-dimensional subspaces instead of lines [BDN17, MZ23a, BR26]. In the plane–versus–plane test of Raz and Safra [RS97], one

[1] These two theorems give agreement $\Omega(\varepsilon^4)$ and ε^2, respectively. The list-decoding step of Section 4.2 makes the agreement linear at the same thresholds.

works with a table of low-degree restrictions to affine planes and checks agreement between two planes intersecting in a line. Bhangale and Richelson [BR26] showed that this test is sound once its acceptance is $\Omega(d/q)$, which is optimal in its dependence on the field size. In the language above, the analogous exponent for this test is $\tau = 1$. Its answers, however, are bivariate polynomials of degree d, with $(d+1)(d+2)/2$ coefficients instead of the $d+1$ of a line. This leaves the following questions.

What is the optimal soundness exponent τ of the line–versus–point test?
Is the test sound at acceptance of order d/q, like the plane–versus–plane test?

1.1 Our results

We show that the optimal exponent is $\tau = 1/3$. In particular, unlike the plane–versus–plane test, the line–versus–point test is not sound at acceptance of order d/q. Theorem 1.1 below shows that acceptance ε gives agreement $\varepsilon - O((d/q)^{1/3})$ with some polynomial of total degree at most d, and Theorem 1.2 gives a matching construction.

Throughout, fix a finite field $\mathbb{F}_q$ of order q, where q is an arbitrary prime power; in particular, $\mathbb{F}_q$ need not be a prime field. Let $\mathcal{L}_{q,m}$ be the set of affine lines in $\mathbb{F}_q^m$, where each affine line is given by a translate $a \in \mathbb{F}_q^m$ and a direction $u \in \mathbb{F}_q^m \setminus \{0\}$, so that $L = \{a + tu \mid t \in \mathbb{F}_q\}$. When we work in two dimensions and the field is clear from context, we instead write $\mathcal{L}$ for $\mathcal{L}_{q,2}$. For every affine line L, fix once and for all an affine parameterization, consisting of translate $a_L \in L$ and direction u_L, and write $L(t) := a_L + tu_L$ to denote the t-th point of L. We slightly abuse notation by using L both for the affine line and, through $L(\cdot)$, for its fixed parameterization.

Formally, in the line–versus–point test we are given a point table $f\colon \mathbb{F}_q^m \to \mathbb{F}_q$ and a degree-d line table $\{P_L\}_{L\in\mathcal{L}_{q,m}}$, where each $P_L : L \to \mathbb{F}_q$ is a univariate polynomial of degree at most d defined over L. Relative to the fixed parameterization of L, let $p_L \in \mathbb{F}_q[t]$ denote the chosen degree-at-most-d polynomial satisfying $p_L(t) = P_L(L(t))$. We use this notation henceforth. The test samples an incident pair (x, L) uniformly at random, where $x \in \mathbb{F}_q^m$ and $L \in \mathcal{L}_{q,m}$ such that $x \in L$, and accepts if and only if $P_L(x) = f(x)$. Given the point and line tables, we define the acceptance probability of the test as $\mathrm{Acc}(f, \{P_L\}) := \Pr_{x\in\mathbb{F}_q^m,\, L\ni x}[P_L(x) = f(x)]$.

Theorem 1.1. *There is an absolute constant $C > 0$ with the following property. Let $m \geq 2$ and $d \geq 1$. For every point table $f\colon \mathbb{F}_q^m \to \mathbb{F}_q$ and degree-d line table $\{P_L\}_{L\in\mathcal{L}_{q,m}}$, there is a polynomial $Q \in \mathbb{F}_q[X_1, \ldots, X_m]$ of total degree at most d such that*

$$\mathrm{agr}(f, Q) := \Pr_{x\in\mathbb{F}_q^m}[Q(x) = f(x)] \geq \mathrm{Acc}(f, \{P_L\}) - C\left(\frac{d}{q}\right)^{1/3}.$$

To complement this upper bound, we give an explicit construction of point and line tables over cubic extensions $q = s^3$. To our knowledge, all previously known tables with low global agreement had acceptance $O(d/q)$. Already at degree one, our tables have acceptance $(1 + o(1))q^{-1/3}$, while every affine polynomial agrees with the point table on only an $O(q^{-2/3})$ fraction of points. The construction extends to every degree $d \leq q$ in every dimension $m \geq 3$.

Theorem 1.2. *For every prime power $s \geq 3$, let $q = s^3$. For all integers $m \geq 3$ and $1 \leq d \leq q$, there is an explicit point table $f_d : \mathbb{F}_q^m \to \mathbb{F}_q$ and a degree-d line table $\{P_L^{(d)}\}$ with acceptance*

$$\varepsilon_{s,d} := \mathrm{Acc}(f_d, \{P_L^{(d)}\}) = \frac{d}{q} + \left(1 - \frac{d}{q}\right) q^{-1/3} + O(q^{-2/3})$$

such that every polynomial $Q \in \mathbb{F}_q[X_1, \ldots, X_m]$ of total degree at most d satisfies

$$\operatorname{agr}(f_d, Q) \le \varepsilon_{s,d} - \left(1 - \frac{d}{q}\right) q^{-1/3} + O(q^{-2/3}).$$

When $d = o(q^{2/3})$, global agreement is therefore a vanishing fraction of acceptance, which gives the following corollary.

Corollary 1.3. *Fix integers $m \ge 2$ and $d \ge 1$, and a prime p. For every $\tau > 1/3$ and every $C, c > 0$, there are arbitrarily large fields $\mathbb{F}_q$ of characteristic p, a point table $f_d : \mathbb{F}_q^m \to \mathbb{F}_q$, and a degree-at-most-$d$ line table $\{P_L^{(d)}\}$ such that*

$$\operatorname{Acc}(f_d, \{P_L^{(d)}\}) \ge C(d/q)^\tau, \qquad \operatorname{agr}(f_d, Q) < c \operatorname{Acc}(f_d, \{P_L^{(d)}\})$$

for every polynomial Q of total degree at most d.

For $m \ge 3$, Corollary 1.3 follows from Theorem 1.2, which we prove in Section 5. For $m = 2$, it follows from the planar construction in the same section, whose acceptance and agreement are bounded in Lemmas 5.1 and 5.3.

Mechanization. Theorems 1.1, 1.2 and 3.1 and Corollary 1.3 have been formalized and machine-checked in Lean 4 with the Mathlib library, with no axioms beyond Lean's standard ones. The formalization also proves Theorem 4.2. Some of the formal proofs reuse arguments from an earlier version of this paper instead of following the proofs given here; the repository lists where they differ. We are grateful to Joachim Neu for his help in mechanizing this paper. The Lean proof is available at `https://github.com/kz99/Line-vs-Point-Paper-and-Lean`.

1.2 Proof overview and technical contributions

Our proof proceeds in two stages. We first establish the result in the bivariate case $m = 2$, which contains most of the work, and then bootstrap it to arbitrary dimension using the results of [BR26]. This overview therefore focuses on the bivariate case.

The bivariate argument follows the weighted-interpolation and Newton–Hensel approach of [HKSS24]. We interpolate a single polynomial through the whole graph of f, at a weighted degree $O((dq^2)^{1/3})$ that does not depend on ε; this degree is what sets the cube-root threshold. Our key new argument then uses uniqueness of lifts from simple roots to propagate agreement throughout a large connected component of the point–line incidence graph, which makes the agreement linear in ε.

Steps 1–3 below prove the multiplicative bound in Theorem 3.1 for $m = 2$. Step 4 extends it to every m, and a standard list-decoding reduction then gives Theorem 1.1. The final part describes the counterexample.

1. Interpolating the point table. Let G_{acc} be the bipartite graph whose two sides are the affine lines $\mathcal{L}$ and the points $\mathbb{F}_q^2$, with an edge (x, L) whenever $P_L(x) = f(x)$. We call these *accepted incidences*. Acceptance probability at least ε means that the accepted-incidences have measure at least ε among all point–line incidences.

Our goal is to turn the agreement counted by the test into a single polynomial relation satisfied identically by the line polynomials, while preserving most of the accepted incidences.

To this end, we construct a nonzero trivariate polynomial $A(X, Y, Z)$ satisfying $A(x, f(x)) = 0$ for every $x \in \mathbb{F}_q^2$. Here, the weighted degree of A, denoted $\operatorname{wdeg}(A)$, is the maximum value of

$i+j+dk$ over all monomials $X^iY^jZ^k$ appearing in A with nonzero coefficient. These weights ensure that $A(L(t), p_L(t))$ has degree at most $\mathrm{wdeg}(A)$ in t.

Each condition $A(x, f(x)) = 0$ is a homogeneous linear constraint on the coefficients of A. There are q^2 such constraints and $\Omega(D^3/d)$ coefficients of weighted degree at most D. Thus we can take

$$\mathrm{wdeg}(A) \le D = O\Big((dq^2)^{1/3}\Big)$$

while ensuring that $A(x, f(x)) = 0$ for every $x \in \mathbb{F}_q^2$.

Now consider any line L. At every accepted point $x = L(t)$, we have $p_L(t) = f(x)$ and hence $A(L(t), p_L(t)) = 0$. If L has more than D accepted points, this polynomial has more roots than its degree, forcing $A(L(t), p_L(t)) \equiv 0$.

Henceforth, we consider only the acceptance of the test captured by these lines. Indeed, every other line has at most D accepted points, so all such lines together contribute at most $D/q = O((d/q)^{1/3})$ to the acceptance probability. Under our hypothesis $\varepsilon \ge C(d/q)^{1/3}$ for sufficiently large C, this is at most $\varepsilon/2$. Thus A captures the agreement between the point and line tables up to a constant factor.

2. Simple roots throughout the incidence graph. Before extracting a bivariate polynomial from the common relation $A(L(t), p_L(t)) \equiv 0$, we arrange that, at every remaining point x, the value $f(x)$ is a simple root of $A(x, Z)$. This gives the later lifting and propagation steps the uniqueness property that we need.

We first replace A by the product of its distinct irreducible factors, preserving its zeros without increasing its weighted degree. We next arrange the simple-root property. Choose $\lambda_X, \lambda_Y, \lambda_Z \in \mathbb{F}_q$ so that A and $\lambda_X A_X + \lambda_Y A_Y + \lambda_Z A_Z$ have no common factor, where A_X, A_Y, and A_Z are the formal partial derivatives of A. After removing a small number of lines and incidences and pruning, the remaining graph has minimum degree $\Omega(\varepsilon q) > D$ on both sides and retains accepted incidence measure $\Omega(\varepsilon)$. In particular, every remaining point x is adjacent to at least two distinct lines and satisfies

$$A(x, f(x)) = 0 \qquad \text{and} \qquad (\lambda_X A_X + \lambda_Y A_Y + \lambda_Z A_Z)(x, f(x)) \neq 0.$$

We justify why the root $f(x)$ is simple. Let L and M be two distinct lines adjacent to x in the remaining graph, with direction vectors $u_L, u_M \in \mathbb{F}_q^2$. Reparameterize both lines so that $t = 0$ corresponds to x, and write

$$\nabla_{X,Y} A(x, f(x)) := \big(A_X(x, f(x)), A_Y(x, f(x))\big).$$

Suppose that $A_Z(x, f(x)) = 0$. Differentiating the two formal line identities at $t = 0$ gives

$$\langle u_L, \nabla_{X,Y} A(x, f(x))\rangle = 0, \qquad \langle u_M, \nabla_{X,Y} A(x, f(x))\rangle = 0.$$

Here $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{F}_q^2$. Because L and M are distinct lines through the same point, u_L and u_M are linearly independent and therefore form a basis of $\mathbb{F}_q^2$. Hence

$$A_X(x, f(x)) = A_Y(x, f(x)) = 0.$$

Together with $A_Z(x, f(x)) = 0$, this would imply

$$(\lambda_X A_X + \lambda_Y A_Y + \lambda_Z A_Z)(x, f(x)) = 0,$$

which is a contradiction because such points were removed from consideration in the pruning process. Thus $A_Z(x, f(x)) \neq 0$ at every remaining point.

