Electronic Colloquium on Computational Complexity, Report No. 163 (2026)

Quantitative Results on Super-Ramanujan Graphs

Gil Cohen* Gal Maor†

September 1, 2026

This work studies super-Ramanujan graphs: d-regular graphs whose spectral expansion lies strictly below the Ramanujan threshold $2\sqrt{d-1}$. Although this threshold is asymptotically optimal for fixed d as n tends to infinity, it leaves open the finer question of how small the spectral expansion can be as a joint function of n and d. Beyond their fundamental mathematical interest and potential applications to pseudorandomness, super-Ramanujan graphs may also have practical significance, since asymptotic analyses can obscure important finite-size effects.

Our first result shows that, for every $d \geq 3$ and every even n, there exists a d-regular graph G on n vertices satisfying

$$\lambda_2(G) \leq 2\sqrt{d-1} - \frac{\sqrt{d}}{4n^{2/3}}.$$

For each fixed d, an asymptotic version of this bound also follows from the recent breakthrough work of Huang, McKenzie, and Yau. Related edge-universality results of Huang and Yau (*The Annals of Probability*, 2026) imply a bound of the same order in the regime $n^{\varepsilon} \leq d \leq n^{1/3-\varepsilon}$, for any fixed sufficiently small $\varepsilon > 0$, while He (*Commun. Math. Phys.*, 2024) obtains a substantially larger advantage in the denser regime $d \gg n^{2/3}$. These works provide a much richer probabilistic description of the spectral edge, whereas our proof is considerably simpler and gives a nonasymptotic bound that is uniform in both n and d.

Turning to explicit constructions, it is folklore that the Lubotzky–Phillips–Sarnak graphs are super-Ramanujan, but their guaranteed advantage over the Ramanujan threshold is only exponentially small in n. Our second result gives a polynomial-time construction of bipartite super-Ramanujan expanders with advantage at least $\frac{\sqrt{d}}{2n}$ over the Ramanujan threshold.

*Tel Aviv University. `gil@tauex.tau.ac.il`. Supported by ERC starting grant 949499 and by the Israel Science Foundation grant 2989/24.

†MIT. `galmaor@mit.edu`. Supported by the MIT–Kalaniyot Postdoctoral Fellows Program.

ISSN 1433-8092

1 Introduction

Expanders are sparse yet highly connected, rapidly mixing graphs. Decades of research have produced constructions drawing on combinatorial [RVW02, BL06, BATS11], group-theoretic [LPS88, Mar88], and analytic [MSS15] techniques. Expander graphs have found myriad applications in coding theory [SS94, AEL95, TS17], computational complexity [INW94, Din07, Rei08], and beyond. For background, see, for example, the survey [HLW06], Chapter 4 of [Vad12], or the book manuscript [Spi25].

This work focuses on spectral expansion. For a d-regular graph G, let $\lambda(G)$ denote the largest absolute value of a nontrivial eigenvalue of the adjacency matrix of G. Famously, the fundamental benchmark for this parameter is the Ramanujan threshold $2\sqrt{d-1}$. The Alon–Boppana theorem [Nil91] states that, for every fixed $d \geq 3$ and every $\varepsilon > 0$, there are only finitely many d-regular graphs for which $\lambda(G) < 2\sqrt{d-1} - \varepsilon$. Together with explicit constructions and existence proofs for Ramanujan graphs [LPS88, Mar88, MSS15], [1] this establishes $2\sqrt{d-1}$ as the asymptotically optimal threshold when d is fixed and the number of vertices tends to infinity.

When talking about expanders, one typically has the constant-degree regime in mind. In applications, however, the degree is often not fixed a priori, but is instead coupled to a parameter of interest, such as error, randomness, locality, or query complexity. In such settings, what matters is not constant degree per se, but rather the tradeoff between degree and spectral expansion.

Moreover, several important applications involve expanders whose degree grows with their size. Among these, we highlight two prominent examples from pseudorandomness and coding theory, which also provided the initial motivation for this work. The first is the Impagliazzo–Nisan–Wigderson generator [INW94], a central primitive in space-bounded derandomization. There, a sequence of expanders is used recursively in the PRG construction. The vertices are associated with the different seeds of the recursively constructed PRG, whereas the degree depends on the error of the construction. This error accumulates throughout the recursion, requiring the degree to be superconstant. We note that advances in PRG constructions within this framework lead to larger degrees relative to the number of vertices. The second example is Ta-Shma's celebrated construction of near-optimal small-bias sets [TS17], in which the degree likewise grows with the size of the underlying graph. In this context, the factor of 2 in the Ramanujan threshold is one of the obstacles to the fundamental goal of explicitly matching the Gilbert–Varshamov bound; see [CC26].

This motivates the following question, which is fundamental in its own right: given n and d, how small can $\lambda(G)$ be? In particular, by how much can it improve upon the Ramanujan

[1] For ease of exposition, throughout the introduction we use *spectral expansion* for both the one-sided parameter $\lambda_2(G)$ and the two-sided parameter given by the largest absolute value of a nontrivial adjacency eigenvalue, specifying the distinction whenever it matters.

threshold $2\sqrt{d-1}$? We refer to the amount by which the relevant spectral parameter lies below $2\sqrt{d-1}$ as its *advantage*. To the best of our knowledge, quantitative improvements over this threshold have not previously been studied systematically, although high-degree graphs beating the Ramanujan threshold are known, as we survey shortly. The phenomenon is mentioned explicitly in only a few places, under terms such as "super-Ramanujan," which we adopt in this work, and "better than Ramanujan"; see, for example, [MMST96, Gla03, CKNV16, Sri24].

Super-Ramanujan graphs may also be relevant to real-world applications. A common way to obtain a good expander in practice is simply to sample a d-regular graph from a natural random model. Sampling alone, however, does not guarantee an advantage: in several natural regimes, the probability of failing to beat the Ramanujan threshold remains bounded away from zero. Moreover, graphs used in practice have finite size, and asymptotic analyses may obscure finite-size effects that are practically significant. Thus, even a modest advantage over the Ramanujan threshold may be beneficial in such settings.

The classical Alon–Boppana bound [Alo86, Nil91] was quantitatively sharpened by Friedman [Fri93, Corollary 3.7], who showed that every connected n-vertex d-regular graph satisfies

$$\lambda(G) \geq 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\log_{d-1}^2 n}\right)\right).$$

In particular, if $d = d(n) \to \infty$ and $d = n^{o(1)}$, then $\lambda(G) \geq (2 - o(1))\sqrt{d}$. Lower bounds for the regime in which d grows polynomially with n were recently obtained by Räty, Sudakov, and Tomon [RST26]. A companion expository note [BRST23] presents a short, self-contained proof, as well as a precise account of earlier contributions by Balla and Ihringer. Specifically, one has $\lambda(G) = \Omega(\sqrt{d})$ for $d \leq n^{2/3}$, $\lambda(G) = \Omega(n/d)$ for $n^{2/3} \leq d \leq n^{3/4}$, and $\lambda(G) = \Omega(d^{1/3})$ for $n^{3/4} \leq d \leq n/3$. Thus, nontrivial spectral lower bounds persist well beyond the range in which the diameter-based Alon–Boppana estimate is informative.

On the positive side, in the $d = \mathrm{poly}(n)$ regime, classical constructions based on the symmetries of finite geometries have long been known. One example is the affine-plane construction of Reingold, Vadhan, and Wigderson [RVW02, Lemma 5.1], which is based on Alon's projective-plane construction [Alo86]. For every prime-power degree d, this construction gives an explicit d-regular graph G on $n = d^2$ vertices with $\lambda(G) = \sqrt{d}$. Another example is the incidence graph of a generalized quadrangle, which has $d \approx n^{1/3}$ and $\lambda(G) \approx \sqrt{2d}$. These finite-geometric examples are interesting and shed light on the geometric structures on which they rely, no less than on the problem of constructing expander graphs. However, they exist only for special parameter choices and are quite rare.

Considering still higher degrees, a classical example is provided by the Paley graphs. For every prime power $q \equiv 1 \pmod 4$, the Paley graph P_q is a $d = (q-1)/2$-regular graph on q vertices and is super-Ramanujan, with spectral parameter $\lambda(P_q) \approx \sqrt{d/2}$. The construction originates in Paley's work on orthogonal matrices [Pal33]; for treatments emphasizing expansion, see [Lub94, KS06]. Paley graphs belong to the broader class of conference graphs, every member of which is super-Ramanujan [BH12]. For the related class of generalized Paley graphs, see [LP09, PV25].

The celebrated Lubotzky–Phillips–Sarnak graphs [LPS88] provide a further example. These highly structured graphs arise from an arithmetic group-theoretic construction and are usually studied in the fixed-degree regime. They are famously known to satisfy the Ramanujan bound; in fact, the inequality is strict, and hence they are super-Ramanujan. This strictness follows from a theorem of Coleman and Edixhoven [CE98], whose proof draws on deep arithmetic geometry. We learned from Ori Parzanchevski that Nina Zubrilina observed that this qualitative strictness can be made quantitative, yielding an exponentially small advantage. Sarnak's lecture notes [Sar23] record a related bound for certain cubic arithmetic Ramanujan graphs and attribute it to Zubrilina. As we are not aware of a written proof of the corresponding quantitative statement for the LPS family, we include one in Appendix A, together with a refinement that improves the guaranteed advantage from $\exp(-O(n))$ to $\exp(-O(n^{2/3}))$.

The examples of super-Ramanujan graphs above are based on particular geometric or algebraic structures. However, by sampling random d-regular graphs according to a natural distribution, one can easily be convinced that, with some constant probability, the resulting graphs are super-Ramanujan. The question is: by how much? Recent breakthrough work of Huang, McKenzie, and Yau [HMY24], together with earlier results for growing-degree regimes by Huang and Yau [HY26] and He [He24], sheds light on this question. Although these results are not formulated in super-Ramanujan terms, they imply not only the existence but also the abundance of super-Ramanujan graphs. Indeed, their edge-universality theorems locate the extreme nontrivial eigenvalues on scales fine enough to detect a nontrivial advantage over the Ramanujan threshold $2\sqrt{d-1}$.

