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Feasible disjunction for random resolution

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Abstract

We show that a (stronger) version of random resolution has the feasible disjunction property. This is the first instance of a proof system not known to have feasible interpolation, which nevertheless has the feasible disjunction property.

1 Introduction

We say that a proof system P has the *feasible interpolation* property if

> given a P refutation of a formula of the form $F \wedge G$, where F and G do not share any variables, we can decide in polynomial time whether F or G is unsatisfiable.

Feasible interpolation is a central object of study in proof complexity. In particular, if P has the feasible interpolation property, then we get conditional (or unconditional in the case of monotone feasible interpolation) lower bounds for P (see e.g. [8, 13]). If on the other hand P does not have the feasible interpolation property, this implies that P is not automatable, that is, short P refutations are hard to find (see e.g. [11, 2, 1]).

A closely related but less studied concept is *feasible disjunction* [16, 14]. We say that a proof system P has the feasible disjunction property if

> whenever $F \wedge G$, where F and G do not share any variables, has a P refutation of size s, then either F or G must have a P refutation of size polynomial in s.

The two properties are related in that typically, systems which have feasible interpolation also have the feasible disjunction property. In fact, the arguments themselves showing feasible interpolation can be straightforwardly modified to not only decide which formula is unsatisfiable, but also produce a refutation of it. Moreover, in some cases, feasible interpolation arguments go through feasible disjunction (see e.g. [5]). In the other direction, it is shown in [4] that feasible disjunction fails for Res(2), which is precisely the point in the bounded-depth Frege hierarchy where feasible interpolation seems to fail too.

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Having said that, there is no reason to believe that either of these two properties implies the other. In one direction, feasible interpolation is downward closed under p-simulation (that is, if P has feasible interpolation and P p-simulates Q, then Q has feasible interpolation), whereas this is not true for feasible disjunction. Indeed, it was recently shown in [6] and [12] that by restricting a proof system with both feasible interpolation and feasible disjunction so that it corresponds to an intersection of TFNP classes, one gets a subsystem for which feasible disjunction fails.

In this paper we give the first example in the opposite direction, in the following weaker sense: we identify a proof system, namely an extension of random resolution, which has the feasible disjunction property, yet is not known to have feasible interpolation. Unlike all known proofs of feasible disjunction, our proof is non-constructive (or, more precisely, not polynomial-time constructive), and is based on the minimax theorem.

2 Random resolution refutations

Let F be a CNF formula and let $V = V(F)$ be the set of variables occurring in F. A *resolution refutation* of F is a sequence of clauses ending with the empty clause and such that each clause in the sequence is either a clause of F or results from earlier clauses by either the *resolution rule* "from $C \vee x$ and $D \vee \neg x$ derive $C \vee D$" or the *weakening rule* "from C derive $C \vee D$". Now, let Π be a resolution refutation of F and let $\alpha : I \to \{0,1\}$ be a truth assignment where $I \subseteq V$. Applying α to every clause of Π and deleting the clauses that get satisfied, we get a resolution refutation of the restricted formula $F|_\alpha$, which we denote by $\Pi|_\alpha$. Finally, for a truth assignment $\alpha : V \to \{0,1\}$, we define the *traversal of* Π *by* α, $\tau_\Pi(\alpha)$, to be the clause of F we reach if we traverse Π starting at the empty clause and always choosing at applications of derivation rules the premise which is falsified by α. Traversals and restrictions by partial assignments interact as follows.

Lemma 2.1. *Let $I \subseteq V$ and let $\alpha : I \to \{0,1\}$ and $\beta : V \setminus I \to \{0,1\}$ be truth assignments. We have that*

$$\tau_{\Pi|_\beta}(\alpha) = \tau_\Pi(\alpha, \beta)|_\beta.$$

Random resolution was introduced in [3] and further studied in [15]. It is a proof system which models the notion of resolution proofs that make mistakes. The papers [3, 15] consider one type of error of random resolution refutations which we call static error. For the purposes of showing feasible disjunction, it is necessary to introduce a different error measure, which we will call dynamic error.

Definition 2.1. A *random resolution refutation* of F is a probability distribution $(B_i, \Pi_i)_{i\sim\Delta}$, where B_i is a CNF with variables from $V = V(F)$ and Π_i is a resolution refutation of $F \wedge B_i$. For an $\alpha : V \to \{0,1\}$, the *static error* of $(B_i, \Pi_i)_{i\sim\Delta}$ on α is defined as the probability $\Pr_{i\sim\Delta}[B_i|_\alpha = 0]$. The *dynamic error* of $(B_i, \Pi_i)_{i\sim\Delta}$ on α is defined as the probability $\Pr_{i\sim\Delta}\left[\tau_{\Pi_i}(\alpha) \in B_i\right]$. The static (dynamic) error of $(B_i, \Pi_i)_{i\sim\Delta}$ is defined as the maximum over all $\alpha : V \to \{0,1\}$ of the static (resp. dynamic) error of $(B_i, \Pi_i)_{i\sim\Delta}$ on α.