3. Componentwise propagation. We can now choose a point b in the remaining incidence graph. The Newton–Hensel lifting argument underlying [HKSS24] produces a degree-d bivariate polynomial Q whose restrictions agree with the remaining line polynomials on sufficiently many lines through b and which satisfies $A(X, Y, Q(X, Y)) \equiv 0$. This gives agreement in one local neighborhood of b, but that local agreement alone is not enough to obtain the desired $\Omega(\varepsilon)$ agreement with f.

This is the key step which goes beyond the analysis of [HKSS24]. There, the final agreement is obtained from the lines through the lifting point. Here, the simple-root property from Step 2 allows the same polynomial to have agreement with the line polynomials P_L for lines L throughout an entire connected component of the incidence graph.

Indeed, we extend this agreement through the component by a breadth-first search, starting from the lines through b on which agreement is already known. At each step, suppose $Q|_L = P_L$ for a line L already reached by the search, and let (x, L) be an edge of the remaining incidence graph. Then $Q(x) = P_L(x) = f(x)$. For any other line M adjacent to x, parameterize M so that $M(0) = x$. Since $A_Z(x, f(x)) \neq 0$, $p_M(t)$ is the unique polynomial of degree at most d satisfying $p_M(0) = f(x)$ and $A(M(t), p_M(t)) \equiv 0$. The restriction $Q(M(t))$ also satisfies these conditions, so $Q|_M = P_M$. The search can therefore extend from L through x to M. Repeating this step extends the agreement to every line and point in the connected component.

Finally, expansion of the point–line incidence graph shows that every connected component of the pruned graph contains an $\Omega(\varepsilon)$ fraction of the points. Since $Q(x) = f(x)$ throughout such a component, the propagation step converts one local lift into $\Omega(\varepsilon)$ global agreement. In [HKSS24], the agreement obtained at this step is $\Omega(\varepsilon^4)$; componentwise propagation improves it to $\Omega(\varepsilon)$.

4. Higher dimensions and additive agreement loss. Our last step is to bootstrap the bivariate result to arbitrary dimension. Here we leverage the recent plane–versus–plane theorem of Bhangale and Richelson [BR26], which has soundness $\Omega(d/q)$. A simple conversion shows that plane–point agreement β induces plane–versus–plane acceptance $\Omega(\beta^2)$; thus their theorem applies once $\beta \geq \Omega((d/q)^{1/2})$.

Most affine planes inherit line–versus–point acceptance $\Omega(\varepsilon)$. Applying the bivariate theorem on each such plane produces a degree-d plane polynomial with $\Omega(\varepsilon)$ agreement with the point table. The plane–versus–plane theorem then glues these plane polynomials into one global polynomial, and point–plane expansion recovers $\Omega(\varepsilon)$ agreement with f.

The result of Bhangale and Richelson allows us to pass directly from the bivariate theorem to a consistent global polynomial while preserving both the cubic threshold and the linear agreement. This step uses well known techniques and we do not view it as our main contribution.

Finally, to obtain the additive bound, we apply the multiplicative bound to find a degree-d polynomial whose agreement with the point table is at least a constant fraction of the acceptance probability. We then try to "delete" this agreement: at previously unchanged points where this polynomial agrees with f, replace the point-table values by x_1^{d+1}. Keeping the line table fixed, we apply the multiplicative bound again to the modified point table.

By Schwartz–Zippel, any degree-d polynomial agrees with x_1^{d+1} on at most a $(d+1)/q$ fraction of all points. Thus the replaced points can contribute very little to the agreement of any polynomial found in a later round. While acceptance is above a sufficiently large constant multiple of $(d/q)^{1/3}$, each application must therefore find agreement with f on an $\Omega((d/q)^{1/3})$ fraction of previously unchanged points. These sets are disjoint, so the process stops after $O((q/d)^{1/3})$ rounds. The original accepted incidences at points that have not been replaced then have measure $O((d/q)^{1/3})$.

Assign each line L to the first recovered polynomial whose restriction to L equals P_L. A root count and mixing over the disjoint point and line sets give one polynomial with agreement at least

acceptance minus $O((d/q)^{1/3})$.

5. A counterexample. For the sake of exposition, we first describe a simpler construction in dimension two, using only degree-one line polynomials. This construction achieves acceptance probability $(1+o(1))q^{-1/3}$, while every polynomial of total degree at most d has agreement $O(dq^{-2/3})$ with the point table. Thus the global agreement is a vanishing fraction of acceptance whenever $d = o(q^{1/3})$, and at every fixed degree this construction shows that the soundness exponent $1/3$ is optimal. We then briefly describe how to lift this construction to degree-d line polynomials, which extends the separation to degrees $d = o(q^{2/3})$.

Our construction works over the field $\mathbb{F}_{s^3}$, where $q = s^3$ and $s \geq 3$ is a prime power. We first describe the line table $\{p_L\}$. We will arrange that, for almost every point x, there are s^2 lines $L \ni x$ for which $p_L(x)$ takes the same value.

For every pair $u, v \in \mathbb{F}_{s^3}$ that is linearly independent over $\mathbb{F}_s$, consider the affine line given by

$$L_{u,v}(t) = (u^s, v^s) + t(u, v)$$

and have $p_L : L \to \mathbb{F}_q$ assign the value t at $L_{u,v}(t)$. Set the remaining line polynomials to zero, i.e. lines that are not of the form above. A simple counting argument shows that only a negligible fraction of lines are assigned zero in this latter way. At each point x, we construct the point table f by setting $f(x)$ to be the most popular value assigned by the line polynomials on lines through x.

The key property of this line construction is that, at every point $x = (x_1, x_2)$ with independent coordinates over $\mathbb{F}_s$ (i.e. neither coordinate is an $\mathbb{F}_s$ multiple of the other), there is a value α such that $p_L(x) = \alpha$ for s^2 distinct $L \ni x$. This is arranged through the following property. A line $L_{u,v}$ through x assigns the value α exactly when

$$u^s + \alpha u = x_1, \qquad v^s + \alpha v = x_2.$$

Hence, we just need to show that there are many solutions (u, v) to the pair of equations above. To this end, note that the map $T_\alpha(z) = z^s + \alpha z$ is $\mathbb{F}_s$-linear, hence each equation has either no solution or exactly s solutions if T_α has a nontrivial kernel. For $\alpha = -w^{s-1}$ with $w \neq 0$, this map has kernel $w\mathbb{F}_s$. If either equation has a solution z_0, its solutions form the affine line $z_0 + w\mathbb{F}_s$, consisting of the s values $z_0 + cw$ with $c \in \mathbb{F}_s$. Thus, whenever both equations have a solution, there are s choices for u and, independently, s choices for v. These s^2 pairs give distinct lines through x, all assigning the same value α. For every other α, there is exactly one solution pair. By a straightforward counting argument on the lines through x we can show that there is one value with s^2 agreeing line polynomials. Moreover, this is the most popular value over lines through x, so it equals $f(x)$.

Altogether, since there are $q + 1$ lines through each point, this gives acceptance at least $(1 - o(1))q^{-1/3}$.

To get low global agreement, we use the following property of the point table. For every $x = (x_1, x_2)$ with coordinates linearly independent over $\mathbb{F}_s$, there is a nonzero constant C_x such that

$$f(tx)^s = C_x t^{s-1} \qquad \forall t \in \mathbb{F}_{s^3}.$$

To see this, fix $t \neq 0$ and let $r = t^{s^2}$, so $r^s = t$. Multiplication by t pairs the lines $L_{u,v}$ through x with the lines $L_{ru,rv}$ through tx, since

$$tL_{u,v}(\alpha) = L_{ru,rv}((t/r)\alpha).$$

Thus a value α assigned at x becomes $(t/r)\alpha$ at tx. The most popular value is therefore multiplied by t/r, giving $f(tx) = (t/r)f(x)$. Taking sth powers gives the identity above with $C_x = f(x)^s$.

We now use this identity to bound agreement. For fixed x, the points tx form the line through 0 and x. In particular, this shows that any polynomial Q of total degree at most d agrees with f on at most a ds/q fraction of this line. Indeed, every agreement $Q(tx) = f(tx)$ implies

$$Q(tx)^s - C_x t^{s-1} = 0.$$

The left-hand side has degree at most ds in t. It is nonzero because every exponent in $Q(tx)^s$ is divisible by s, whereas $s-1$ is not. Thus there are at most ds values of t giving agreement.

Every nonzero point lies on exactly one line through the origin. Only $s+1$ of these lines have directions with dependent coordinates over $\mathbb{F}_s$. Together they contain an $O(s/q)$ fraction of all points. Even allowing agreement at all of these points gives global agreement $O(ds/q)+O(s/q) = O(dq^{-2/3})$.

To extend the separation to higher degrees, we use the freedom to choose degree-d line polynomials. We add a coordinate z and let $f_d(x, z) = f(x)H(z)$, where H is monic of degree $d-1$ with distinct roots in $\mathbb{F}_q$. On each line whose projection onto the first two coordinates is a line L, multiply P_L by $H(z)$. This has degree at most d. On lines with constant projection, use the exact restriction of f_d. At every root of H, the new point and line tables both take the value zero. On lines with nonconstant projection, agreement away from these roots is exactly the same as in the original construction. The resulting acceptance is

$$d/q + (1 - d/q)q^{-1/3} + O(q^{-2/3}).$$

To bound global agreement, fix a polynomial $Q(x_1, x_2, z)$ of total degree at most d, and let $B(x_1, x_2)$ be its coefficient of z^{d-1}. Then B has degree at most one, so the above shows that $B(x) = f(x)$ on at most an $O(q^{-2/3})$ fraction of points x. For every other x, the coefficients of z^{d-1} in $Q(x, z)$ and $f(x)H(z)$ differ, since H is monic. Thus their difference is a nonzero polynomial in z of degree at most d, and they agree for at most d values of z. Counting every z as an agreement when $B(x) = f(x)$ gives

$$\operatorname{agr}(f_d, Q) \le d/q + O(q^{-2/3}).$$

This is a vanishing fraction of acceptance whenever $d = o(q^{2/3})$, as desired.