1.1 Our Results

In this paper, we make two contributions to the study of super-Ramanujan graphs. Our first theorem establishes the existence of one-sided super-Ramanujan graphs with an advantage of order $\sqrt{d}/n^{2/3}$.

Theorem 1.1. *For every even n and integer $3 \le d \le n$, there exists a d-regular graph G on n vertices satisfying*

$$\begin{aligned}\lambda_2(G) &\le 2\sqrt{d-1} - \frac{3}{4^{1/3}}\left(1-\frac{2}{d}\right)^{5/3}\frac{\sqrt{d}}{n^{2/3}}\\ &\le 2\sqrt{d-1} - \frac{\sqrt{d}}{4n^{2/3}}.\end{aligned}$$

For each fixed d, an asymptotic version of our bound also follows from the recent breakthrough work of Huang, McKenzie, and Yau [HMY24]. Earlier edge-universality results of Huang and Yau [HY26] imply a bound of the same order in the regime $n^{\varepsilon} \le d \le n^{1/3-\varepsilon}$, for any fixed sufficiently small $\varepsilon > 0$, while He [He24] obtains a substantially larger advantage in the dense regime $d \gg n^{2/3}$. These works establish much finer probabilistic results and, moreover, control both spectral edges, thus yielding two-sided bounds, whereas our result is one-sided. Interestingly, the agreement extends beyond the common $\sqrt{d}/n^{2/3}$ scaling: the exact d-dependent factor arising in our analysis, which we approximate by $\sqrt{d}$ in the statement of Theorem 1.1, is

$$\sigma_d \triangleq \frac{(d-2)^{4/3}}{d^{2/3}(d-1)^{1/6}}.$$

This exactly matches the edge-fluctuation scale identified in [HMY24]. Their result is stronger at the level of the leading constant, however: for every fixed $c > 0$, it yields an advantage of at least $c\sigma_d/n^{2/3}$ for all sufficiently large n. Our proof, however, is considerably simpler and gives a nonasymptotic bound uniformly for all d and n; in particular, it also covers the intermediate regime $n^{1/3} \le d \le n^{2/3}$.

Our proof builds on the work of Marcus, Spielman, and Srivastava and on an observation identifying a certain finite free convolution with a Jacobi polynomial. We then apply a known bound on the largest root of this polynomial due to Krasikov [Kra06]. Although finite free probability is itself a deep theory, our argument is considerably simpler than the random-matrix machinery underlying [HMY24]. The agreement between the resulting scales is not entirely coincidental: in both cases, the $n^{-2/3}$ scale reflects the square-root behavior of the Kesten–McKay law at the spectral edge, manifested through Tracy–Widom universality in [HMY24] and through the largest zero of a Jacobi polynomial in our argument.

Interestingly, an advantage of order $\frac{d}{n}$ within the framework of Marcus, Spielman, and Srivastava appears to be folklore; see, for example, Srivastava [Sri24] and Sarnak's lecture notes [Sar23]. To the best of our knowledge, however, a proof has not appeared in the literature, and so we provide one in Section 3. The bound in Theorem 1.1 is stronger whenever $d = o(n^{2/3})$.

Our second main result gives an explicit construction of super-Ramanujan graphs in the bipartite setting.

Theorem 1.2. *There is an algorithm that, given an even integer n and an integer d satisfying $3 \le d \le n/2$, runs in time $\mathrm{poly}(n)$ and outputs a d-regular bipartite graph G on n vertices such that*

$$\lambda(G) \le 2\sqrt{d-1} - \frac{\sqrt{d}}{2n}.$$

Although the construction in Theorem 1.2 does not attain the existential advantage guaranteed by Theorem 1.1, it achieves an advantage of order $\sqrt{d}/n$. In contrast, as noted above, the known advantage of the LPS construction is exponentially small in n, whereas the advantage here has only inverse-linear dependence on n. Our proof is based on Cohen's construction of bipartite Ramanujan graphs [Coh16]. As discussed above, for certain choices of n and d, finite-geometric constructions achieve a larger advantage over the Ramanujan threshold—indeed, even larger than the advantage guaranteed by Theorem 1.1. The significance of Theorem 1.2 lies in its applicability to every degree $d \geq 3$ and every n.

Organization. We begin in Section 3 with a short quantitative proof of the folklore $\Omega(d/n)$ advantage. We present this proof first because it sets the stage for our two main results. In Section 4, we use a similar perspective to obtain the explicit bipartite construction, thereby proving Theorem 1.2. Finally, in Section 5, we prove Theorem 1.1. As mentioned above, the analysis of the advantage of LPS graphs is given in Appendix A.

2 Preliminaries

Throughout, graphs may have parallel edges and self-loops. Adjacency matrices count edge multiplicities, with each self-loop contributing 2 to the corresponding diagonal entry, and hence 2 to the degree.

Let G be an undirected d-regular graph on n vertices, with adjacency matrix $\mathbf{A}$. Since G is undirected, $\mathbf{A}$ is symmetric and so its spectrum is real-valued. We denote the eigenvalues of $\mathbf{A}$ by $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -d$, where $\lambda_n = -d$ for bipartite graphs. The *random walk matrix* of a d-regular graph G is $\mathbf{W} = \frac{1}{d}\mathbf{A}$. The *spectral expansion* of G, denoted as $\lambda(G)$, is the largest absolute nontrivial eigenvalue.

2.1 Interlacing Families and Finite Free Probability

We use the following standard notation: for a symmetric matrix $\mathbf{A}$, $\chi_x(\mathbf{A})$ is the characteristic polynomial of $\mathbf{A}$ with variable x. We denote by $\lambda_k(\mathbf{A})$ the k-th largest eigenvalue of $\mathbf{A}$ (as $\mathbf{A}$ is symmetric, all eigenvalues are real and hence can be ordered as above).

For a real-rooted polynomial $p(x)$, we denote by $\alpha_k(p)$ the k-th largest root of $p(x)$. For a distribution P over polynomials, we denote by $\mathbf{E}_{p\sim P}[p(x)]$ the *expected* polynomial over this distribution, where the expectation is taken in coefficient space, namely, for every k, the coefficient of x^k in $\mathbf{E}_{p\sim P}[p(x)]$ is the expectation over coefficients corresponding to x^k in $p(x)$ drawn according to P.

The following lemma is a simplified version of a more general statement from [MSS18], where it appeared as the main tool enabling us to analyze the expected characteristic polynomial over a distribution, and deduce an existence result.

Lemma 2.1. *Suppose $\mathbf{A}_1, \dots, \mathbf{A}_d$ are symmetric $m \times m$ matrices and let $(\mathbf{P}_i)_{i\in[d]}$ be independent uniformly random $m \times m$ permutation matrices. Let $\mathbf{A}_{\mathbf{P}} = \sum_{i=1}^{d} \mathbf{P}_i \mathbf{A}_i \mathbf{P}_i^{\mathsf{T}}$. Then, for every $k \le m$ there exist permutation matrices $(\mathbf{R}_i)_{i\in[d]}$ such that*

$$\lambda_k(\mathbf{A}_{\mathbf{R}}) \le \alpha_k\left(\mathop{\mathbf{E}}_{\mathbf{P}} \chi_x(\mathbf{A}_{\mathbf{P}})\right), \tag{1}$$

where we recall that $\alpha_k(p)$ is the k-th largest root of $p(x)$.

The multiplicative analog of Lemma 2.1 was proved in [CM23, Theorem 6.5 and Lemma 6.3]. A simplified version for our needs is stated below.

Lemma 2.2. *Suppose that $\mathbf{A}$ and $\mathbf{B}$ are symmetric $m \times m$ matrices, at least one of which is positive semidefinite, and let $\mathbf{P}$ be a uniformly random $m \times m$ permutation matrix. Then, for every $k \le m$ there exists a permutation matrix $\mathbf{S}$ such that*

$$\lambda_k\left(\mathbf{ASBS}^{\mathsf{T}}\right) \le \alpha_k\left(\mathop{\mathbf{E}}_{\mathbf{P}} \chi_x\left(\mathbf{APBP}^{\mathsf{T}}\right)\right). \tag{2}$$

Definition 2.3 (Haar distribution on the orthogonal group)**.** *Denote the group of $m \times m$ orthogonal matrices by $\mathcal{O}(m)$. The* Haar distribution *is the unique distribution over $\mathcal{O}(m)$ which is invariant under multiplication (from the right or from the left) with any orthogonal matrix. We call a matrix drawn from this distribution a* Haar random matrix*.*

An important characteristic of the Haar distribution, upon which Definition 2.4 below relies, is the following. Let $\mathbf{A}, \mathbf{B}$ be two arbitrary $m \times m$ symmetric matrices, and $\mathbf{Q}$ a Haar random matrix of the same dimensions. Informally, the random rotation of $\mathbf{B}$ according to $\mathbf{Q}$ removes any dependence between the respective eigenvectors of $\mathbf{A}$ and $\mathbf{B}$. More formally, if $\chi_x(\mathbf{A}) = a(x)$ and $\chi_x(\mathbf{B}) = b(x)$, then both expected characteristic polynomials $\mathbf{E}_{\mathbf{Q}} \chi_x\left(\mathbf{A} + \mathbf{QBQ}^{\mathsf{T}}\right)$ and $\mathbf{E}_{\mathbf{Q}} \chi_x\left(\mathbf{AQBQ}^{\mathsf{T}}\right)$ depend only on $a(x)$ and $b(x)$, and not on the eigenvectors of either $\mathbf{A}$ or $\mathbf{B}$.