An intuitive way to understand a random resolution refutation of F is as a probabilistic procedure solving the search task Search(F) associated with F: given an $\alpha : V \to \{0,1\}$, find a clause of F falsified by α. Namely, given an α, we choose i according to Δ, and carry out the traversal of Π_i by α to find a falsified clause. Dynamic error is the failure probability of this procedure. Notice that for every random resolution refutation, its dynamic error is at most its static error, and therefore random resolution with static error ε a weaker system than random resolution with dynamic error ε. In particular, if the latter system has the feasible interpolation property, then the former has it as well.

The *size* of a random resolution refutation $(B_i, \Pi_i)_{i\sim\Delta}$ is the maximum size of the refutations Π_i. We can assume without loss of generality that Δ always has finite support, so that a random resolution refutation is a finite object and its size is well defined (see [15, Lemma 2.2]).

We end this section by briefly summarizing what is known about interpolation for random resolution with static error. Given a random resolution refutation $(B_i, \Pi_i)_{i\sim\Delta}$ of a formula $F(x,u) \wedge G(x,v)$, it is straightforward to construct a DAG-like randomized communication protocol for the disjointness of the sets $A = \{\alpha : F(\alpha, u) \text{ is satisfiable}\}$ and $B = \{\beta : G(\beta, v) \text{ is satisfiable}\}$ (that is, a protocol where Alice gets an $\alpha \in A$, Bob gets a $\beta \in B$ and the two players find a coordinate j for which $\alpha_j \neq \beta_j$). Based on this, Pudlák and Thapen [15] get an exponential lower bound for tree-like random resolution refutations from a lower bound for tree-like randomized communication protocols. Krajíček [9] shows that one can get a circuit that separates $A' \subseteq A$ and $B' \subseteq B$, where the sizes of A' and B' depend on the static error ε. Provided that $\varepsilon \leq 1/d^2$, where $d = \max_i |B_i|$, this yields an exponential lower bound for the clique-coloring formulas. Finally, Krajíček [10] considers a circuit model separating A and B, where rectangles in which the protocol errs are replaced by oracles separating these rectangles. However, none of the above models gives a polynomial time interpolating algorithm.

3 Feasible disjunction for random resolution with dynamic error

Let F and G be two CNF formulas not sharing any variables such that $F \wedge G$ is unsatisfiable, and let $(B_i, \Pi_i)_{i\sim\Delta}$ be a random resolution refutation of $F \wedge G$ of dynamic error ε. For a truth assignment α to the variables of F and a truth assignment β to the variables of G, we set

$$f(\alpha,\beta) \coloneqq \Pr_{i\sim\Delta}\left[\tau_{\Pi_i}(\alpha,\beta) \in F\right],$$
$$g(\alpha,\beta) \coloneqq \Pr_{i\sim\Delta}\left[\tau_{\Pi_i}(\alpha,\beta) \in G\right].$$

Notice that for all α, β, $f(\alpha,\beta) + g(\alpha,\beta) \geq 1 - \varepsilon$. Now, we define

$$V_F \coloneqq \max_{\mu} \min_{\alpha} \mathop{\mathrm{E}}_{\beta\sim\mu}\left[f(\alpha,\beta)\right],$$
$$V_G \coloneqq \max_{\nu} \min_{\beta} \mathop{\mathrm{E}}_{\alpha\sim\nu}[g(\alpha,\beta)],$$

where μ ranges over all distributions on assignments β and ν ranges over all distributions on assignments α.

Lemma 3.1. *We have that $V_F + V_G \geq 1 - \varepsilon$.*

Proof. By von Neumann's minimax theorem, we get $V_F = \min_\nu \max_\beta \mathrm{E}_{\alpha\sim\nu}\left[f(\alpha,\beta)\right]$. Fix ν_0 attaining the minimum, so that $\mathrm{E}_{\alpha\sim\nu_0}\left[f(\alpha,\beta)\right] \leq V_F$ for every β. Since $f(\alpha,\beta) + g(\alpha,\beta) \geq 1 - \varepsilon$ for every α, β, we get that $\mathrm{E}_{\alpha\sim\nu_0}\left[g(\alpha,\beta)\right] \geq 1 - \varepsilon - V_F$ for every β, and therefore $V_F + V_G \geq 1 - \varepsilon$. $\square$

Now fix μ_1 and ν_1 attaining the maxima of $\max_\mu \min_\alpha \mathrm{E}_{\beta\sim\mu}\left[f(\alpha,\beta)\right]$ and $\max_\nu \min_\beta \mathrm{E}_{\alpha\sim\nu}\left[g(\alpha,\beta)\right]$ respectively. Consider the distributions

$$\Delta_F \coloneqq ((B_i|_\beta \wedge G|_\beta) \setminus F, \Pi_i|_\beta)_{i\sim\Delta,\beta\sim\mu_1}\,,$$
$$\Delta_G \coloneqq ((B_i|_\alpha \wedge F|_\alpha) \setminus G, \Pi_i|_\alpha)_{i\sim\Delta,\alpha\sim\nu_1}\,.$$