1.3 Further directions

Our results characterize the optimal soundness exponent of the line–versus–point test when the bound is required to hold uniformly over degrees and finite fields. Better soundness may still be possible in the cases left open by our construction, and for higher-dimensional variants such as the plane–versus–point test.

Line versus point. The main cases left open are prime fields, their quadratic extensions, and degrees $d = \Omega(q^{2/3})$, or equivalently rates $d/q = \Omega(q^{-1/3})$. Even over cubic extensions, for growing degrees $d = q^\gamma$ with $0 < \gamma < 2/3$, the soundness threshold is only known to lie between order $q^{-1/3}$ and order $(d/q)^{1/3}$. Can soundness below the cube-root threshold be shown in any of these cases?

Higher-dimensional subspace versus point. It would be interesting to characterize the optimal soundness exponent when the line table is replaced by a table of degree-d polynomials on k-dimensional affine subspaces, for $2 \le k < m$. The best general result we know is a soundness of $O(\sqrt{d/q})$, which follows from the plane–versus–plane analysis of Bhangale and Richelson [BR26].

1.4 Organization

Section 2 records the algebraic and incidence tools. Section 3 proves the bivariate multiplicative bound. Section 4 handles higher dimensions and proves Theorem 1.1 by the list-decoding reduction. Section 5 gives the counterexample and its agreement bounds.

2 Preliminaries

2.1 Basic notation

Throughout, we fix $\mathbb{F}_q$ to be a finite field of size q and characteristic p. For variables $X_1, \dots, X_m$, we write $\mathbb{F}_q[X_1, \dots, X_m]$ for the ring of formal polynomials over $\mathbb{F}_q$. The total degree of a monomial $X_1^{e_1} \cdots X_m^{e_m}$ is $e_1 + \cdots + e_m$, and the total degree $\deg A$ of a polynomial A is the maximum total degree of a monomial appearing in A with nonzero coefficient. We write $\deg_{X_i} A$ for the maximum exponent of X_i appearing in such a monomial.

Derivatives always mean formal derivatives. Thus $\partial_X A$, $\partial_Y A$, and $\partial_Z A$ denote the usual formal partial derivatives of $A \in \mathbb{F}_q[X, Y, Z]$. Two polynomials are *coprime* if their only common divisors are nonzero constants. A nonconstant polynomial is *irreducible* if it cannot be written as a product of two nonconstant polynomials. A polynomial is *squarefree* if no irreducible factor occurs with multiplicity greater than one.

We identify polynomial functions on $\mathbb{F}_q^m$ with their unique reduced representatives, whose degree in each variable is at most $q - 1$. Thus $Z^q = Z$ as a function on $\mathbb{F}_q$, although not as a formal polynomial identity. Factorization, coprimality, irreducibility, and derivatives always refer to the corresponding formal polynomials.

An affine k-dimensional subspace of $\mathbb{F}_q^m$ is a translate $a + W$ of a k-dimensional linear subspace $W \subseteq \mathbb{F}_q^m$. It has the form $\{a + u_1 t_1 + \cdots + u_k t_k : t_1, \dots, t_k \in \mathbb{F}_q\}$, where $u_1, \dots, u_k$ are linearly independent. Affine subspaces of dimensions 0, 1, and 2 are called points, lines, and planes, respectively. For each line L, we use the fixed parameterization $L(t)$ introduced in the introduction. We write $\mathcal{L}$ for the set of affine lines in $\mathbb{F}_q^2$.

2.2 Measure and incidence

Throughout we use $\mu(S)$ to denote the measure of a set S with respect to its appropriate ambient set. For example, when working over $\mathbb{F}_q^2$, we write

$$\mu(X) = \frac{|X|}{q^2}, \qquad \mu(T) = \frac{|T|}{q(q+1)}, \qquad \mu(E) = \frac{|E|}{q^2(q+1)},$$

when X, T, E are sets of points, lines, and point-line incidences in $\mathbb{F}_q^2$ respectively.

We use the following specialization of the expander-mixing lemma for points and affine subspaces, which we cite from a special case of [LS16, Theorem 1].

Lemma 2.1. *Let $1 \le k < m$, let $X \subseteq \mathbb{F}_q^m$, and let $\mathcal{T}$ be a family of affine k-dimensional subspaces. Then*

$$\left| \Pr_{W,\, x \in W}[x \in X,\ W \in \mathcal{T}] - \mu(X)\mu(\mathcal{T}) \right| \le \lambda_{m,k} \sqrt{\mu(X)\mu(\mathcal{T})}, \qquad \lambda_{m,k}^2 = \frac{q^{m-k} - 1}{q^m - 1}.$$

In particular, $\lambda_{2,1} = 1/\sqrt{q+1}$, $\lambda_{m,1} \le q^{-1/2}$, and $\lambda_{m,2} \le q^{-1}$.

2.3 Weighted degree and monomial counts

We assign weights $1, 1, d$ to X, Y, Z, respectively. Thus the weighted degree of the monomial $X^iY^jZ^k$ is $i + j + dk$. Henceforth, "weighted degree" always refers to this choice of weights, and for $A \in \mathbb{F}_q[X, Y, Z]$ we write

$$\mathrm{wdeg}(A) := \max\{i + j + dk : [X^iY^jZ^k]A \neq 0\}.$$

In particular, if $\deg p_L \leq d$, then $\deg_t A(L(t), p_L(t)) \leq \mathrm{wdeg}(A)$.

Let $\mathcal{V}_D$ be the space of trivariate polynomials of weighted degree at most D. We use the following coefficient count.

Lemma 2.2. *For every $D \geq 0$ and $d \geq 1$,*

$$\dim \mathcal{V}_D > \frac{D^3}{6d}.$$

Proof. The simplex $x, y, z \geq 0$, $x + y + dz \leq D$ has volume $D^3/(6d)$. Taking coordinatewise floors places each point in a unit box indexed by a monomial of weighted degree at most D. These boxes cover the simplex and a positive volume outside it, proving the bound. □

2.4 Algebraic tools

Here we record some standard algebraic facts about polynomials over finite fields.

The following is the finite-field version of the standard generic-direction argument. We include the proof because the direction must be chosen over $\mathbb{F}_q$ itself.

Lemma 2.3. *Let $B \in \mathbb{F}_q[X, Y, Z]$ be nonzero and squarefree with $\deg B < q$. There is a vector $\lambda = (\lambda_X, \lambda_Y, \lambda_Z) \in \mathbb{F}_q^3$ such that*

$$\mathcal{D}_\lambda := \lambda_X \partial_X + \lambda_Y \partial_Y + \lambda_Z \partial_Z$$

satisfies

$$\gcd(B, \mathcal{D}_\lambda B) = 1.$$

Proof. Factor

$$B = c \prod_{i=1}^{s} P_i,$$

where $c \in \mathbb{F}_q^\times$ and the P_i are distinct nonconstant irreducible polynomials.

For every i, at least one of

$$(P_i)_X, \qquad (P_i)_Y, \qquad (P_i)_Z$$

is nonzero. Otherwise every exponent appearing in P_i would be divisible by p. Since every coefficient in $\mathbb{F}_q$ has a p-th root, P_i would then be a nontrivial p-th power, contradicting irreducibility.

For each i, the set

$$H_i := \{\lambda \in \mathbb{F}_q^3 : \mathcal{D}_\lambda P_i = 0\}$$

is therefore a proper linear subspace of $\mathbb{F}_q^3$, and hence has size at most q^2. Since

$$s \leq \deg B < q,$$

we have

$$\left|\bigcup_{i=1}^{s} H_i\right| \le sq^2 < q^3.$$

Choose λ outside this union. Then $\mathcal{D}_\lambda P_i \neq 0$ for every i. Moreover,

$$\deg(\mathcal{D}_\lambda P_i) < \deg P_i,$$

so $P_i \nmid \mathcal{D}_\lambda P_i$.

By the product rule,

$$\mathcal{D}_\lambda B = c\sum_{i=1}^{s}(\mathcal{D}_\lambda P_i)\prod_{j\neq i} P_j.$$

Reducing modulo P_i gives

$$\mathcal{D}_\lambda B \equiv c(\mathcal{D}_\lambda P_i)\prod_{j\neq i} P_j \pmod{P_i}.$$

The right-hand side is nonzero modulo P_i, since the P_j are distinct and $P_i \nmid \mathcal{D}_\lambda P_i$. Hence $P_i \nmid \mathcal{D}_\lambda B$ for every i. Since the P_i are exactly the irreducible factors of B, the two polynomials are coprime. □

The next lemma is a standard extension of the Schwartz–Zippel lemma.

Lemma 2.4. *Let $R \in \mathbb{F}_q[X,Y]$ have total degree at most D. If $R(L(t)) \equiv 0$ for more than D distinct affine lines L, then $R = 0$.*

Proof. The linear equation defining each such line divides R, so a nonzero R would have more than D distinct linear factors. □

We next recall the resultant. Write

$$U(X,Y,Z) = a_0(X,Y) + a_1(X,Y)Z + \cdots + a_r(X,Y)Z^r,$$

$$V(X,Y,Z) = b_0(X,Y) + b_1(X,Y)Z + \cdots + b_s(X,Y)Z^s,$$

where $r = \deg_Z U$ and $s = \deg_Z V$. The Z-Sylvester matrix of U and V, denoted $\mathrm{Syl}_Z(U,V)$, is the $(r+s)\times(r+s)$ matrix whose rows are the coefficient vectors of the polynomials

$$Z^{s-1}U, \ldots, ZU, U,\ Z^{r-1}V, \ldots, ZV, V.$$

Specifically, it has the form

$$\mathrm{Syl}_Z(U,V) = \begin{pmatrix} a_r & a_{r-1} & \cdots & a_0 & & & \\ & a_r & a_{r-1} & \cdots & a_0 & & \\ & & \ddots & & & \ddots & \\ & & & a_r & a_{r-1} & \cdots & a_0 \\ b_s & b_{s-1} & \cdots & b_0 & & & \\ & b_s & b_{s-1} & \cdots & b_0 & & \\ & & \ddots & & & \ddots & \\ & & & b_s & b_{s-1} & \cdots & b_0 \end{pmatrix},$$

The Z-resultant is

$$\mathrm{Res}_Z(U,V) := \det \mathrm{Syl}_Z(U,V) \in \mathbb{F}_q[X,Y].$$

We use the following standard facts about the resultant.

(i) If U and V have positive Z-degree and are coprime in $\mathbb{F}_q[X, Y, Z]$, then

$$\mathrm{Res}_Z(U, V) \neq 0.$$

(ii) Let

$$\varphi : \mathbb{F}_q[X, Y] \longrightarrow R$$

be any ring homomorphism. Applying φ entrywise to the Sylvester matrix gives

$$\varphi(\mathrm{Res}_Z(U, V)) = \det(\varphi(\mathrm{Syl}_Z(U, V))).$$

In particular, if $r + s > 0$ and the specialized polynomials

$$\varphi(U), \varphi(V) \in R[Z]$$

have a common root in a field containing R, then

$$\varphi(\mathrm{Res}_Z(U, V)) = 0.$$

Indeed, if α is such a common root, then the specialized Sylvester matrix is singular.