Definition 2.4 (Additive and multiplicative convolutions)**.** *Let $\mathbf{A}, \mathbf{B}$ be real symmetric matrices of dimension m, with characteristic polynomials $a(x) = \chi_x(\mathbf{A})$ and $b(x) = \chi_x(\mathbf{B})$. The* additive convolution *$a \boxplus_m b$ and the* multiplicative convolution *$a \boxtimes_m b$ are the polynomials defined by*

$$(a \boxplus_m b)(x) = \mathop{\mathbf{E}}_{\mathbf{Q}} \chi_x\left(\mathbf{A} + \mathbf{QBQ}^{\mathsf{T}}\right),$$

and

$$(a \boxtimes_m b)(x) = \mathop{\mathbf{E}}_{\mathbf{Q}} \chi_x\left(\mathbf{AQBQ}^{\mathsf{T}}\right),$$

where $\mathbf{Q}$ is a Haar random orthogonal matrix. We will use $\boxplus$ and $\boxtimes$ instead of $\boxplus_m$ and $\boxtimes_m$ to simplify notation when the dimension is clear from the context.

Although Definition 2.4 involves the matrices $\mathbf{A}$ and $\mathbf{B}$, it in fact depends only on $a(x)$ and $b(x)$, due to the properties of the Haar measure.[2] It is important to note that $(a \boxplus b)(x)$ is real-rooted, and so is $(a \boxtimes b)(x)$ if either $a(x)$ or $b(x)$ has only nonnegative roots. Interestingly, there are explicit formulas for both $(a \boxplus b)(x)$ and $(a \boxtimes b)(x)$ as functions of the coefficients of $a(x)$ and $b(x)$. We refer the reader to [MSS22] for more details.

Working with graph matrices, an issue with the above definition is that the Haar measure does not have a meaningful combinatorial interpretation. Therefore, we need a way to relate *permutation* matrices - which do have such an interpretation - to Haar random matrices. To this end we state the following lemma (which appeared first in [MSS18] for the additive case, and later in [CM23] for the multiplicative case, whose proof uses similar ideas) which is a quadrature identity, translating an infinite (continuous) measure space to a finite one.

Lemma 2.5 (*Quadrature*; Corollary 4.9 from [MSS18] and Lemma 2.3 from [CM23])**.** *Let $\mathbf{A}, \mathbf{B}$ be real symmetric $m \times m$ matrices such that $\mathbf{A}\mathbf{1} = a\mathbf{1}$ and $\mathbf{B}\mathbf{1} = b\mathbf{1}$. Let $p_{\mathbf{A}}, p_{\mathbf{B}}$ be the polynomials satisfying $\chi_x(\mathbf{A}) = (x-a)p_{\mathbf{A}}(x)$ and $\chi_x(\mathbf{B}) = (x-b)p_{\mathbf{B}}(x)$. Let $\mathbf{P}$ be a uniformly random $m \times m$ permutation matrix. Then*

$$\mathop{\mathbf{E}}_{\mathbf{P}} \chi_x\left(\mathbf{A} + \mathbf{P}\mathbf{B}\mathbf{P}^{\mathsf{T}}\right) = (x - (a+b))\left(p_{\mathbf{A}} \boxplus p_{\mathbf{B}}\right)(x), \tag{3}$$

and

$$\mathop{\mathbf{E}}_{\mathbf{P}} \chi_x\left(\mathbf{A}\mathbf{P}\mathbf{B}\mathbf{P}^{\mathsf{T}}\right) = (x - ab)\left(p_{\mathbf{A}} \boxtimes p_{\mathbf{B}}\right)(x). \tag{4}$$

Moreover, $p_{\mathbf{A}} \boxplus p_{\mathbf{B}}$ is real-rooted. If at least one of $p_{\mathbf{A}}$ and $p_{\mathbf{B}}$ has only nonnegative roots, then $p_{\mathbf{A}} \boxtimes p_{\mathbf{B}}$ is also real-rooted.

2.1.1 Transforms

Let μ be a probability distribution over $\mathbb{R}$. The *Cauchy transform* of μ is defined as the function

$$\mathcal{G}_\mu(x) = \int_{\mathbb{R}} \frac{1}{x-t}\mu(t)dt.$$

We remark that in many settings it is instructive to study the Cauchy transform as a function whose domain is $\mathbb{C}^+$. However, we will consider the Cauchy transform as a function on $\mathbb{R}$, supported outside the support of μ.

Let $p(x)$ be a degree m real-rooted polynomial with roots $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. To $p(x)$ we associate the uniform distribution over its roots, and therefore its Cauchy transform (defined for $x > \alpha_1$) is

$$\mathcal{G}_p(x) = \frac{1}{m}\sum_{i=1}^{m} \frac{1}{x-\alpha_i} = \frac{1}{m} \cdot \frac{p'(x)}{p(x)}. \tag{5}$$

[2]It is easily seen that the convolution is well defined for any real-rooted polynomials $a(x)$ and $b(x)$ by choosing $\mathbf{A}, \mathbf{B}$ to be diagonal matrices with their respective roots on the diagonal.

When $p(x)$ is the characteristic polynomial of an $m \times m$ matrix $\mathbf{A}$, it holds that

$$\mathcal{G}_p(x) = \frac{1}{m} \mathsf{Tr}\left((x\mathbf{I} - \mathbf{A})^{-1}\right).$$

Note that when the Cauchy transform of a polynomial p is restricted to (α_1, ∞), its range is $(0, \infty)$. Additionally, $\mathcal{G}_p(x)$ is monotonically decreasing within this domain. With this in mind, one can define $\mathcal{K}_p : (0, \infty) \to (\alpha_1, \infty)$ as the inverse of $\mathcal{G}_p(x)$ when restricted to the latter domain. In other words, $\mathcal{K}_p$ is the max-inverse of $\mathcal{G}_p$. In particular, for every $y \in (0, \infty)$, $\mathcal{K}_p(y)$ provides an upper bound on α_1, the largest root of p, which we state for later reference in the following claim:

Claim 2.6. *Let $p(x)$ be a degree m real-rooted polynomial with roots $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. Then for every $y \in (0, \infty)$, $\alpha_1 \leq \mathcal{K}_p(y)$.*

For a symmetric matrix $\mathbf{A}$, we use the notation $\mathcal{G}_{\mathbf{A}}$ for $\mathcal{G}_{\chi_x(\mathbf{A})}$, and $\mathcal{G}_{p_{\mathbf{A}}}$ for the Cauchy transform of $p_{\mathbf{A}}$ as used in Lemma 2.5, which will be relevant for all matrices throughout this paper (that is, there will always exist a such that $\mathbf{A}\mathbf{1} = a\mathbf{1}$). We will similarly denote $\mathcal{K}_{\mathbf{A}}$ and $\mathcal{K}_{p_{\mathbf{A}}}$.
For example, when $\mathbf{M}$ is the adjacency matrix of a perfect matching of dimension $2m$, we have that for $x > 1$

$$\mathcal{G}_{\mathbf{M}}(x) = \frac{1}{2m}\left(\frac{m}{x+1} + \frac{m}{x-1}\right) = \frac{x}{x^2-1}, \tag{6}$$

and for $y > 0$,

$$\mathcal{K}_{\mathbf{M}}(y) = \frac{1 + \sqrt{4y^2+1}}{2y}. \tag{7}$$

The key feature of the $\mathcal{K}$ transform is that it behaves very well under the free additive convolution, as was shown in [MSS22, Theorem 1.11] and stated below.

Lemma 2.7 (*Convolution bounds*)**.** *For real-rooted polynomials $p(x)$ and $q(x)$ of the same degree, and for any $y > 0$,*

$$\mathcal{K}_{p \boxplus_m q}(y) \leq \mathcal{K}_p(y) + \mathcal{K}_q(y) - \frac{1}{y}.$$

The inequality is strict in case both polynomials have at least two distinct roots.

Claim 2.8 ([CCM24], Claim 4.12)**.** *Let $\mathbf{B}$ be a symmetric matrix such that $\mathbf{B}\mathbf{1} = b\mathbf{1}$ and b is its largest eigenvalue. Then, for every $x > b$, $\mathcal{G}_{p_{\mathbf{B}}}(x) \leq \mathcal{G}_{\mathbf{B}}(x)$, and for every $y > 0$, $\mathcal{K}_{p_{\mathbf{B}}}(y) \leq \mathcal{K}_{\mathbf{B}}(y)$. Moreover, when $\mathbf{B} \neq b\mathbf{I}$, all inequalities are strict.*

3 Sharpening the MSS Proof

In this section, we prove the folklore $\Omega(d/n)$ lower bound on the advantage. More precisely, we prove the following theorem.

Theorem 3.1. *For all integers $3 \le d < n$ such that n is even there exists a d-regular graph G on n vertices such that*

$$\lambda_2(G) \le 2\sqrt{d-1} - \frac{d-2}{n-1}.$$

Our proof follows the approach of Marcus, Spielman, and Srivastava and should therefore be easier to follow for readers familiar with their work. The proof is, however, self-contained.

The advantage over the Ramanujan threshold arises at the *quadrature* step of the MSS proof, in which the uniform measure on the permutation group is replaced by Haar measure on the orthogonal group. The original work and subsequent works [CCMP25, CM23, CCM24, CM25] use only the fact that this replacement incurs no loss. We observe that it in fact yields a quantifiable gain. We believe that this observation may be of independent interest.

Proof. To prove the theorem, we follow the steps of the proof in [MSS18]. Let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_d$ be $n \times n$ permutation matrices, and write $\mathbf{P} = (\mathbf{P}_1, \ldots, \mathbf{P}_d)$. We denote by $G_{\mathbf{P}}$ the d-regular graph whose adjacency matrix is

$$\mathbf{A}_{\mathbf{P}} = \sum_{i=1}^{d} \mathbf{P}_i \mathbf{M} \mathbf{P}_i^{\mathsf{T}}, \tag{8}$$

where $\mathbf{M}$ is the adjacency matrix of a fixed perfect matching on the n vertices.