These are random resolution refutations of F and G respectively. For every α, we get, using Lemma 2.1, that the success rate (i.e. one minus its dynamic error) of Δ_F on α is

$$\Pr_{i\sim\Delta,\beta\sim\mu_1}\left[\tau_{\Pi_i|_\beta}(\alpha) \in F\right] \geq \Pr_{i\sim\Delta,\beta\sim\mu_1}\left[\tau_{\Pi_i}(\alpha,\beta) \in F\right] \geq V_F,$$

hence its overall success rate is at least V_F. Similarly, we get that the success rate of Δ_G is at least V_G. Since $V_F + V_G \geq 1 - \varepsilon$ by Lemma 3.1, either V_F or V_G must be at least $(1-\varepsilon)/2$. We have thus proved:

Theorem 3.2. *If $F \wedge G$ has a random resolution refutation of size s and dynamic error ε, then either F or G has a random resolution refutation of size s and dynamic error at most $(1+\varepsilon)/2$.*

Notice that for $\varepsilon < 1$, the condition $V_F \geq (1-\varepsilon)/2$ is an interpolant, that is, it correctly decides whether F or G is unsatisfiable. Unfortunately, it is not a polynomial-time computable interpolant. Let us also note that the fact that the success rate of the proof given by Theorem 3.2 drops by half, does not really make the result weaker. As it is the case for static error [15, Lemma 2.3], dynamic error can always be reduced by repeating the refutation at error clauses. Specifically, we get:

Corollary 3.3. *If $F \wedge G$ has a random resolution refutation of size s and dynamic error $\varepsilon \in (0,1)$, then either F or G has a random resolution refutation of dynamic error at most ε and size $s^{O(1+\log(1/\varepsilon))}$.*

In particular, for every $\varepsilon \in (0,1)$, random resolution with dynamic error ε has the feasible disjunction property.

4 Concluding remarks

We have identified a proof system, namely random resolution with dynamic error, which has the feasible disjunction property, yet the question of whether it also has the feasible interpolation property remains open. This partially answers a question raised by Pudlák [14], who asks whether feasible disjunction holds for any of the systems that do not have feasible interpolation.

Whether random resolution has feasible interpolation is an important question for one more reason. If random resolution had feasible interpolation, then resolution over parities (see [7]), also known as Res($\oplus$) or R(LIN/$\mathbb{F}_2$), would have feasible interpolation. This follows from the fact that equality has small randomized communication protocols. Res($\oplus$) is an important proof system, right beyond the boundary of proof systems for which lower bounds can be proved. Closer to the concerns of this paper, it also lies at the boundary beyond which feasible interpolation is believed to fail. In particular, it can be seen that the interpolation problem for Res($\oplus$) can be reduced to the interpolation problem for Res(2), which, as we noted before, is where, in the bounded-depth Frege hierarchy, feasible interpolation seems to fail.

It is worth noting that our main theorem, Theorem 3.2, is quite general. Namely, it holds for the random version of any proof system satisfying Lemma 2.1, and for example Frege systems (under an appropriate formalization) satisfy this lemma. However, dynamic error makes random versions of proof systems more general than resolution become trivial very quickly. Specifically, let $F = C_1 \wedge \cdots \wedge C_m$ be an unsatisfiable CNF. Consider the pair $(\neg C_1, \Pi)$, where in Π we start from $\neg C_1$, derive $\neg C_1 \vee \cdots \vee \neg C_m$, and then use m successive cuts on the clauses $C_1, \ldots, C_m$ to derive the empty clause. This is a depth-2 Frege refutation of F of dynamic error 0 (but static error 1).

Considering that dynamic and static error behave so differently for random depth-2 Frege (this is a non-trivial system for static error, see [15]), we believe it is worth investigating how random resolution with dynamic error ε compares to random resolution with static error ε. A first observation is that the tree-like versions of the two systems coincide. A second observation is that the lower bounds of [15] for DAG-like random resolution do not seem to immediately transfer to random resolution with dynamic error. Briefly, the key ingredient in [15] for proving lower bounds is a "fixing lemma" showing that an appropriately chosen random partial assignment α either falsifies B_i or admits no legal extension falsifying B_i, with high probability. Combined with the static error condition, this yields an α no legal extension of which falsifies B_i, and static error seems necessary for this step. So an interesting problem is first to establish exponential lower bounds for random resolution with dynamic error, say $1/2$, and secondly to see whether the two systems, random resolution with static error ε and random resolution with dynamic error ε, are polynomially equivalent.

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