(iii) If U and V have total degrees at most u and v, respectively, then

$$\deg \mathrm{Res}_Z(U, V) \leq uv.$$

For a line L with associated line-polynomial p_L, its polynomial graph is

$$\{(L(t), p_L(t)) : t \in \mathbb{F}_q\} \subseteq \mathbb{F}_q^3.$$

The next lemma bounds the number of affine lines whose polynomial graphs can lie simultaneously in the zero sets of two coprime polynomials.

Lemma 2.5. *Let $U, V \in \mathbb{F}_q[X, Y, Z]$ be coprime nonzero polynomials of total degrees u and v. Let $\mathcal{T} \subseteq \mathcal{L}$ be a family of affine lines, each with a univariate polynomial P_L of degree at most d, such that*

$$U(L(t), p_L(t)) \equiv V(L(t), p_L(t)) \equiv 0 \qquad \forall L \in \mathcal{T}.$$

Then

$$|\mathcal{T}| \leq uv.$$

Proof. If one of U, V is a nonzero constant, then $\mathcal{T}$ is empty. If one has Z-degree zero but is nonconstant, the claim follows directly from Lemma 2.4.

Assume now that both have positive Z-degree, and let $W(X, Y) := \mathrm{Res}_Z(U, V)$. Since U and V are coprime, property (i) gives $W \neq 0$, and property (iii) gives $\deg W \leq uv$.

Fix $L \in \mathcal{T}$, and consider the substitution homomorphism

$$\varphi_L : \mathbb{F}_q[X, Y] \longrightarrow \mathbb{F}_q[t], \qquad F(X, Y) \longmapsto F(L(t)).$$

Viewed as polynomials in Z over the field $\mathbb{F}_q(t)$, the specializations $U(L(t), Z)$ and $V(L(t), Z)$ have the common root $Z = p_L(t)$. Hence property (ii) gives

$$W(L(t)) = \varphi_L(W) = \det(\varphi_L(\mathrm{Syl}_Z(U, V))) \equiv 0.$$

Thus W vanishes identically on every line in $\mathcal{T}$. Since $W \neq 0$, Lemma 2.4 implies $|\mathcal{T}| \leq \deg W \leq uv$. □

The following standard statement follows from Newton iteration; see [Bür00]. The form stated below appears as Lemma 2.10 of [HKSS24].

Lemma 2.6. *Let $B \in \mathbb{F}_q[X, Y, Z]$ and $\alpha \in \mathbb{F}_q$ satisfy*

$$B(0, 0, \alpha) = 0, \qquad B_Z(0, 0, \alpha) \neq 0.$$

For every integer $k \geq 0$, there is a unique polynomial $\Phi_k \in \mathbb{F}_q[X, Y]$ of total degree at most k such that

$$\Phi_k(0, 0) = \alpha, \qquad B(X, Y, \Phi_k(X, Y)) \equiv 0 \pmod{(X, Y)^{k+1}}.$$

The same statement, with the same proof, holds for $B \in \mathbb{F}_q[T, Z]$ with one variable T in place of X, Y; we call this the one-variable form of Lemma 2.6.

The next lemma is the lifting step from [HKSS24, Lemma 3.1], in the form needed here. We include its short proof.

Lemma 2.7. *Suppose $B \in \mathbb{F}_q[X, Y, Z]$ has weighted degree at most D. Let $b \in \mathbb{F}_q^2$ and $\alpha \in \mathbb{F}_q$ satisfy*

$$B(b, \alpha) = 0, \qquad B_Z(b, \alpha) \neq 0.$$

Suppose $\mathcal{U}$ contains more than D distinct lines through b, and each $L \in \mathcal{U}$ carries a polynomial P_L of degree at most d such that

$$P_L(b) = \alpha, \qquad B(L(t), p_L(t)) \equiv 0.$$

Then there is a polynomial $Q \in \mathbb{F}_q[X, Y]$ of total degree at most d such that

$$B(X, Y, Q(X, Y)) \equiv 0 \qquad \text{and} \qquad Q|_L = P_L \qquad \forall L \in \mathcal{U}.$$

Proof. After translating, assume $b = 0$ and write $L(t) = tu_L$ for each $L \in \mathcal{U}$. Let $Q = \Phi_d$ be the polynomial given by Lemma 2.6 with $k = d$. For each $L \in \mathcal{U}$, the condition $B_Z(0, 0, \alpha) \neq 0$ and the one-variable form of Lemma 2.6 imply that $p_L(t)$ is the unique polynomial of degree at most d satisfying

$$p_L(0) = \alpha, \qquad B(tu_L, p_L(t)) \equiv 0 \pmod{t^{d+1}}.$$

The restriction $Q(tu_L)$ satisfies the same conditions, so $Q|_L = P_L$.

Let $R(X, Y) = B(X, Y, Q(X, Y))$. Since $\deg Q \leq d$ and $\mathrm{wdeg}(B) \leq D$, we have $\deg R \leq D$. Moreover, $R(L(t)) \equiv 0$ for every $L \in \mathcal{U}$. As $|\mathcal{U}| > D$, Lemma 2.4 gives $R = 0$, as desired. □

3 The Bivariate Soundness Bound

Towards the proof of Theorem 1.1, we first prove the following weaker version, in which global agreement is a constant fraction of acceptance. We prove the two-dimensional case here and the higher-dimensional case in the next section. We then obtain Theorem 1.1 by a straightforward list-decoding reduction.

Theorem 3.1. *There are absolute constants $C, c > 0$ with the following property. Let $m \geq 2$ and $d \geq 1$, and let $\varepsilon \geq C\left(\frac{d}{q}\right)^{1/3}$. Suppose that $f \colon \mathbb{F}_q^m \to \mathbb{F}_q$ and a degree-d line table $\{P_L\}_{L \in \mathcal{L}_{q,m}}$ satisfy $\mathrm{Acc}(f, \{P_L\}) \geq \varepsilon$. Then there is a polynomial $Q \in \mathbb{F}_q[X_1, \ldots, X_m]$ of total degree at most d such that $\mathrm{agr}(f, Q) \geq c\varepsilon$.*

An earlier version of this paper [KTZ26] proved Theorem 3.1 with a more involved interpolation step.

We first prove the case $m = 2$. Throughout this section, assume $\mathrm{Acc}(f, \{P_L\}) \geq \varepsilon \geq C(d/q)^{1/3}$, where C is a sufficiently large absolute constant. Let

$$D = \left\lceil 4(dq^2)^{1/3} \right\rceil, \qquad k = \left\lfloor \frac{\varepsilon q}{1000} \right\rfloor.$$

For sufficiently large C, these parameters satisfy

$$D < k < q, \qquad k \geq \frac{\varepsilon q}{2000}, \qquad \frac{1}{\sqrt{q+1}} \leq \frac{\varepsilon}{8000} \leq \frac{k}{4q}.$$

3.1 Step 1: Interpolating the point table

We now construct a squarefree interpolant of low weighted degree. This polynomial will vanish identically on every line with more than D accepted points.

Lemma 3.2. *There is a nonzero squarefree polynomial $A \in \mathbb{F}_q[X, Y, Z]$ with $\mathrm{wdeg}(A) \leq D$ such that*

$$A(x, f(x)) = 0 \qquad \forall x \in \mathbb{F}_q^2.$$

Proof. Write $\widetilde{A} \in \mathcal{V}_D$ with unknown coefficients. For each $x \in \mathbb{F}_q^2$, the condition $\widetilde{A}(x, f(x)) = 0$ imposes one homogeneous linear condition. By Lemma 2.2, $\dim \mathcal{V}_D > q^2$, so there is a nonzero solution.

Now to find the squarefree polynomial, let A be the product of the distinct irreducible factors of $\widetilde{A}$. Then A is squarefree, $\mathrm{wdeg}(A) \leq D$, and A vanishes at the same points as $\widetilde{A}$, giving the desired squarefree polynomial. □

3.2 Step 2: Removing exceptional incidences

Fix A from Lemma 3.2. To apply Lemma 2.6, we need every remaining $p_L(t)$ to solve $A(L(t), Z) = 0$ and every remaining value $f(x) = P_L(x)$ to be a simple root of $A(x, Z)$.

Since $\deg A \leq D < q$, Lemma 2.3 lets us choose

$$S = \lambda_X A_X + \lambda_Y A_Y + \lambda_Z A_Z, \qquad \gcd(A, S) = 1.$$

First delete every accepted incidence on a line L with $A(L(t), p_L(t)) \not\equiv 0$. For each such line, let $g_L(t) = A(L(t), p_L(t))$, so $\deg g_L \leq D$. At each accepted point $x \in L$, we have $P_L(x) = f(x)$, so $A(x, P_L(x)) = 0$. Thus each such line has at most D accepted points, and these deletions lose incidence measure at most D/q.

For each remaining line with $S(L(t), p_L(t)) \equiv 0$, both A and S vanish on the same line graph, so Lemma 2.5 bounds the number of such lines by $D(D-1)$. Delete every accepted incidence on these lines and every remaining incidence (x, L) with $S(x, f(x)) = 0$. The total measure deleted is at most

$$\frac{2D}{q} + \frac{D(D-1)}{q(q+1)} \leq \frac{\varepsilon}{4},$$

because the restriction of S to every other line is a nonzero polynomial of degree at most D. The last inequality holds for sufficiently large C, since $D/q = O((d/q)^{1/3})$.

After these deletions, repeatedly remove vertices of degree below k.

Lemma 3.3. *The pruning process leaves a nonempty graph G, with point side X, line side T, and edge set E. It has minimum degree at least k and $\mu(E) \geq \varepsilon/2$. Moreover:*

(i) for every $x \in X$,

$$A(x, f(x)) = 0, \qquad A_Z(x, f(x)) \neq 0;$$

(ii) for every $L \in T$,

$$A(L(t), p_L(t)) \equiv 0.$$

Proof. The first deletion leaves at least $3\varepsilon/4$ fraction of incidences as edges in the graph. Then deleting the low degree vertices removes less than

$$\frac{k}{q+1} + \frac{k}{q} < \frac{\varepsilon}{16},$$

fraction of incidences so we end up with $\mu(E) \geq \varepsilon/2$.

Every remaining line satisfies $A(L(t), p_L(t)) \equiv 0$, and every remaining point x satisfies $S(x, f(x)) \neq 0$. By Lemma 3.2, $A(x, f(x)) = 0$ at every point. To prove $A_Z(x, f(x)) \neq 0$, choose two distinct incident lines, parameterized so that $L(0) = M(0) = x$, with independent directions u_L, u_M. Differentiating their two identities at x gives

$$\langle u_L, \nabla_{X,Y} A \rangle + p_L'(0) A_Z = 0, \qquad \langle u_M, \nabla_{X,Y} A \rangle + p_M'(0) A_Z = 0.$$

If $A_Z = 0$, independence forces $A_X = A_Y = 0$, contradicting $S(x, f(x)) \neq 0$. □

3.3 Step 3: Lifting and propagation

Next, we show that no connected component of G is too small. This follows from the minimum-degree conditions in G and the expansion properties of the point versus line incidence graph.