By Lemma 2.1, it suffices to bound the second-largest root, denoted by $\alpha_2(p)$, of the expected characteristic polynomial

$$p(x) \triangleq \mathop{\mathbf{E}}_{\mathbf{P}} \chi_x(\mathbf{A}_{\mathbf{P}}),$$

where the expectation is over d independent uniformly distributed permutation matrices.

To this end, we invoke Lemma 2.5, observing that $\mathbf{1}$ is indeed an eigenvector of $\mathbf{M}$ with eigenvalue 1. Hence,

$$p(x) = (x-d)p_{\mathbf{M}}^{\boxplus d}(x),$$

where $p_{\mathbf{M}}(x)$ is defined by $\chi_x(\mathbf{M}) = (x-1)p_{\mathbf{M}}(x)$, and $\boxplus$ denotes the additive finite free convolution defined in Definition 2.4. Thus, bounding $\alpha_2(p)$ reduces to bounding

$$\alpha_1 \triangleq \alpha_1\left(p_{\mathbf{M}}^{\boxplus d}\right).$$

By Claim 2.6, for every $y > 0$,

$$\alpha_1 < \mathcal{K}_{p_{\mathbf{M}}^{\boxplus d}}(y). \tag{9}$$

The convolution bound (Lemma 2.7) implies that, for every $y > 0$,

$$\begin{aligned}\mathcal{K}_{p_{\mathbf{M}}^{\boxplus d}}(y) &\leq d \cdot \mathcal{K}_{p_{\mathbf{M}}}(y) - \frac{d-1}{y} \\ &< d \cdot \mathcal{K}_{\mathbf{M}}(y) - \frac{d-1}{y} \\ &= \frac{2 - d + d\sqrt{4y^2+1}}{2y},\end{aligned} \tag{10}$$

where the second inequality follows from Claim 2.8, and the final equality follows from Equation (7). The right-hand side of Equation (10), which we denote by $\mathcal{K}(y)$, is minimized at

$$y_0 = \frac{\sqrt{d-1}}{d-2}, \tag{11}$$

where it attains the value

$$\mathcal{K}(y_0) = 2\sqrt{d-1}. \tag{12}$$

This completes the original proof of [MSS18].

To prove Theorem 3.1, we obtain a lower bound on the advantage in terms of the difference between $\mathcal{K}_{\mathbf{M}}$ and $\mathcal{K}_{p_{\mathbf{M}}}$, evaluated at y_0. More precisely, we prove the following claim.

Claim 3.2.

$$\mathcal{K}_{p_{\mathbf{M}}^{\boxplus d}}(y_0) \leq 2\sqrt{d-1} - d \cdot \left(\mathcal{K}_{\mathbf{M}}(y_0) - \mathcal{K}_{p_{\mathbf{M}}}(y_0)\right).$$

Proof. First, a straightforward calculation using Equations (7) and (11) gives

$$x_0 \triangleq \mathcal{K}_{\mathbf{M}}(y_0) = \sqrt{d-1}. \tag{13}$$

By the convolution bound (Lemma 2.7),

$$\mathcal{K}_{p_{\mathbf{M}}^{\boxplus d}}(y_0) \leq d \cdot \mathcal{K}_{p_{\mathbf{M}}}(y_0) - \frac{d-1}{y_0}. \tag{14}$$

Instead of invoking Claim 2.8, we now use the definition of y_0 from Equation (11), together with Equation (13), to obtain

$$-\frac{d-1}{y_0} = (2-d)\sqrt{d-1} = 2\sqrt{d-1} - d \cdot \mathcal{K}_{\mathbf{M}}(y_0).$$

Substituting this identity into Equation (14) proves the claim. □

In view of Claim 3.2, we prove the following claim.

Claim 3.3.

$$\mathcal{K}_{\mathbf{M}}(y_0) - \mathcal{K}_{p_{\mathbf{M}}}(y_0) = \frac{d\left(1-\sqrt{1-\eta}\right)}{2\sqrt{d-1}} \quad \textit{where} \quad \eta = \frac{4(d-2)\sqrt{d-1}}{d^2(n-1)}. \tag{15}$$

Proof of Claim 3.3. Notice that

$$\mathcal{G}_{p_{\mathbf{M}}}(x) = \frac{n}{n-1} \cdot \mathcal{G}_{\mathbf{M}}(x) - \frac{1}{(n-1)(x-1)}.$$

Let $x_1 = \mathcal{K}_{p_{\mathbf{M}}}(y_0)$. By the above equation applied at $x = x_1$, we have that

$$y_0 = \mathcal{G}_{p_{\mathbf{M}}}(x_1) = \frac{n}{n-1}\frac{x_1}{x_1^2-1} - \frac{1}{(n-1)(x_1-1)} = \frac{(n-1)x_1 - 1}{(n-1)(x_1^2-1)}.$$

Therefore,

$$y_0 x_1^2 - x_1 + \frac{1}{n-1} - y_0 = 0.$$

By definition, $x_1 = \mathcal{K}_{p_{\mathbf{M}}}(y_0)$ lies to the right of the largest root of $p_{\mathbf{M}}$. We must therefore take the larger of the two solutions of the above quadratic equation, which gives

$$x_1 = \frac{1+\sqrt{1+4y_0^2-\frac{4y_0}{n-1}}}{2y_0}. \tag{16}$$

Recall from Equation (11) that $y_0 = \frac{\sqrt{d-1}}{d-2}$. Substituting this value and using $(d-2)^2 + 4(d-1) = d^2$, we can write the discriminant as

$$\begin{aligned}
1 + 4y_0^2 - \frac{4y_0}{n-1} &= 1 + \frac{4(d-1)}{(d-2)^2} - \frac{4\sqrt{d-1}}{(d-2)(n-1)} \\
&= \frac{d^2}{(d-2)^2}\left(1 - \frac{4(d-2)\sqrt{d-1}}{d^2(n-1)}\right) \\
&= \frac{d^2}{(d-2)^2}(1-\eta).
\end{aligned}$$

It now follows from Equation (16) that

$$x_1 = \frac{d-2+d\sqrt{1-\eta}}{2\sqrt{d-1}}.$$

On the other hand, by Equation (13), $x_0 = \sqrt{d-1}$. Therefore,

$$\mathcal{K}_{\mathbf{M}}(y_0) - \mathcal{K}_{p_{\mathbf{M}}}(y_0) = x_0 - x_1 = \frac{d\left(1 - \sqrt{1-\eta}\right)}{2\sqrt{d-1}},$$

which proves the claim. □

Using that $1 - \sqrt{1-x} \geq x/2$ for every $x \in [0,1]$ (and noting that $\eta \in [0,1]$), Claim 3.3 gives

$$\mathcal{K}_{\mathbf{M}}(y_0) - \mathcal{K}_{p_{\mathbf{M}}}(y_0) \geq \frac{d-2}{d(n-1)}.$$

The proof of Theorem 3.1 then follows by Claim 3.2.

□

4 Explicit Bipartite Super-Ramanujan Graphs

In this section, we prove the following theorem, from which Theorem 1.2 follows. Our argument follows the proof of [MSS18, Theorem 1.1] and invokes Cohen's technique [Coh16] as a black box to obtain an explicit construction of bipartite super-Ramanujan graphs.

Theorem 4.1. *There is an algorithm that, given an even integer n and an integer d satisfying $3 \leq d \leq n/2$, runs in time $\mathrm{poly}(n)$ and outputs a d-regular bipartite graph G on n vertices such that*

$$\begin{aligned}\lambda(G) &< 2\sqrt{d-1}\sqrt{1 - \frac{2(d-2)}{nd-4}} \\ &< 2\sqrt{d-1} - \frac{\sqrt{d}}{2n}.\end{aligned}$$

The proof of Theorem 4.1 parallels that of Theorem 3.1. To state it formally, however, we require slightly different notions of asymmetric convolution and polynomial symmetrization. The construction is again a union of d perfect matchings, but now each matching connects the two sides of a fixed balanced bipartition.

For two square matrices $\mathbf{A}$ and $\mathbf{B}$, not necessarily symmetric, define

$$p(x) = \chi_x\left(\mathbf{A}\mathbf{A}^\intercal\right) \qquad \text{and} \qquad q(x) = \chi_x\left(\mathbf{B}\mathbf{B}^\intercal\right).$$

The asymmetric free convolution of p and q is defined by

$$(p \boxplus\!\!\boxplus\, q)(x) = \mathop{\mathbf{E}}_{\mathbf{Q},\mathbf{R}} \chi_x\left(\left(\mathbf{A} + \mathbf{Q}\mathbf{B}\mathbf{R}^\intercal\right)\left(\mathbf{A} + \mathbf{Q}\mathbf{B}\mathbf{R}^\intercal\right)^\intercal\right), \tag{17}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are independent Haar-random orthogonal matrices. Marcus, Spielman, and Srivastava [MSS22] also define $p \boxplus\!\!\boxplus q$ directly in terms of the coefficients of p and q, without reference to the underlying matrices, and prove that their definition agrees with the one above whenever p and q arise from matrices in this way.

As observed in [MSS18], the above can alternatively be written as

$$\mathbb{S}\left(p(x) \boxplus\!\!\boxplus q(x)\right) = \mathop{\mathbf{E}}_{\mathbf{Q},\mathbf{R}} \chi_x \left(\begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{A}^\mathsf{T} & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{R} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{B} \\ \mathbf{B}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{R} \end{bmatrix}^\mathsf{T} \right) \tag{18}$$

where $(\mathbb{S}h)(x) \triangleq h(x^2)$ for every polynomial $h(x)$.

We will also use the quadrature identity established in [MSS18], which is the analogue of Lemma 2.5 for bipartite matrices.