Lemma 3.4. *Let (X', T') be the point and line sides of a connected component of G. Then*

$$\mu(X'), \mu(T') \geq \frac{k}{4q} \geq \frac{\varepsilon}{8000}.$$

Proof. Let E' be the set of edges in the connected component. By assumption on the minimum degree in G, we have

$$\mu(E') \geq \frac{k}{2q} \mu(X'), \qquad \mu(E') \geq \frac{k}{q} \mu(T').$$

By Lemma 2.1,

$$\mu(E') \leq \mu(X')\mu(T') + \frac{\sqrt{\mu(X')\mu(T')}}{\sqrt{q+1}}.$$

Substituting $1/\sqrt{q+1} \leq \varepsilon/8000 \leq k/(4q)$ gives both desired bounds. □

It remains to show that a single degree-d polynomial governs all of the line and point labels in such a component. Starting from one point, Lemma 2.7 constructs the polynomial on the incident lines, and simple-root uniqueness propagates it through the component.

Lemma 3.5. *For every connected component (X', T') of G, there is a polynomial $Q \in \mathbb{F}_q[X, Y]$ of total degree at most d such that $Q(x) = f(x)$ for all $x \in X'$ and $Q|_L = P_L$ for all $L \in T'$.*

Proof. We construct Q from the lines adjacent to one point, then extend agreement through the component by a breadth-first search. At each step, we pass from a line to its adjacent points and then to the other lines adjacent to those points, and show that the agreement propagates along lines in this way.

Choose $b \in X'$ arbitrarily. By Lemmas 2.7 and 3.3, there is a degree-d polynomial Q with $A(X, Y, Q(X, Y)) \equiv 0$ and $Q|_L = P_L$ for every line adjacent to b.

Suppose $Q|_L = P_L$ for a line already reached, and let $(x, L) \in E$. Then $Q(x) = f(x)$. If M is another line adjacent to x, parameterize it with $M(0) = x$. The polynomials $p_M(t)$ and $Q(M(t))$ have degree at most d, take the value $f(x)$ at $t = 0$, and satisfy

$$A(M(t), p_M(t)) \equiv 0, \qquad A(M(t), Q(M(t))) \equiv 0.$$

Since $A_Z(x, f(x)) \neq 0$, the uniqueness case in the one-variable version of Lemma 2.6 gives $Q|_M = P_M$. Connectivity propagates this identity to every line of T', and gives $Q(x) = f(x)$ for every $x \in X'$. □

Proof of Theorem 3.1 for $m = 2$. Let A be the interpolating polynomial from Lemma 3.2, and let G be the graph from Lemma 3.3. Choose a connected component (X', T') of G. By Lemma 3.5, there is a polynomial Q of total degree at most d such that $Q(x) = f(x)$ for every $x \in X'$. By Lemma 3.4,

$$\operatorname{agr}(f, Q) \geq \mu(X') \geq \frac{k}{4q} \geq \frac{\varepsilon}{8000}.$$

Thus the theorem holds for $m = 2$ with $c = 1/8000$. □

4 Bootstrapping from Planes to Arbitrary Dimension

Through Section 4.1 let $m \geq 3$; Section 4.2 again allows every $m \geq 2$. Let $\mathcal{P}_{m,q}$ denote the affine planes in $\mathbb{F}_q^m$. We use the following standard sampler estimates for affine Grassmann graphs; see, for example, [BDN17, BR26].

Lemma 4.1. *For $m \geq 3$ and $q \geq 2$, the following hold.*

(i) *If $a \colon \mathcal{L}_{q,m} \to [0, 1]$, $\bar{a} = \mathbb{E}_L[a(L)]$, and $a(P) = \mathbb{E}_{L \subseteq P}[a(L)]$, then*

$$\mathbb{E}_P[(a(P) - \bar{a})^2] \leq \frac{4}{q}.$$

(ii) *If $\mathcal{B} \subseteq \mathcal{P}_{m,q}$ and $X \subseteq \mathbb{F}_q^m$, then*

$$\left| \Pr_{P \in \mathcal{P}_{m,q},\, x \in P}[P \in \mathcal{B},\ x \in X] - \mu(\mathcal{B})\mu(X) \right| \leq \frac{2}{q}\sqrt{\mu(\mathcal{B})\mu(X)}.$$

(iii) *Fix $x \in \mathbb{F}_q^m$, let $\mathcal{P}_x := \{P \in \mathcal{P}_{m,q} : x \in P\}$, and let G_x be the uniform edge distribution on the graph on $\mathcal{P}_x$ in which P and P' are adjacent when $P \cap P'$ is a line. If $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{P}_x$, and μ_x denotes uniform density in $\mathcal{P}_x$, then*

$$\left| \Pr_{(P,P') \sim G_x}[P \in \mathcal{B}_1,\ P' \in \mathcal{B}_2] - \mu_x(\mathcal{B}_1)\mu_x(\mathcal{B}_2) \right| \leq \frac{1}{q}\sqrt{\mu_x(\mathcal{B}_1)\mu_x(\mathcal{B}_2)}.$$

Proof. For (i), let $b(x) = \mathbb{E}_{L \ni x}[a(L)] - \bar{a}$. The spectral bound in [LS16, Lemma 10] gives

$$\mathbb{E}_x[b(x)^2] \le \frac{1}{q}\mathbb{E}_L[(a(L) - \bar{a})^2] \le \frac{1}{q}.$$

Choose a random plane P and two independent random lines L, L' in P. With probability $q/(q+1)$ they are distinct and intersect. Conditioned on that event, their intersection x is uniform and the lines form a uniform ordered pair of distinct lines through x. Sampling without replacement gives

$$\mathbb{E}[a(L)a(L') \mid x,\ L \neq L'] \le \big(\bar{a} + b(x)\big)^2.$$

On the complementary event the product is at most one. Hence

$$\mathbb{E}_P[a(P)^2] \le \frac{q}{q+1}\left(\bar{a}^2 + \frac{1}{q}\right) + \frac{1}{q+1} \le \bar{a}^2 + \frac{2}{q+1}.$$

Subtracting $\bar{a}^2$ proves (i). Part (ii) follows from Lemma 2.1 with $k = 2$, and (iii) is [BR26, Claim 5]. □

For a plane table T, let G denote the uniform edge distribution on the affine Grassmann graph on $\mathcal{P}_{m,q}$, where P and P' are adjacent when $P \cap P'$ is a line, and define

$$\mathrm{Acc}_{\mathrm{PVP}}(T) := \Pr_{(P,P') \sim G}[T(P)|_{P \cap P'} = T(P')|_{P \cap P'}].$$

We use the following theorem of Bhangale and Richelson [BR26].

Theorem 4.2. *There is an absolute constant $C_{\mathrm{BR}} > 0$ such that, if a degree-d plane table T satisfies*

$$\mathrm{Acc}_{\mathrm{PVP}}(T) \ge C_{\mathrm{BR}}\frac{d}{q},$$

then there is a global polynomial $Q \in \mathbb{F}_q[X_1, \ldots, X_m]$ of total degree at most d satisfying

$$\Pr_{P \in \mathcal{P}_{m,q}}[T(P) = Q|_P] \ge \frac{1}{10}\,\mathrm{Acc}_{\mathrm{PVP}}(T).$$

The consequence we need is the following plane–versus–point statement.

Lemma 4.3. *There are absolute constants $C_{\mathrm{PP}}, c_{\mathrm{PP}} > 0$ such that the following holds for every agreement threshold $\beta \ge C_{\mathrm{PP}}\sqrt{d/q}$. Let $\mathcal{G} \subseteq \mathcal{P}_{m,q}$ have measure at least $1/2$, and suppose that for every $P \in \mathcal{G}$ we are given a degree-d polynomial $T(P)$ satisfying*

$$\Pr_{x \in P}[T(P)(x) = f(x)] \ge \beta.$$

Then there exists a global degree-d polynomial Q such that

$$\Pr_{x \in \mathbb{F}_q^m}[Q(x) = f(x)] \ge c_{\mathrm{PP}}\beta.$$

Proof. Complete the table outside $\mathcal{G}$ by partitioning $\mathcal{P}_{m,q} \setminus \mathcal{G}$ into q classes whose sizes differ by at most one, and assigning a different constant polynomial to each class. Thus every constant is assigned to either

$$\left\lfloor \frac{|\mathcal{P}_{m,q} \setminus \mathcal{G}|}{q} \right\rfloor \quad \text{or} \quad \left\lceil \frac{|\mathcal{P}_{m,q} \setminus \mathcal{G}|}{q} \right\rceil$$

planes. Consequently, if Q is constant, the measure of planes in $\mathcal{P}_{m,q} \setminus \mathcal{G}$ on which $T(P) = Q|_P$ is at most

$$\frac{1}{q} + \frac{1}{|\mathcal{P}_{m,q}|} \leq \frac{1}{q} + \frac{1}{q^3} \leq \frac{2d}{q}.$$

If Q is nonconstant, then $T(P) = Q|_P$ forces $Q|_P$ to be constant. To bound how often this happens, pick a uniformly random plane P and two distinct uniformly random points $y, y' \in P$, so that (y, y') is a uniformly random pair of distinct points of $\mathbb{F}_q^m$. If $Q|_P$ is constant, then $Q(y') = Q(y)$. For each fixed y, the polynomial $Q(Y) - Q(y)$ is nonzero of degree at most d, so by the Schwartz–Zippel lemma $Q(y') = Q(y)$ for at most a $(dq^{m-1} - 1)/(q^m - 1) \leq d/q$ fraction of the points $y' \neq y$. Thus $Q|_P$ is constant on at most a d/q fraction of planes.

In both cases, every global degree-d polynomial Q satisfies $T(P) = Q|_P$ on a set of planes $P \in \mathcal{P}_{m,q} \setminus \mathcal{G}$ of measure at most $2d/q$.

The completed table passes the plane–versus–point test with probability at least $\beta/2$. Let

$$a := \Pr_{P \in \mathcal{P}_{m,q},\, x \in P}[T(P)(x) = f(x)] \geq \frac{\beta}{2}.$$

For each $x \in \mathbb{F}_q^m$, let

$$\mathcal{B}_x := \{P \in \mathcal{P}_x : T(P)(x) = f(x)\}, \qquad \rho_x := \mu_x(\mathcal{B}_x).$$

Double counting gives $\mathbb{E}_x[\rho_x] = a$.