Lemma 4.2 (Theorem 4.10 in [MSS18])**.** *Let $\mathbf{A}, \mathbf{B}$ be matrices of dimension m and let a, b be real numbers such that $\mathbf{A1} = \mathbf{A}^\mathsf{T}\mathbf{1} = a\mathbf{1}$ and $\mathbf{B1} = \mathbf{B}^\mathsf{T}\mathbf{1} = b\mathbf{1}$. Let $p_\mathbf{A}(x), p_\mathbf{B}(x)$ be degree $m-1$ polynomials such that*

$$\chi_x\left(\mathbf{A}\mathbf{A}^\mathsf{T}\right) = (x - a^2)p_\mathbf{A}(x), \quad \chi_x\left(\mathbf{B}\mathbf{B}^\mathsf{T}\right) = (x - b^2)p_\mathbf{B}(x).$$

Then, for uniformly random and independent permutation matrices $\mathbf{P}, \mathbf{S}$ we have that

$$\begin{aligned} \mathop{\mathbf{E}}_{\mathbf{P},\mathbf{S}} \chi_x \left(\begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{A}^\mathsf{T} & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{B} \\ \mathbf{B}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S} \end{bmatrix}^\mathsf{T} \right) &= \mathbb{S}\left(\left(x - (a+b)^2\right) p_\mathbf{A}(x) \boxplus\!\!\boxplus p_\mathbf{B}(x)\right) \\ &= \left(x^2 - (a+b)^2\right) \mathbb{S}\left(p_\mathbf{A}(x) \boxplus\!\!\boxplus p_\mathbf{B}(x)\right). \end{aligned}$$

We will use the following analogue of Lemma 2.7 for the asymmetric additive convolution:

Lemma 4.3 (Theorem 1.13 and Theorem 4.12 in [MSS22])**.** *Let $p(x), q(x)$ be real-rooted monic polynomials of equal degree with nonnegative roots. Then for every $w > 0$,*

$$\mathcal{K}_{\mathbb{S}p \boxplus\!\!\boxplus q}(w) \leq \mathcal{K}_{\mathbb{S}p}(w) + \mathcal{K}_{\mathbb{S}q}(w) - \frac{1}{w},$$

where equality holds if and only if either polynomial is of the form x^m.

Using these ingredients, we combine the arguments of [MSS18] and [Coh16] to obtain a polynomial-time algorithm that constructs a bipartite super-Ramanujan graph on $n = 2m$ vertices for every admissible degree $d \leq m$. To formulate the argument rigorously, we first introduce the *root characteristic polynomial.*

Definition 4.4. *The (n, d)-root characteristic polynomial is the expected characteristic* *polynomial*

$$p_{\emptyset} \triangleq \mathop{\mathbf{E}}_{\substack{\mathbf{P}_1,\ldots,\mathbf{P}_d\\ \mathbf{S}_1,\ldots,\mathbf{S}_d}} \chi_x\left(\sum_{i=1}^{d}\begin{bmatrix}\mathbf{P}_i & 0\\ 0 & \mathbf{S}_i\end{bmatrix}\begin{bmatrix}0 & \mathbf{I}\\ \mathbf{I} & 0\end{bmatrix}\begin{bmatrix}\mathbf{P}_i & 0\\ 0 & \mathbf{S}_i\end{bmatrix}^{\mathsf{T}}\right),$$

where all the P_i's and S_i's are independent and uniformly distributed.

The (n,d)-root characteristic polynomial can be interpreted as the expected symmetrized characteristic polynomial of the union of d independent uniformly random perfect matchings across a fixed balanced bipartition of the n vertices. The main theorem of Cohen [Coh16] converts a bound on the second-largest root of this polynomial into an efficient construction. Although Cohen states the result for $b = 2\sqrt{d-1}$, his proof applies without modification to every $b \in \mathbb{R}$, yielding the following slightly more general formulation.

Theorem 4.5 (Implicit in [Coh16])**.** *Let $b \in \mathbb{R}$ be such that the second-largest root of the (n,d)-root characteristic polynomial $p_{\emptyset}(x)$ is smaller than b; that is, $\alpha_2(p_{\emptyset}) \leq b$. Then there is an algorithm, running in time* $\mathrm{poly}(n)$*, that outputs a bipartite graph G satisfying $\lambda(G) \leq b$.*

Proof of Theorem 4.1. By Theorem 4.5, it suffices to improve the bound of [MSS18] on the second-largest root of $p_{\emptyset}$. Applying Lemma 4.2 $d-1$ times gives

$$\begin{aligned}p_{\emptyset} &= \mathop{\mathbf{E}}_{\substack{\mathbf{P}_1,\ldots,\mathbf{P}_d\\ \mathbf{S}_1,\ldots,\mathbf{S}_d}} \chi_x\left(\sum_{i=1}^{d}\begin{bmatrix}\mathbf{P}_i & 0\\ 0 & \mathbf{S}_i\end{bmatrix}\begin{bmatrix}0 & \mathbf{I}\\ \mathbf{I} & 0\end{bmatrix}\begin{bmatrix}\mathbf{P}_i & 0\\ 0 & \mathbf{S}_i\end{bmatrix}^{\mathsf{T}}\right)\\ &= \left(x^2-d^2\right)\mathbb{S}\left(p^{\boxplus\boxplus d}(x)\right),\end{aligned}$$

where $p(x) = (x-1)^{m-1}$, and the expectation is over independent uniformly random permutation matrices $\mathbf{P}_1,\ldots,\mathbf{P}_d,\mathbf{S}_1,\ldots,\mathbf{S}_d$. Set

$$q(x) \triangleq \mathbb{S}\left(p^{\boxplus\boxplus d}(x)\right).$$

Together with the above factorization, we have that $\alpha_2(p_{\emptyset}) = \alpha_1(q)$, and henceforth denote this common value by α.

By Claim 2.6, for every $y > 0$,

$$\alpha < \mathcal{K}_q(y). \tag{19}$$

Applying Lemma 4.3 $d-1$ times, we obtain, for every $y > 0$,

$$\mathcal{K}_q(y) < d \cdot \mathcal{K}_{\mathbb{S}p}(y) - \frac{d-1}{y} = \frac{2-d+d\sqrt{4y^2+1}}{2y}. \tag{20}$$

Here the equality uses

$$\mathbb{S}p(x) = (x-1)^{m-1}(x+1)^{m-1}$$

and the formula for its $\mathcal{K}$-transform from Equation (7). The right-hand side of Equation (20) is minimized at $y_0 = \frac{\sqrt{d-1}}{d-2}$, where it attains the value $x_0 = 2\sqrt{d-1}$. Consequently,

$$\alpha < \mathcal{K}_q(y_0) < x_0 < d.$$

The estimate $\alpha < 2\sqrt{d-1}$ recovers the bound of [MSS18] and, through Theorem 4.5, Cohen's explicit construction. We now quantify the improvement in this bound.

Since $\mathcal{K}_q(y_0) < x_0$, and since $\mathcal{G}_q$ is strictly decreasing to the right of the largest root of q with inverse $\mathcal{K}_q$, we have

$$\mathcal{G}_q(x_0) < y_0. \tag{21}$$

The polynomial q is an even real-rooted polynomial of degree $r = n-2$, where its evenness follows from the definition of the $\mathbb{S}$ operator. Write its roots as

$$\pm a_1, \ldots, \pm a_{r/2}, \qquad a_1 = \alpha \geq a_2 \geq \cdots \geq a_{r/2} \geq 0.$$

It follows from Equation (21) that

$$r\mathcal{G}_q(x_0) = \sum_{i=1}^{r/2} \left(\frac{1}{x_0 - a_i} + \frac{1}{x_0 + a_i} \right) < ry_0. \tag{22}$$

For every i,

$$\frac{1}{x_0 - a_i} + \frac{1}{x_0 + a_i} = \frac{2x_0}{x_0^2 - a_i^2} \geq \frac{2}{x_0}.$$

Isolating the term corresponding to $a_1 = \alpha$ in Equation (22) therefore yields

$$\begin{aligned} \frac{1}{x_0 - \alpha} + \frac{1}{x_0 + \alpha} &< ry_0 - \left(\frac{r}{2} - 1\right)\frac{2}{x_0} \\ &= ry_0 - \frac{r-2}{x_0} = \frac{nd-4}{2(d-2)\sqrt{d-1}}. \end{aligned} \tag{23}$$

Substituting $x_0 = 2\sqrt{d-1}$ into Equation (23) and solving for α gives

$$\alpha < 2\sqrt{d-1}\sqrt{1 - \frac{2(d-2)}{nd-4}}.$$

Using that $1 - \sqrt{1-x} \geq x/2$ for all $x \in [0, 1]$, we obtain

$$2\sqrt{d-1} - \alpha \geq \frac{2\sqrt{d-1}}{n}\left(1 - \frac{2}{d}\right) \geq \frac{\sqrt{d}}{2n},$$

which completes the proof. □

5 The $n^{-2/3}$ Advantage

In this section, we prove Theorem 1.1. Rather than working, as in MSS, with a union of d perfect matchings, we use the *configuration model*, defined below. This is a classical and extensively studied model for random regular graphs; see, e.g., [Bol80, Wor99]. It has also played an important role in the spectral analysis of random regular graphs, including Bordenave's proof of Friedman's theorem [Bor20]. In [CM25], the model was studied in connection with free projections and the analysis of non-regular graphs. Here, it allows us to reduce the problem of bounding the second-largest eigenvalue to the study of the roots of Jacobi polynomials.

Let $G = (V, E)$ be a d-regular undirected graph on n vertices, and assume that each vertex v assigns distinct labels from $\{1, \ldots, d\}$ to its d incident edges, so that every label appears exactly once. Note that each edge receives two labels—one from each of its endpoints—and these labels need not coincide. Let $v[i]$ denote the i-th neighbor of v according to this labeling.

Definition 5.1 (Edge rotation map)**.** *Let G be a graph labeled as above. The* edge rotation map *(or simply* rotation map*) of G, denoted $\mathrm{Rot}_G : V \times [d] \to V \times [d]$, is defined by*

$$\mathrm{Rot}_G(v, i) = (u, j) \iff v[i] = u \ \wedge \ u[j] = v.$$

In other words, $\mathrm{Rot}_G(v, i) = (u, j)$ if the i-th neighbor of v is u, and the j-th neighbor of u is v. Observe that Rot_G is an involution. We note that, unlike in some other random graph models, self-loops are allowed: it is possible that $\mathrm{Rot}_G(v, i) = (v, j)$ for some vertex v and indices $i, j \in [d]$, in which case v is its own i-th and j-th neighbor. We also define the associated *rotation matrix* $\dot{\mathbf{G}}$, which is an $N \times N$ Boolean matrix with $N = nd$, where $\dot{\mathbf{G}}_{(v,i),(u,j)} = 1$ if and only if $\mathrm{Rot}_G(v, i) = (u, j)$.

We now define the *configuration model* for sampling a random d-regular graph on n vertices, where $n, d \in \mathbb{N}$ and at least one of them is even. One way to visualize the process is to imagine that each vertex is initially connected to d "half-edges", and the graph is formed by pairing up the N half-edges at random.

Definition 5.2 (Configuration model for regular graphs)**.** *A random undirected d-regular graph on n vertices is said to be sampled by the* configuration model *if it is generated by choosing a uniformly random perfect matching on the nd* half-edges.

The model can also be described more formally in matrix form. To describe this, let $\mathbf{U}$ be the $N \times n$ *up* matrix, defined by its action on vectors $x \in \mathbb{R}^n$ as $\mathbf{U}x = x \otimes \mathbf{1}_d$, where $\otimes$ denotes the tensor product and $\mathbf{1}_d$ is the all-ones vector of dimension d. Let $\mathbf{M}$ denote the adjacency matrix of an arbitrary perfect matching on N vertices (recall that $N = nd$). Then, sampling a graph G from the configuration model is equivalent to choosing a random

$N \times N$ permutation matrix $\mathbf{P}$ and setting

$$\mathbf{A_P} \triangleq \mathbf{U}^\mathsf{T}\mathbf{P}\mathbf{M}\mathbf{P}^\mathsf{T}\mathbf{U}. \tag{24}$$

Thus, $\mathbf{A_P}$ is the adjacency matrix of the resulting multigraph $G_\mathbf{P}$.

Note that the middle operator $\mathbf{PMP}^\mathsf{T}$ is precisely $\dot{\mathbf{G}}$, which encodes the rotation map Rot_G. We denote the d-regular n-vertex graph whose adjacency matrix is $\mathbf{A_P}$ by $G_\mathbf{P}$.

With the configuration model in place, we are ready to prove Theorem 1.1. We will use the following simple technical claim in the proof. We will also use $f^{(k)}(x)$ to denote the k-th derivative of $f(x)$.

Claim 5.3. *Let f be a real-rooted polynomial, and let θ be the largest root of f. Define $g(x) = \frac{f(x)}{x-\theta}$. Then, for every $k \geq 0$, the largest root of $g^{(k)}$ is at most the largest root of $f^{(k)}$.*

Proof. The roots of g are obtained from the roots of f by deleting one occurrence of the largest root θ. Thus, the roots of g interlace the roots of f in the trivial sense. Since differentiation preserves interlacing, it follows that, for every $k \geq 0$, the roots of $g^{(k)}$ interlace the roots of $f^{(k)}$. In particular, the largest root of $g^{(k)}$ is at most the largest root of $f^{(k)}$. □

Proof of Theorem 1.1. For every matrix $\mathbf{P}$,

$$x^{n(d-1)} \cdot \chi_x\left(\frac{1}{d}\mathbf{U}^\mathsf{T}\mathbf{P}\mathbf{M}\mathbf{P}^\mathsf{T}\mathbf{U}\right) = \chi_x\left(\frac{1}{d}\mathbf{U}\mathbf{U}^\mathsf{T}\mathbf{P}\mathbf{M}\mathbf{P}^\mathsf{T}\right).$$

Indeed, matrices of the form AB and BA have the same nonzero eigenvalues, counted with multiplicity, while the difference in their dimensions accounts for the factor $x^{n(d-1)}$.

Recall that, for a real-rooted polynomial $p(x)$, we denote its i-th largest root by $\alpha_i(p)$. By Lemma 2.2, there exists a permutation matrix $\mathbf{P}$ such that

$$\frac{\lambda_2(G_\mathbf{P})}{d} \leq \alpha_2,$$

where

$$\alpha_2 \triangleq \alpha_2\left(\mathop{\mathbf{E}}_{\mathbf{P}} \chi_x\left(\frac{1}{d}\mathbf{U}\mathbf{U}^\mathsf{T}\mathbf{P}\mathbf{M}\mathbf{P}^\mathsf{T}\right)\right). \tag{25}$$

It therefore remains to analyze the expected characteristic polynomial on the right-hand side.

We first note that

$$\mathbf{V} \triangleq \frac{1}{d}\mathbf{U}\mathbf{U}^\mathsf{T} = \frac{1}{d}\mathbf{I}_n \otimes \mathbf{J}_d,$$

where $\mathbf{J}_d$ denotes the $d \times d$ all-ones matrix. The characteristic polynomials of $\mathbf{M}$ and $\mathbf{V}$ are therefore

$$\begin{aligned}\chi_x(\mathbf{M}) &= (x-1)^{N/2}(x+1)^{N/2},\\ \chi_x(\mathbf{V}) &= x^{n(d-1)}(x-1)^n.\end{aligned}$$

Applying Lemma 2.5, we obtain

$$\mathop{\mathbf{E}}_{\mathbf{P}} \chi_x\left(\mathbf{VPMP}^\mathsf{T}\right) = (x-1)\left(p_\mathbf{V} \boxtimes p_\mathbf{M}\right)(x), \tag{26}$$

where $p_\mathbf{V}$ and $p_\mathbf{M}$ are defined by $\chi_x(\mathbf{V}) = (x-1)p_\mathbf{V}(x)$ and $\chi_x(\mathbf{M}) = (x-1)p_\mathbf{M}(x)$, respectively. We now invoke the following lemma of Arizmendi, Garza-Vargas, and Perales, which uses the MSS framework developed in [MSS22] to give a clean characterization of the relevant special case of the multiplicative convolution.

Lemma 5.4 ([AGVP23, Lemma 3.5])**.** *Let $p(x)$ be a degree L polynomial and let $j \in [L]$. Define $q(x) = x^j(x-1)^{L-j}$. Then*

$$\left(p \boxtimes_L q\right)(x) = \frac{1}{(L)_j} x^j \cdot p^{(j)}(x), \tag{27}$$

where $(L)_j$ is the falling factorial.

Set $L = N-1$ and $j = n(d-1)$. Then

$$p_\mathbf{V}(x) = x^{n(d-1)}(x-1)^{n-1} = x^j(x-1)^{L-j}.$$

Using the commutativity of the multiplicative convolution and Lemma 5.4, we obtain

$$\left(p_\mathbf{V} \boxtimes_L p_\mathbf{M}\right)(x) = \left(p_\mathbf{M} \boxtimes_L p_\mathbf{V}\right)(x) = \frac{x^j}{(L)_j} p_\mathbf{M}^{(j)}(x).$$

Together with Equation (26), this implies that $\alpha_2 \le \max\{0, \alpha_1(p_\mathbf{M}^{(j)})\}$, where the zero accounts for the factor x^j. Recall that $\chi_x(\mathbf{M}) = (x-1)p_\mathbf{M}(x)$. Since 1 is the largest root of $\chi_x(\mathbf{M})$, Claim 5.3 gives

$$\alpha_1\left(p_\mathbf{M}^{(j)}\right) \le \alpha_1\left(\frac{\partial^j}{\partial x^j}\chi_x(\mathbf{M})\right).$$

We therefore define

$$Q_{n,d}(x) \triangleq \frac{\partial^j}{\partial x^j}\chi_x(\mathbf{M}) = \frac{\partial^j}{\partial x^j}(x^2-1)^m, \qquad m = \frac{nd}{2}.$$

The polynomial $Q_{n,d}$ is real-rooted and its roots are symmetric about zero, so its largest root is nonnegative. Consequently, $\alpha_2 \le \alpha_1 (Q_{n,d})$. Thus, it remains to bound the largest root of $Q_{n,d}$. To this end, we identify $Q_{n,d}$, up to a nonzero scalar factor, with a symmetric Jacobi polynomial. We begin by recalling the basic properties of these polynomials.

For $\gamma > -1$, the symmetric Jacobi polynomials $P_r^{(\gamma,\gamma)}(x)$ form a family of polynomials orthogonal on $[-1,1]$ with respect to the weight $(1-x^2)^\gamma$. Thus, whenever $r \ne s$,

$$\int_{-1}^{1} P_r^{(\gamma,\gamma)}(x) P_s^{(\gamma,\gamma)}(x)(1-x^2)^\gamma \, dx = 0.$$

We will use the following standard equivalent characterization: up to multiplication by a nonzero scalar, $P_r^{(\gamma,\gamma)}$ is the unique polynomial of degree r satisfying

$$(1-x^2)y'' - 2(\gamma+1)xy' + r(r+2\gamma+1)y = 0.$$

We now establish the desired identification of $Q_{n,d}$ with a symmetric Jacobi polynomial.