We now compare this plane–point agreement with the plane–versus–plane acceptance distribution. If $(P, P') \sim G$ and then x is chosen uniformly from the line $P \cap P'$, the resulting distribution on triples (x, P, P') is the same as first choosing $x \in \mathbb{F}_q^m$ uniformly and then choosing $(P, P') \sim G_x$. Therefore, by Lemma 4.1(iii),

$$\begin{aligned}
\Pr_{(P,P') \sim G,\, x \in P \cap P'}[T(P)(x) = T(P')(x) = f(x)] &= \mathbb{E}_x \Pr_{(P,P') \sim G_x}[P, P' \in \mathcal{B}_x] \\
&\geq \mathbb{E}_x \left[\rho_x^2 - \frac{\rho_x}{q}\right] \\
&\geq a^2 - \frac{a}{q} \\
&\geq \frac{\beta^2}{4} - \frac{1}{q}.
\end{aligned}$$

If $T(P)|_{P \cap P'}$ and $T(P')|_{P \cap P'}$ are distinct, their difference is a nonzero univariate polynomial of degree at most d, so the two restrictions agree on at most d points of $P \cap P'$. Hence

$$\mathrm{Acc}_{\mathrm{PVP}}(T) \geq \frac{\beta^2}{4} - \frac{d+1}{q},$$

and because $d \geq 1$, we have $(d+1)/q \leq 2d/q$. Since $\beta \geq C_{\mathrm{PP}}\sqrt{d/q}$, taking C_{PP} sufficiently large therefore gives,

$$\mathrm{Acc}_{\mathrm{PVP}}(T) \geq \frac{\beta^2}{8} = \Omega(\beta^2),$$

and also ensures the hypothesis $\mathrm{Acc}_{\mathrm{PVP}}(T) \geq C_{\mathrm{BR}} d/q$ of Theorem 4.2. The theorem therefore gives a global polynomial Q agreeing with T on a plane set $\mathcal{M}$ of measure $\Omega(\beta^2)$. By choosing C_{PP} sufficiently large, the $O(d/q)$ agreement from planes outside $\mathcal{G}$ is a sufficiently small constant multiple of β^2. Thus $\mathcal{M} \cap \mathcal{G}$ still has measure $\Omega(\beta^2)$.

Let $X_Q = \{x : Q(x) = f(x)\}$. Every plane in $\mathcal{M} \cap \mathcal{G}$ contains at least a β-fraction of points from X_Q. Applying Lemma 4.1(ii) gives

$$\beta \le \mu(X_Q) + O\Big(\frac{1}{q\beta}\Big).$$

The error is at most $\beta/2$ once C_{PP} is sufficiently large, which proves the claim. □

4.1 Proof in higher dimensions

Proof of Theorem 3.1 for $m \ge 3$. Let $C_0, c_0 > 0$ be the constants established for the bivariate case, and enlarge the constant C in Theorem 3.1 as needed.

1. Decode almost every plane. For an affine plane P, let

$$\alpha_P := \Pr_{x \in P,\, L \subseteq P,\, x \in L}[P_L(x) = f(x)].$$

Then $\mathbb{E}_P[\alpha_P] \ge \varepsilon$. By Lemma 4.1(i), $\Pr_P[\alpha_P < \varepsilon/2] \le \frac{16}{q\varepsilon^2}$. Thus, after increasing C, the set

$$\mathcal{G} := \{P : \alpha_P \ge \varepsilon/2\}$$

has measure at least $1/2$. Moreover, every $P \in \mathcal{G}$ satisfies $\alpha_P \ge C_0(d/q)^{1/3}$. Applying the bivariate case of Theorem 3.1 inside P, choose a degree-d polynomial $T(P)$ such that

$$\Pr_{x \in P}[T(P)(x) = f(x)] \ge c_0 \alpha_P \ge \frac{c_0 \varepsilon}{2}.$$

2. Complete the plane table and apply the plane–versus–point theorem. Set $\beta = c_0\varepsilon/2$. Since

$$\varepsilon \ge C\left(\frac{d}{q}\right)^{1/3} \ge C\left(\frac{d}{q}\right)^{1/2},$$

we have $\beta \ge C_{\mathrm{PP}}\sqrt{d/q}$ after increasing C once more. Apply Lemma 4.3 to the plane polynomials constructed on $\mathcal{G}$. It produces a global degree-d polynomial Q satisfying

$$\Pr_{x \in \mathbb{F}_q^m}[Q(x) = f(x)] \ge c_{\mathrm{PP}}\beta = \Omega(\varepsilon),$$

which proves Theorem 3.1 for $m \ge 3$. □

4.2 From multiplicative to additive agreement

Now we extend from Theorem 3.1 to prove Theorem 1.1 via a standard list-decoding argument.

Proof of Theorem 1.1. Let $C_0, c_0 > 0$ be the constants in Theorem 3.1, and write

$$\varepsilon = \mathrm{Acc}(f, \{P_L\}), \qquad \rho = \frac{d}{q}, \qquad \alpha = \rho^{1/3}.$$

Choose $K \ge \max\{C_0, 4/c_0, 1\}$ and let $\delta = K\alpha$. We prove the theorem with $C = K + 2$. We may assume $\varepsilon \ge \delta$, since otherwise the desired bound is trivial. In particular, $\alpha \le 1$.

Let $f^{(0)} = f$. We keep the line table $\{P_L\}$ fixed and apply Theorem 3.1 repeatedly while the acceptance is at least δ. Each application gives a degree-d polynomial agreeing with the current

point table on at least a $c_0\delta$ fraction of points. We attempt to delete this agreement by replacing the values at all previously unremoved points where this polynomial agrees with the original f. We replace these values by $g(x) = x_1^{d+1}$. For every polynomial Q of total degree at most d, $X_1^{d+1} \neq Q$, so

$$\Pr_x[g(x) = Q(x)] \leq \frac{d+1}{q} \leq 2\rho$$

by Schwartz–Zippel.

Let $U_0 = \mathbb{F}_q^m$. At round i, if

$$\mathrm{Acc}(f^{(i-1)}, \{P_L\}) \geq \delta,$$

then Theorem 3.1 gives a degree-d polynomial Q_i with $\mathrm{agr}(f^{(i-1)}, Q_i) \geq c_0\delta$. Let

$$X_i = \{x \in U_{i-1} : f(x) = Q_i(x)\}, \qquad U_i = U_{i-1} \setminus X_i.$$

Update the point table by setting

$$f^{(i)}(x) = \begin{cases} f(x), & x \in U_i, \\ g(x), & x \notin U_i. \end{cases}$$

The current table $f^{(i-1)}$ equals g outside U_{i-1}, so Q_i agrees with it there on a set of measure at most 2ρ. Hence

$$\mu(X_i) \geq c_0\delta - 2\rho \geq \frac{c_0\delta}{2},$$

where the last inequality uses $\rho \leq \alpha$ and $c_0 K \geq 4$. The sets X_i are disjoint, so this process stops after

$$k \leq \frac{2}{c_0\delta}$$

rounds, with $\mathrm{Acc}(f^{(k)}, \{P_L\}) < \delta$. Since $f^{(k)} = f$ on U_k, the original accepted incidence measure on U_k is less than δ.

We assign each line L to the first polynomial in the list whose restriction to L is equal to p_L. Write

$$T_i = \{L : P_L = Q_i|_L,\ P_L \neq Q_j|_L \text{ for every } j < i\}.$$

If $x \in X_i$ is accepted on a line with $P_L = Q_i|_L$, then $L \in T_i$: an earlier matching Q_j would give $f(x) = Q_j(x)$, so x would already have been removed. On every line with $P_L \neq Q_i|_L$, there are at most d points with $P_L(x) = Q_i(x)$. Summing over i gives

$$\varepsilon \leq \delta + k\rho + \sum_{i=1}^{k} \Pr_{x, L \ni x}[x \in X_i,\ L \in T_i]. \tag{1}$$

Both the point sets X_i and the line sets T_i are disjoint, so we can sum their mixing errors without a factor depending on k. By Lemma 2.1 with $k = 1$, in every dimension $m \geq 2$ we have

$$\Pr_{x, L \ni x}[x \in X,\ L \in T] \leq \mu(X)\mu(T) + q^{-1/2}\sqrt{\mu(X)\mu(T)}.$$

Let $G = \max_{\deg Q \leq d} \mathrm{agr}(f, Q)$. Since $\mu(X_i) \leq G$, Cauchy–Schwarz gives

$$\sum_{i=1}^{k} \Pr_{x, L \ni x}[x \in X_i,\ L \in T_i] \leq G \sum_i \mu(T_i) + q^{-1/2}\sqrt{\Big(\sum_i \mu(X_i)\Big)\Big(\sum_i \mu(T_i)\Big)}$$

$$\leq G + q^{-1/2}.$$

Substituting in (1), and using $q^{-1/2} \leq \alpha$ since $d \geq 1$, gives

$$\varepsilon \leq G + K\alpha + \frac{2\rho}{c_0 K\alpha} + q^{-1/2} \leq G + (K+2)\alpha.$$

This proves Theorem 1.1. □

5 The Lower-Bound Construction

In this section we prove Theorem 1.2. We work over $\mathbb{F}_{s^3}$ for a prime power $s \geq 3$ and set $q := s^3$. We first construct a point table on $\mathbb{F}_{s^3}^2$ with degree-one line functions, local agreement of order $q^{-1/3}$, and agreement of order at most $q^{-2/3}$ with every affine polynomial. We then add a third coordinate to obtain degree-d line polynomials and global agreement at most $d/q + O(q^{-2/3})$. This gives a vanishing ratio of global to local agreement whenever $d = o(q^{2/3})$.

5.1 The construction and local agreement

Let $\mathcal{L}$ be the set of affine lines in $\mathbb{F}_{s^3}^2$. To start, we assign a linear function $p_L : L \to \mathbb{F}_q$ to each $L \in \mathcal{L}$ so that, at almost every point $x = (x_1, x_2) \in \mathbb{F}_q^2$, many of the lines through x assign the same value at that point.

For each $u, v \in \mathbb{F}_{s^3}$ that are linearly independent over $\mathbb{F}_s$, meaning $u \neq 0$ and $v/u \notin \mathbb{F}_s$, take the line $L_{u,v}$ consisting of the points

$$L_{u,v}(t) = (u^s, v^s) + t(u, v), \qquad \forall t \in \mathbb{F}_{s^3},$$

and assign it the linear function $p_{L_{u,v}}(t) = t$. In words, this is the line through (u^s, v^s) in the direction (u, v), and its line function assigns the value t to the point $(u^s, v^s) + t(u, v)$. Write $\mathcal{L}^\star \subseteq \mathcal{L}$ for this family of lines.

We first check that different pairs give different lines, so that each line is assigned a unique linear function. Suppose (u, v) and (u', v') are independent pairs with $L_{u,v} = L_{u',v'}$. Then the directions of these lines are proportional, so $(u', v') = c(u, v)$ for some $c \in \mathbb{F}_{s^3}^\times$. Since (u'^s, v'^s) lies on $L_{u,v}$, for some $\alpha \in \mathbb{F}_{s^3}$,

$$(c^s - 1)(u^s, v^s) = \alpha(u, v).$$

If $c^s \neq 1$, dividing the two coordinate equations gives $(v/u)^s = v/u$, contradicting $v/u \notin \mathbb{F}_s$. Hence $c^s = 1$, so $c = 1$ and $(u', v') = (u, v)$.