Claim 5.5. *$Q_{n,d}$ is a nonzero scalar multiple of $P_n^{(\alpha,\alpha)}$, where $\alpha = \frac{n(d-2)}{2}$.*

Proof. Let $F(x) = (1-x^2)^m$. Since $(x^2-1)^m = (-1)^m F(x)$, the polynomial $Q_{n,d}$ is a nonzero scalar multiple of $F^{(j)}$. The function F satisfies

$$\left(1-x^2\right) F'(x) + 2mxF(x) = 0.$$

Applying D^{j+1}, where $D = \frac{d}{dx}$, to this identity and using the generalized product rule gives

$$D^{j+1}\left(\left(1-x^2\right)F'\right) = \left(1-x^2\right)F^{(j+2)} - 2(j+1)xF^{(j+1)} - j(j+1)F^{(j)},$$

and

$$D^{j+1}(xF) = xF^{(j+1)} + (j+1)F^{(j)}.$$

Thus, setting $y = F^{(j)}$, we obtain

$$\left(1-x^2\right)y'' - 2(j-m+1)xy' + (j+1)(2m-j)y = 0.$$

Since $j - m = \alpha$ and $2m - j = n$, this becomes

$$\left(1-x^2\right)y'' - 2\left(\alpha+1\right)xy' + n\left(n+2\alpha+1\right)y = 0.$$

Moreover, y has degree n. By the differential-equation characterization above, $Q_{n,d}$ is a nonzero scalar multiple of $P_n^{(\alpha,\alpha)}$. □

With Claim 5.5 in place, our next step is to prove the following bound on the largest zero of a Jacobi polynomial.

Claim 5.6 (Jacobi root bound)**.** *Let $d \geq 3$ and $n \geq 2$, and suppose that nd is even. Let $x_{n,d}$ denote the largest root of $Q_{n,d}$. Then*

$$\frac{2\sqrt{d-1}}{d} - x_{n,d} > \frac{3}{4^{1/3}} n^{-2/3} \cdot \frac{\left(d-2+\frac{2}{n}\right)^{4/3}}{\left(d-1+\frac{1}{n}\right)^{1/6}\left(d^2+\frac{4(d-1)}{n}+\frac{4}{n^2}\right)^{5/6}}.$$

To prove Claim 5.6, we will use the following symmetric specialization of [Kra06, Theorem 2].

Theorem 5.7 (Krasikov's bound, symmetric case)**.** *Let $z_{r,\gamma}$ be the largest zero of $P_r^{(\gamma,\gamma)}$, where $\gamma > -1$. Define $s = 2\gamma+1$, $\rho = 2r+2\gamma+1$, and*

$$R = \sqrt{(\rho^2+2s+1)(\rho^2-s^2)}, \qquad B = \frac{R}{\rho^2+2s+1}.$$

Then

$$z_{r,\gamma} < B - 3\left(1-B^2\right)^{2/3}(2R)^{-1/3}.$$

Proof of Claim 5.6. By Claim 5.5, $x_{n,d}$ is the largest zero of $P_n^{(\alpha,\alpha)}$, where recall $\alpha = \frac{n(d-2)}{2}$. We apply Theorem 5.7 with $r = n$ and $\gamma = \alpha$. In the notation of that theorem, $s = n(d-2)+1$ and $\rho = nd+1$. For notational convenience, define

$$A = n\left(n(d-1)+1\right), \qquad C = n^2d^2+4n(d-1)+4.$$

A direct calculation gives $\rho^2 - s^2 = (\rho-s)(\rho+s) = 4A$, and $\rho^2+2s+1 = C$. Consequently,

$$R = 2\sqrt{AC}, \qquad B = 2\sqrt{\frac{A}{C}}.$$

Krasikov's bound (Theorem 5.7) therefore gives

$$x_{n,d} < B - 3\left(1-B^2\right)^{2/3}\left(4\sqrt{AC}\right)^{-1/3}.$$

Moreover,

$$1-B^2 = 1-\frac{4A}{C} = \frac{\left(n(d-2)+2\right)^2}{C}.$$

It follows that

$$3\left(1-B^2\right)^{2/3}\left(4\sqrt{AC}\right)^{-1/3} = \frac{3}{4^{1/3}}\frac{\left(n(d-2)+2\right)^{4/3}}{\left(n\left(n(d-1)+1\right)\right)^{1/6}\left(n^2d^2+4n(d-1)+4\right)^{5/6}}.$$

It remains to compare B with $L_d = \frac{2\sqrt{d-1}}{d}$. Since both quantities are positive, it suffices to compare their squares. We have

$$L_d^2 - B^2 = 4\left(\frac{d-1}{d^2} - \frac{A}{C}\right) = 4\frac{(d-1)\,C - d^2 A}{d^2 C}.$$

The numerator satisfies

$$(d-1)\,C - d^2 A = n(d-2)(3d-2) + 4(d-1) > 0.$$

Hence $B < L_d$. Together with the previously obtained inequality we get

$$\frac{2\sqrt{d-1}}{d} - x_{n,d} > \frac{3}{4^{1/3}} n^{-2/3} \cdot \frac{\left(d-2+\frac{2}{n}\right)^{4/3}}{\left(d-1+\frac{1}{n}\right)^{1/6}\left(d^2+\frac{4(d-1)}{n}+\frac{4}{n^2}\right)^{5/6}},$$

completing the proof of Claim 5.6. □

Returning to the proof of Theorem 1.1, we have that

$$d-1+\frac{1}{n} \leq d \qquad \text{and} \qquad d^2 + \frac{4(d-1)}{n} + \frac{4}{n^2} \leq \left(d+\frac{2}{n}\right)^2.$$

Consequently, the right-hand side of the inequality in Claim 5.6 is lower bounded by

$$\frac{3}{4^{1/3}} \cdot \frac{\left(1-\frac{2}{d}+\frac{2}{nd}\right)^{4/3}}{\left(1+\frac{2}{nd}\right)^{5/3}} \cdot \frac{1}{\sqrt{d}\cdot n^{2/3}} \geq \frac{3}{4^{1/3}}\left(1-\frac{2}{d}\right)^{5/3} \frac{1}{\sqrt{d}\cdot n^{2/3}}.$$

This completes the proof of Theorem 1.1.

□

References

[AEL95] Noga Alon, Jeff Edmonds, and Michael Luby. Linear time erasure codes with nearly optimal recovery (extended abstract). In *Proceedings of the 36th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 512–519. IEEE Computer Society, 1995.

[AGVP23] Octavio Arizmendi, Jorge Garza-Vargas, and Daniel Perales. Finite free cumulants: multiplicative convolutions, genus expansion and infinitesimal distributions. *Transactions of the American Mathematical Society*, 376(06):4383–4420, 2023.

[Alo86] Noga Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986.

[BATS11] Avraham Ben-Aroya and Amnon Ta-Shma. A combinatorial construction of almost-Ramanujan graphs using the zig-zag product. *SIAM J. Comput.*, 40(2):267–290, 2011.

[BH12] Andries E. Brouwer and Willem H. Haemers. *Spectra of Graphs*. Universitext. Springer, New York, 2012.

[BL06] Yonatan Bilu and Nathan Linial. Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 26(5):495–519, 2006.

[Bol80] Béla Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European Journal of Combinatorics*, 1(4):311–316, 1980.

[Bor20] Charles Bordenave. A new proof of Friedman's second eigenvalue theorem and its extension to random lifts. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(6):1393–1439, 2020.

[BRST23] Igor Balla, Eero Räty, Benny Sudakov, and István Tomon. Note on the second eigenvalue of regular graphs, 2023.

[CC26] Gil Cohen and Itay Cohen. Wide replacement products meet gray codes: toward optimal small-bias sets. In *41st Computational Complexity Conference*, volume 383 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Paper No. 21, 27. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2026.

[CCM24] Gil Cohen, Itay Cohen, and Gal Maor. Tight bounds for the Zig-Zag product. In *2024 IEEE 65th Annual Symposium on Foundations of Computer Science—FOCS 2024*, pages 1470–1499. IEEE Computer Soc., Los Alamitos, CA, 2024.

[CCMP25] Gil Cohen, Itay Cohen, Gal Maor, and Yuval Peled. Derandomized squaring: an analytical insight into its true behavior. In *16th Innovations in Theoretical Computer Science Conference*, volume 325 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 40, 24. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2025.

[CE98] Robert F. Coleman and Bas Edixhoven. On the semi-simplicity of the U_p-operator on modular forms. *Mathematische Annalen*, 310(1):119–127, 1998.

[CKNV16] Sebastian M. Cioabă, Jack H. Koolen, Hiroshi Nozaki, and Jason R. Vermette. Maximizing the order of a regular graph of given valency and second eigenvalue. *SIAM Journal on Discrete Mathematics*, 30(3):1509–1525, 2016.

[CM23] Gil Cohen and Gal Maor. Random walks on rotating expanders. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*, 2023.

[CM25] Gil Cohen and Gal Maor. Ultra-sparse expanders and the free method. In *Electron. Colloquium Comput. Complex.*, 2025.

[Coh16] Michael B. Cohen. Ramanujan graphs in polynomial time. In *57th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2016*, pages 276–281. IEEE Computer Soc., Los Alamitos, CA, 2016.

[Din07] Irit Dinur. The PCP theorem by gap amplification. *J. ACM*, 54(3):Art. 12, 44, 2007.

[DSV03] Giuliana Davidoff, Peter Sarnak, and Alain Valette. *Elementary number theory, group theory, and Ramanujan graphs*, volume 55 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2003.

[Fri93] Joel Friedman. Some geometric aspects of graphs and their eigenfunctions. *Duke Mathematical Journal*, 69(3):487–525, 1993.

[Gla03] Yair Glasner. Ramanujan graphs with small girth. *Combinatorica*, 23(3):487–502, 2003.

[He24] Yukun He. Spectral gap and edge universality of dense random regular graphs. *Communications in Mathematical Physics*, 405(8):181, 2024.

[HLW06] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4):439–561, 2006.

[HMY24] Jiaoyang Huang, Theo McKenzie, and Horng-Tzer Yau. Ramanujan property and edge universality of random regular graphs. *arXiv preprint arXiv:2412.20263*, 2024.

[HY26] Jiaoyang Huang and Horng-Tzer Yau. Edge universality of random regular graphs of growing degrees. *The Annals of Probability*, 54(5):2377–2435, 2026.

[INW94] Russell Impagliazzo, Noam Nisan, and Avi Wigderson. Pseudorandomness for network algorithms. In *Proceedings of the 26th Annual ACM Symposium on Theory of Computing (STOC)*, pages 356–364. ACM, 1994.

[Kra06] Ilia Krasikov. On extreme zeros of classical orthogonal polynomials. *Journal of computational and applied mathematics*, 193(1):168–182, 2006.

[KS06] Michael Krivelevich and Benny Sudakov. Pseudo-random graphs. In Ervin Győri, Gyula O. H. Katona, and László Lovász, editors, *More Sets, Graphs and Numbers*, volume 15 of *Bolyai Society Mathematical Studies*, pages 199–262. Springer, 2006.

[LP09] Tian Khoon Lim and Cheryl E. Praeger. On generalised Paley graphs and their automorphism groups. *Michigan Mathematical Journal*, 58(1):293–308, 2009.

[LPS88] Alexander Lubotzky, Ralph Phillips, and Peter Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[Lub94] Alexander Lubotzky. *Discrete Groups, Expanding Graphs and Invariant Measures*, volume 125 of *Progress in Mathematics*. Birkhäuser, Basel, 1994.

[Mar88] Grigorii Aleksandrovich Margulis. Explicit group-theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators. *Problemy peredachi informatsii*, 24(1):51–60, 1988.

[MMST96] A. Medrano, P. Myers, H. M. Stark, and A. Terras. Finite analogues of Euclidean space. *Journal of Computational and Applied Mathematics*, 68(1–2):221–238, 1996.

[MSS15] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava. Interlacing families I: Bipartite Ramanujan graphs of all degrees. *Ann. of Math. (2)*, 182(1):307–325, 2015.

[MSS18] Adam W Marcus, Daniel A Spielman, and Nikhil Srivastava. Interlacing families IV: Bipartite ramanujan graphs of all sizes. *SIAM Journal on Computing*, 47(6):2488–2509, 2018.

[MSS22] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava. Finite free convolutions of polynomials. *Probab. Theory Related Fields*, 182(3-4):807–848, 2022.

[Nil91] A. Nilli. On the second eigenvalue of a graph. *Discrete Math.*, 91(2):207–210, 1991.

[Pal33] R. E. A. C. Paley. On orthogonal matrices. *Journal of Mathematics and Physics*, 12(1–4):311–320, 1933.

[PV25] Ricardo A. Podestá and Denis E. Videla. Spectral properties of generalized Paley graphs. *Australasian Journal of Combinatorics*, 91(3):326–365, 2025.

[Rei08] Omer Reingold. Undirected connectivity in log-space. *J. ACM*, 55(4):Art. 17, 24, 2008.

[RST26] Eero Räty, Benny Sudakov, and István Tomon. Positive discrepancy, MaxCut, and eigenvalues of graphs. *Transactions of the American Mathematical Society*, 379(3):2111–2140, 2026.

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Ann. of Math. (2)*, 155(1):157–187, 2002.

[Sar23] Peter Sarnak. Prescribing the spectra of locally uniform geometries: Cubic graphs. Notes for Lecture 2 of the 2023 Chern Lectures, University of California, Berkeley, 2023.

[Spi25] Daniel A. Spielman. Spectral and algebraic graph theory. Unpublished book manuscript, 2025.

[Sri24] Nikhil Srivastava. Ramanujan graphs and interlacing families. *arXiv preprint arXiv:2412.20721*, 2024.

[SS94] Michael Sipser and Daniel A. Spielman. Expander codes. In *Proceedings of the 35th IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 566–576, 1994.

[TS17] Amnon Ta-Shma. Explicit, almost optimal, epsilon-balanced codes. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 238–251. ACM, 2017.

[Vad12] Salil P. Vadhan. Pseudorandomness. *Foundations and Trends in Theoretical Computer Science*, 7(1–3):1–336, 2012.

[Wor99] N. C. Wormald. Models of random regular graphs. In *Surveys in combinatorics, 1999 (Canterbury)*, volume 267 of *London Math. Soc. Lecture Note Ser.*, pages 239–298. Cambridge Univ. Press, Cambridge, 1999.

A The LPS Construction is Super-Ramanujan

In this appendix, we record that the Lubotzky–Phillips–Sarnak graphs are, in fact, super-Ramanujan, and quantify their advantage over the Ramanujan threshold. We assume some familiarity with the LPS construction; for an accessible introductory treatment, see the excellent monograph of Davidoff, Sarnak, and Valette [DSV03].

Let $X^{p,q}$ be an LPS graph, which is $(p+1)$-regular. By the Hecke-theoretic realization used in the proof of [LPS88, Theorem 4.1], every nontrivial eigenvalue of $X^{p,q}$ is of the form $a_p(f)$, where f is a normalized weight-2 cuspidal Hecke eigenform of level coprime to

p. Coleman and Edixhoven [CE98] proved that every such eigenvalue satisfies the strict inequality $|a_p| < 2\sqrt{p}$.

The following elementary norm argument turns the qualitative strictness into a quantitative lower bound on the advantage. It first gives a lower bound of the form $\exp(-O_p(n))$. We then exploit the large eigenvalue multiplicities forced by the symmetries of the LPS construction to improve this to $\exp(-O_p(n^{2/3}))$. We emphasize that the exponent $2/3$ appearing here is unrelated to the exponent $2/3$ in our main result. Here it arises solely from the facts that $n = \Theta(q^3)$ and that every nontrivial eigenvalue has multiplicity $\Omega(q)$; the numerical coincidence is accidental.

Theorem A.1. *Let $G = X^{p,q}$ be an LPS graph on n vertices. Then*

$$\lambda(G) \leq 2\sqrt{p} - (4p)^{-(n-1)}.$$

Proof. Let $\mathbf{A}$ be the adjacency matrix of G, and fix a nontrivial eigenvalue λ of $\mathbf{A}$. By the preceding Hecke-theoretic realization and the strict Ramanujan inequality of Coleman and Edixhoven [CE98, Theorem 4.1],

$$|\lambda| < 2\sqrt{p}. \tag{28}$$

Let $f \in \mathbb{Z}[x]$ be the minimal polynomial of λ, and denote its degree by r. Since $\mathbf{A}$ is an integer matrix, λ is an algebraic integer, and hence f is monic. Moreover, f divides $\chi_x(\mathbf{A})$. Since $\lambda \neq p+1$, the polynomial f in fact divides $\frac{\chi_x(\mathbf{A})}{x-(p+1)}$, and therefore $r \leq n-1$.

Let $\lambda = \lambda_1, \ldots, \lambda_r$ be the algebraic conjugates of λ. Since f divides $\chi_x(\mathbf{A})$, every λ_i is also an eigenvalue of $\mathbf{A}$. Furthermore, these eigenvalues are all nontrivial. Indeed, if some λ_i were equal to $p+1$, or to $-(p+1)$ when G is bipartite, then the irreducibility of f would imply that f is linear, contradicting the fact that λ is nontrivial. The Ramanujan property of the LPS construction thus gives

$$|\lambda_i| \leq 2\sqrt{p} \qquad \text{for every } i \in [r]. \tag{29}$$

Consider the algebraic integer $\delta \triangleq 4p - \lambda^2$. By Equation (28), δ is nonzero. Its norm over $\mathbb{Q}$ is therefore a nonzero integer, and consequently

$$1 \leq \left|\mathrm{N}_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\delta)\right| = \prod_{i=1}^{r} \left|4p - \lambda_i^2\right| \leq \left(4p - \lambda^2\right)(4p)^{r-1},$$

where the final inequality follows from Equation (29). Hence,

$$4p - \lambda^2 \geq (4p)^{-(r-1)} \geq (4p)^{-(n-2)}.$$

It follows that

$$2\sqrt{p} - |\lambda| = \frac{4p - \lambda^2}{2\sqrt{p} + |\lambda|} \geq \frac{1}{4\sqrt{p}\,(4p)^{n-2}} \geq (4p)^{-(n-1)}.$$

Since λ was an arbitrary nontrivial eigenvalue of G, the proof follows. □

As mentioned, the exponent can be improved further by exploiting the large multiplicities of the eigenvalues of LPS graphs as stated in the proposition below. Our limited numerical experiments suggest, however, that the true advantage may be substantially larger, possibly even inverse-polynomial. The numerical evidence is not yet sufficient to support a precise conjecture, and we leave a more systematic investigation of this fascinating question to future work.

Proposition A.2. *Let $G = X^{p,q}$ be an LPS graph on n vertices. Then*

$$\lambda(G) \leq 2\sqrt{p} - (4p)^{-\frac{2n}{q-1}}.$$

In particular, for fixed p,

$$2\sqrt{p} - \lambda(G) \geq \exp\bigl(-O_p(n^{2/3})\bigr).$$

Proof. We refine the bound on the degree r of the minimal polynomial used in the proof of Theorem A.1. Recall that $X^{p,q}$ is a Cayley graph of either $\mathrm{PSL}_2(\mathbb{F}_q)$ or $\mathrm{PGL}_2(\mathbb{F}_q)$. Its adjacency operator commutes with the left-regular action of the corresponding group. Moreover, every irreducible representation that does not give rise to one of the trivial eigenvalues has dimension at least $(q-1)/2$; see, e.g., [DSV03]. Consequently, every nontrivial eigenvalue of $X^{p,q}$ has multiplicity at least $(q-1)/2$.

The r algebraic conjugates of λ are distinct nontrivial eigenvalues. Therefore,

$$r \cdot \frac{q-1}{2} \leq n.$$

Repeating the norm calculation from the proof of Theorem A.1 gives

$$2\sqrt{p} - |\lambda| \geq \frac{(4p)^{-(r-1)}}{4\sqrt{p}} \geq (4p)^{-r} \geq (4p)^{-\frac{2n}{q-1}}.$$

Finally, since n is either $q(q^2-1)$ or $\frac{1}{2}q(q^2-1)$, we have $\frac{2n}{q-1} = O(n^{2/3})$, completing the proof. □

ECCC ISSN 1433-8092

https://eccc.weizmann.ac.il