Counting independent pairs gives $|\mathcal{L}^\star| = (q-1)(q-s)$. Each $L_{u,v}$ has slope $k = v/u \notin \mathbb{F}_s$ and nonzero intercept $b = (k^s - k)u^s$. Since there are exactly $(q-s)(q-1)$ lines $x_2 = kx_1 + b$ with $k \notin \mathbb{F}_s$ and $b \neq 0$, these are precisely the lines in $\mathcal{L}^\star$. They form a $1 - O(q^{-2/3})$ fraction of all $q(q+1)$ lines in $\mathcal{L}$. For every $L \in \mathcal{L} \setminus \mathcal{L}^\star$, set $p_L = 0$. This completes the definition of the line polynomials $\{p_L\}$.

From these line polynomials, we define the point table $f : \mathbb{F}_{s^3}^2 \to \mathbb{F}_{s^3}$ to maximize consistency. For each point x, take $f(x)$ to be the most popular value $p_L(x)$ among all lines L containing x. The following lemma establishes local agreement between the point and line tables.

Lemma 5.1. *For every point $x = (x_1, x_2) \in \mathbb{F}_{s^3}^2$ such that x_1, x_2 are linearly independent over $\mathbb{F}_s$, there is a unique value α such that exactly s^2 lines L through x satisfy $p_L(x) = \alpha$. Moreover, this is the most popular value over lines through x.*

Consequently,

$$\mathrm{Acc}(f, \{P_L\}) \geq \frac{(q-1)(q-s)s^2}{q^2(q+1)} = (1 - o(1))q^{-1/3}. \tag{2}$$

Proof. Fix $x = (x_1, x_2)$ with x_1, x_2 independent over $\mathbb{F}_s$. Let $\mathcal{L}^\star_x$ be the set of lines in $\mathcal{L}^\star$ through x. For each $\alpha \in \mathbb{F}_{s^3}$, define

$$B_\alpha = \{L_{u,v} \in \mathcal{L}^\star_x : u^s + \alpha u = x_1,\ v^s + \alpha v = x_2\}.$$

Equivalently, $B_\alpha = \{L \in \mathcal{L}^\star_x : p_L(x) = \alpha\}$, so the sets B_α partition $\mathcal{L}^\star_x$. We will show that $|B_\alpha| = s^2$ for exactly one α.

To count B_α, we solve

$$T_\alpha(u) = x_1, \qquad T_\alpha(v) = x_2, \qquad T_\alpha(z) := z^s + \alpha z. \tag{3}$$

Since T_α is $\mathbb{F}_s$-linear, we look for a nontrivial kernel. Setting $\alpha = -w^{s-1}$ for $w \neq 0$, note that

$$\ker T_\alpha = w\mathbb{F}_s = \{cw : c \in \mathbb{F}_s\}.$$

Indeed, the kernel contains $w\mathbb{F}_s$, and it has at most s elements because T_α is a nonzero polynomial of degree s. Thus each equation in (3) has either no solutions or exactly s solutions. Write

$$\mathcal{A} = \{-w^{s-1} : w \in \mathbb{F}^\times_{s^3}\}, \qquad |\mathcal{A}| = \frac{q-1}{s-1} = s^2 + s + 1.$$

For $\alpha \notin \mathcal{A}$, the map T_α is invertible: a nonzero kernel element w would give $\alpha = -w^{s-1} \in \mathcal{A}$.

Next, we check that every solution u, v defines a line in $\mathcal{L}^\star$, so that solutions indeed correspond to lines in $\mathcal{L}^\star$. Since $x_1 \neq 0$, we have $u \neq 0$. If $v = cu$ for some $c \in \mathbb{F}_s$, then

$$x_2 = T_\alpha(v) = cT_\alpha(u) = cx_1,$$

contradicting the independence of x_1, x_2. Thus u, v are independent over $\mathbb{F}_s$, and distinct solution pairs give distinct lines. It follows that

$$|B_\alpha| = \begin{cases} 1, & \alpha \notin \mathcal{A}, \\ 0 \text{ or } s^2, & \alpha \in \mathcal{A}. \end{cases}$$

We now sum over sizes to show there is an α with $|B_\alpha| = s^2$. First, there are $s+2$ lines through x in $\mathcal{L} \setminus \mathcal{L}^\star$: the s lines with directions $(1, k)$ for $k \in \mathbb{F}_s$, the vertical line, and the line through the origin. Thus $|\mathcal{L}^\star_x| = q - s - 1$. Since the sets B_α partition $\mathcal{L}^\star_x$,

$$\sum_{\alpha \in \mathcal{A}} |B_\alpha| = |\mathcal{L}^\star_x| - \sum_{\alpha \notin \mathcal{A}} |B_\alpha| = (q - s - 1) - (q - |\mathcal{A}|) = s^2.$$

Each summand is either zero or s^2, so there is exactly one $\alpha \in \mathcal{A}$ with $|B_\alpha| = s^2$.

We finally check that this value is the one chosen by f. It is nonzero, and every other nonzero value occurs at most once. Zero is assigned at x by exactly one of the line polynomials on lines of $\mathcal{L}^\star$ through x, and by those on all $s+2$ remaining lines, for $s+3$ votes in total. Since $s^2 > s+3$ for $s \geq 3$, the value α is the unique most popular value.

Finally, to obtain the acceptance probability, recall that there are $(q-1)(q-s)$ points with independent coordinates. At each point, exactly s^2 of the $q+1$ lines containing it have line polynomial agreeing with f. Hence, the acceptance probability is at least

$$\frac{(q-1)(q-s)s^2}{q^2(q+1)},$$

as desired. □

5.2 Global agreement

To bound global agreement, we will count agreements separately on each line through the origin. The only property of f we need is the identity below.

Lemma 5.2. *For every $x = (x_1, x_2)$ with $\mathbb{F}_s$-independent coordinates, there is a constant $C_x \neq 0$ such that*

$$f(tx)^s = C_x t^{s-1} \qquad \forall t \in \mathbb{F}_{s^3}. \tag{4}$$

Proof. Fix $t \neq 0$. We compare the lines through x with those through tx. Let $r = t^{s^2}$, so $r^s = t$. For every $L_{u,v} \in \mathcal{L}^\star$ and $\alpha \in \mathbb{F}_{s^3}$,

$$L_{ru,rv}((t/r)\alpha) = t(u^s, v^s) + t\alpha(u, v) = tL_{u,v}(\alpha).$$

Multiplication by r preserves independence over $\mathbb{F}_s$, so $L_{u,v} \mapsto L_{ru,rv}$ is a bijection between the lines in $\mathcal{L}^\star$ through x and those through tx. By Lemma 5.1, $f(x) \neq 0$, and the s^2 lines assigning $f(x)$ at x correspond to exactly s^2 lines assigning $(t/r)f(x)$ at tx. Since tx also has independent coordinates, Lemma 5.1 gives

$$f(tx) = (t/r)f(x).$$

Raising to the sth power and using $r^s = t$ gives

$$f(tx)^s = (t/r)^s f(x)^s = t^{s-1} f(x)^s.$$

Thus $C_x = f(x)^s \neq 0$ satisfies the identity for $t \neq 0$. It remains to check $t = 0$. No line in $\mathcal{L}^\star$ passes through the origin. Indeed, if $L_{u,v}(\alpha) = 0$, then

$$u^s + \alpha u = 0, \qquad v^s + \alpha v = 0.$$

Thus $u, v \in \ker T_\alpha$, which is either $\{0\}$ or $w\mathbb{F}_s$ for some $w \neq 0$, so u, v cannot be linearly independent over $\mathbb{F}_s$. Therefore every line through the origin lies outside $\mathcal{L}^\star$ and hence $f(0) = 0$ and the identity holds at $t = 0$ as well. □

Lemma 5.3. *Every $Q \in \mathbb{F}_{s^3}[X, Y]$ of total degree at most d satisfies*

$$\operatorname{agr}(f, Q) \leq \frac{ds(q-s) + (s+1)(q-1) + 1}{q^2}.$$

For fixed d, the right-hand side is $(d + 1 + o(1))q^{-2/3}$.

Proof. On a line through the origin with direction x whose coordinates are independent over $\mathbb{F}_s$, Lemma 5.2 implies that every t with $Q(tx) = f(tx)$ satisfies

$$Q(tx)^s - C_x t^{s-1} = 0.$$

As a formal polynomial in t, the left side has degree at most ds. It is nonzero: all exponents in $Q(tx)^s$ are divisible by s, while $C_x \neq 0$ and $s - 1$ is not divisible by s. Thus Q has at most ds agreements with f on this line.

The lines through the origin partition the nonzero points of the plane. Of these lines, $q - s$ have independent directions. The remaining $s + 1$ have slopes in $\mathbb{F}_s$ or are vertical, and contain $(s + 1)(q - 1)$ nonzero points. Counting all these exceptional points and the origin as possible agreements gives the bound. □

5.3 The higher-degree construction

The planar tables above have acceptance $(1 - o(1))q^{-1/3}$ (Lemma 5.1) and agreement $O(dq^{-2/3})$ with every polynomial of total degree at most d (Lemma 5.3). All of the line polynomials in this construction are linear. For $d > 1$, we can use the extra flexibility of degree-d line polynomials to increase acceptance. In particular, we introduce an extra coordinate z and multiply the point table and line polynomials by a degree-$(d-1)$ polynomial $H(z)$. At roots of H, the new point and line tables agree automatically.

Proof of Theorem 1.2. We first prove the result for $m = 3$. Choose distinct $\beta_1, \ldots, \beta_{d-1} \in \mathbb{F}_{s^3}$ and let

$$H(Z) = \prod_{i=1}^{d-1}(Z - \beta_i), \qquad f_d(x, z) = f(x)H(z),$$

for $x = (x_1, x_2) \in \mathbb{F}_{s^3}^2$ and $z \in \mathbb{F}_{s^3}$.

We define the line table by projecting onto the first two coordinates and multiplying by H in the third coordinate. Let $\pi : \mathbb{F}_{s^3}^3 \to \mathbb{F}_{s^3}^2$ be given by $\pi(x_1, x_2, z) = (x_1, x_2)$, and let $\mathcal{V} \subseteq \mathcal{L}_{q,3}$ be the lines parallel to the third coordinate axis. For each line L, write

$$L(t) = (a_1, a_2, a_3) + t(b_1, b_2, b_3), \qquad z_L(t) = a_3 + tb_3.$$

For $L \notin \mathcal{V}$, let $\ell_L = \pi(L)$ with parameterization $\ell_L(t) = \pi(L(t))$. Define

$$P_L^{(d)}(L(t)) = \begin{cases} P_{\ell_L}(\ell_L(t))H(z_L(t)), & L \notin \mathcal{V}, \\ f(a_1, a_2)H(z_L(t)), & L \in \mathcal{V}. \end{cases}$$

For $L \notin \mathcal{V}$, this has degree at most $1 + (d-1) = d$. For $L \in \mathcal{V}$, it is the exact restriction of f_d and has degree at most $d - 1$.

For $L \notin \mathcal{V}$ and $(x, z) \in L$, the definitions give

$$P_L^{(d)}(x, z) = f_d(x, z) \quad \iff \quad H(z) = 0 \text{ or } P_{\ell_L}(x) = f(x). \tag{5}$$

Let $\varepsilon_s = \mathrm{Acc}(f, \{P_\ell\})$. By Lemma 5.1, at every point with independent coordinates exactly s^2 of its $q + 1$ incident lines agree with f. The remaining points have density $O(q^{-2/3})$, so

$$\varepsilon_s = q^{-1/3} + O(q^{-2/3}).$$

Write

$$\theta = \Pr_z[H(z) = 0] = \frac{d-1}{q}, \qquad \nu = \frac{|\mathcal{V}|}{|\mathcal{L}_{q,3}|} = \frac{1}{q^2 + q + 1}.$$

For a uniformly random incident point-line pair $((x, z), L)$, conditioning on $L \notin \mathcal{V}$ makes (x, ℓ_L) a uniform incident point–line pair in $\mathbb{F}_{s^3}^2$, with z independent and uniform. By (5), the conditional acceptance is $\theta + (1 - \theta)\varepsilon_s$. Every incidence with $L \in \mathcal{V}$ is accepted. Hence, uniformly for $1 \le d \le q$,

$$\varepsilon_{s,d} = \nu + (1 - \nu)[\theta + (1 - \theta)\varepsilon_s] = \frac{d}{q} + \left(1 - \frac{d}{q}\right) q^{-1/3} + O(q^{-2/3}),$$

giving the desired acceptance probability for the point table f_d and line functions $\{P_L^{(d)}\}$.

Next we show the global agreement with degree d is small. We fix x and count the number of roots in the third coordinate. Let $Q \in \mathbb{F}_{s^3}[X_1, X_2, Z]$ have total degree at most d. Let c and

$B(X_1, X_2)$ be its coefficients of Z^d and Z^{d-1}, respectively. Since Q has total degree at most d, c is constant and B is affine. For each $x = (x_1, x_2)$, define

$$R_x(Z) = Q(x_1, x_2, Z) - f(x)H(Z).$$

Thus $Q(x_1, x_2, z) = f_d(x, z)$ exactly when $R_x(z) = 0$. If $c \neq 0$, then $\deg R_x = d$ for every x, so R_x has at most d roots and $\operatorname{agr}(f_d, Q) \leq d/q$.

Suppose $c = 0$. Since H is monic, the coefficient of Z^{d-1} in R_x is $B(x) - f(x)$. Hence R_x has degree exactly $d-1$ when $B(x) \neq f(x)$, and has at most $d-1$ roots. When $B(x) = f(x)$, we use the trivial bound of q roots. Averaging over x gives

$$\operatorname{agr}(f_d, Q) \leq \theta + (1-\theta)\operatorname{agr}(f, B). \tag{6}$$

Since B is affine, Lemma 5.3 with $d = 1$ gives

$$\operatorname{agr}(f, B) \leq \frac{s(q-s) + (s+1)(q-1) + 1}{q^2} = (2 + o(1))q^{-2/3}. \tag{7}$$

Combining (6) and (7) gives

$$\operatorname{agr}(f_d, Q) \leq \frac{d-1}{q} + (2 + o(1))q^{-2/3}.$$

Together with the case $c \neq 0$, this gives $\operatorname{agr}(f_d, Q) \leq d/q + (2 + o(1))q^{-2/3}$. Comparing with the acceptance formula gives

$$\operatorname{agr}(f_d, Q) \leq \varepsilon_{s,d} - \left(1 - \frac{d}{q}\right) q^{-1/3} + O(q^{-2/3}),$$

as required.

For $m > 3$, extend the point table by ignoring the last $m-3$ coordinates. Project each line onto the first three coordinates and use its assigned polynomial. If the projection is a point, use the constant value of the point table there. Conditional on a line having nonconstant projection, the projected point–line incidence is uniform. Lines with constant projection have density $(q^{m-3} - 1)/(q^m - 1) = O(q^{-3})$, so the acceptance changes by at most $O(q^{-3})$. Finally, fixing the last $m-3$ coordinates of any degree-d polynomial gives a degree-at-most-d polynomial in three variables. Averaging the three-dimensional agreement bound over these coordinates gives the same global agreement bound in dimension m. □

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AI declaration

In the earlier ECCC version [KTZ26], the authors developed a more convoluted version of the arguments in Sections 3.1 and 3.2 – which required interpolating over a specially chosen set of "random looking" lines – and conjectured the cubic soundness threshold. GPT-5.6 Sol suggested the ideas for Section 3.3, whose details the authors completed in interaction with it. Section 4 uses standard techniques and was written with the help of GPT-5.6 Sol.

Subsequent to this original version, the authors used GPT-6 Astra to discover the soundness lower bound construction in Section 5 and to obtain the more direct variant of Sections 3.1 and 3.2 presented here.

For this revision, we also used GPT-5.6 Sol, GPT-6 Astra, GPT-6.1 Sol, GPT-6.1 Terra, and Claude Opus 5.5, including for exposition and checking the formalization. The authors verified the correctness and originality of all content, including references, and take responsibility for the results, proofs, and exposition.

References

[AKK+05] Noga Alon, Tali Kaufman, Michael Krivelevich, Simon Litsyn, and Dana Ron. Testing Reed–Muller codes. *IEEE Transactions on Information Theory*, 51(11):4032–4039, 2005.

[ALM+98] Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. Proof verification and the hardness of approximation problems. *Journal of the ACM*, 45(3):501–555, 1998.

[AS98] Sanjeev Arora and Shmuel Safra. Probabilistic checking of proofs: A new characterization of NP. *Journal of the ACM*, 45(1):70–122, 1998.

[AS03] Sanjeev Arora and Madhu Sudan. Improved low-degree testing and its applications. *Combinatorica*, 23(3):365–426, 2003.

[BDN17] Amey Bhangale, Irit Dinur, and Inbal Livni Navon. Cube versus cube low-degree test. In *Proceedings of the 8th Innovations in Theoretical Computer Science Conference (ITCS)*, volume 67 of *LIPIcs*, pages 40:1–40:31, 2017.

[BKS+10] Arnab Bhattacharyya, Swastik Kopparty, Grant Schoenebeck, Madhu Sudan, and David Zuckerman. Optimal testing of Reed–Muller codes. In *Proceedings of the 51st Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 488–497, 2010.

[BR26] Amey Bhangale and Silas Richelson. Plane vs. plane low degree test. In *Proceedings of the 2026 Annual ACM–SIAM Symposium on Discrete Algorithms (SODA)*, pages 5172–5198, 2026.

[Bür00] Peter Bürgisser. *Completeness and Reduction in Algebraic Complexity Theory*, volume 7 of *Algorithms and Computation in Mathematics*. Springer, 2000.

[DKK+21] Irit Dinur, Subhash Khot, Guy Kindler, Dor Minzer, and Muli Safra. On non-optimally expanding sets in Grassmann graphs. *Israel Journal of Mathematics*, 243(1):377–420, 2021.

[DKK+25] Irit Dinur, Subhash Khot, Guy Kindler, Dor Minzer, and Muli Safra. Towards a proof of the 2-to-1 games conjecture. *Theory of Computing*, 21(11):1–50, 2025.

[FS95] Katalin Friedl and Madhu Sudan. Some improvements to total degree tests. In *Proceedings of the 3rd Israel Symposium on the Theory of Computing Systems*, pages 190–198, 1995.

[HKSS24] Prahladh Harsha, Mrinal Kumar, Ramprasad Saptharishi, and Madhu Sudan. An improved line-point low-degree test. In *Proceedings of the 65th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 1883–1892, 2024.

[HSS13] Elad Haramaty, Amir Shpilka, and Madhu Sudan. Optimal testing of multivariate polynomials over small prime fields. *SIAM Journal on Computing*, 42(2):536–562, 2013.

[KM22] Tali Kaufman and Dor Minzer. Improved optimal testing results from global hypercontractivity. In *Proceedings of the 63rd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 98–109. IEEE, 2022.

[KMS23] Subhash Khot, Dor Minzer, and Muli Safra. Pseudorandom sets in Grassmann graph have near-perfect expansion. *Annals of Mathematics*, 198(1):1–92, 2023.

[KMS25] Subhash Khot, Dor Minzer, and Muli Safra. On independent sets, 2-to-2 games, and Grassmann graphs. *Theory of Computing*, 21(10):1–55, 2025.

[KR06] Tali Kaufman and Dana Ron. Testing polynomials over general fields. *SIAM Journal on Computing*, 36(3):779–802, 2006.

[KS13] Subhash Khot and Muli Safra. A two-prover one-round game with strong soundness. *Theory of Computing*, 9:863–887, 2013.

[KTZ26] Scott Duke Kominers, Justin Thaler, and Kai Zhe Zheng. Improved soundness for the line–versus–point test. Technical Report TR26-147, Electronic Colloquium on Computational Complexity, 2026. Revision 1, August 16, 2026.

[LS16] Ben Lund and Shubhangi Saraf. Incidence bounds for block designs. *SIAM Journal on Discrete Mathematics*, 30(4):1997–2010, 2016.

[MR10] Dana Moshkovitz and Ran Raz. Two-query PCP with subconstant error. *Journal of the ACM*, 57(5):29:1–29:29, 2010.

[MZ23a] Dor Minzer and Kai Zhe Zheng. Approaching the soundness barrier: A near optimal analysis of the cube versus cube test. In *Proceedings of the 34th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA)*, pages 2761–2776, 2023.

[MZ23b] Dor Minzer and Kai Zhe Zheng. Optimal testing of generalized Reed–Muller codes in fewer queries. In *Proceedings of the 64th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 206–233. IEEE, 2023.

[MZ25] Dor Minzer and Kai Zhe Zheng. Improved round-by-round soundness IOPs via Reed–Muller codes. In *Proceedings of the 66th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 1286–1294, 2025.

[MZ26a] Dor Minzer and Kai Zhe Zheng. Near optimal alphabet–soundness tradeoff PCPs. *Journal of the ACM*, 73(4):1–98, 2026.

[MZ26b] Dor Minzer and Kai Zhe Zheng. Near optimal hardness of approximating k-CSP. In Aditya Bhaskara and Artur Czumaj, editors, *Proceedings of the 58th Annual ACM Symposium on Theory of Computing (STOC)*, pages 72–83. ACM, 2026.

[PS94] Alexander Polishchuk and Daniel A. Spielman. Nearly-linear size holographic proofs. In *Proceedings of the 26th Annual ACM Symposium on Theory of Computing (STOC)*, pages 194–203, 1994.

[RS96] Ronitt Rubinfeld and Madhu Sudan. Robust characterizations of polynomials with applications to program testing. *SIAM Journal on Computing*, 25(2):252–271, 1996.

[RS97] Ran Raz and Shmuel Safra. A sub-constant error-probability low-degree test, and a sub-constant error-probability PCP characterization of NP. In *Proceedings of the 29th Annual ACM Symposium on Theory of Computing (STOC)*, pages 475–484, 1997.

[RZS13] Noga Ron-Zewi and Madhu Sudan. A new upper bound on the query complexity of testing generalized Reed–Muller codes. *Theory of Computing*, 9(25):783–807, 2013.

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