

Almost-Everywhere Near-Cubic Wire Lower Bounds for $\text{SYM} \circ \text{THR}$ and $\text{THR} \circ \text{THR}$

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August 17, 2026

Abstract

Kane and Williams proved average-case wire lower bounds at the $n^{5/2}/\text{polylog } n$ scale for an explicit function against depth-two linear-threshold circuits. We prove an almost-everywhere near-cubic wire lower bound for a language in E^{NP} . For every fixed $c > 0$, there is one language F_c and positive constants $b_{\text{S},c}$ and $b_{\text{T},c}$ such that, at every sufficiently large input length n , no $\text{SYM} \circ \text{THR}$ circuit with at most $b_{\text{S},c}n^3/\log^{10} n$ wires and no $\text{THR} \circ \text{THR}$ circuit with at most $b_{\text{T},c}n^3/\log^{12} n$ wires has agreement at least $1/2 + n^{-c}$ with $(F_c)_n$. The same language works for both classes, and the gates may have arbitrary real weights. At any fixed positive advantage, the logarithmic denominators improve to $\log^5 n$ and $\log^9 n$.

The algorithmic core is a deterministic circuit-acceptance-probability algorithm that charges a restricted circuit by the number of input wires touching the live variables, rather than by its number of bottom gates. Exact residualization removes both constant-zero and constant-one bottom gates from the algebraic population. Multiscale counting polynomials and exact signed rectangular multiplication then give inverse-polynomially accurate analysis with an arbitrary fixed polynomial saving. A componentwise Chen–Lyu–Williams transfer, its XOR contrapositive, and an exact length schedule yield the stated correlation lower bounds.

Keywords: circuit lower bounds; threshold circuits; wire complexity; circuit analysis; hardness amplification.

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AI Disclosure

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1 Introduction

1.1 Result and contribution

Wire complexity is the natural resource when a depth-two circuit may contain many bottom gates with very small support. Gate-count arguments treat all bottom gates alike; a wire bound must instead account for how often input variables actually occur. The strongest previous average-case lower bound in this unrestricted simple-edge setting is the $n^{5/2}/\text{polylog } n$ -scale theorem of Kane and Williams. We raise the wire exponent to $3 - o(1)$, at the cost of placing the hard language in E^{NP} rather than linear time.

More precisely, for every fixed $c > 0$, one language is hard at every sufficiently large length against both $\text{SYM} \circ \text{THR}$ and $\text{THR} \circ \text{THR}$. Agreement $1/2 + n^{-c}$ requires, respectively, more than a fixed positive multiple of $n^3/\log^{10} n$ and $n^3/\log^{12} n$ wires. The circuit gates may have arbitrary real weights, and there is no separate bound on the number or support sizes of the bottom gates. [Theorem 2.3](#) gives the formal statement.

The main algorithmic contribution is a deterministic *circuit-acceptance-probability problem* (CAPP) algorithm for a conjunction of at most four input circuits; we abbreviate this task by $\text{AND}_{\leq 4}\text{-CAPP}$. After keeping $K = \Theta(\log n)$ variables live, the algorithm charges only bottom supports that touch the live set. Exact residualization removes both constant-zero and constant-one restricted gates. The resulting polynomial degree depends on the square root of the touched population, enabling the near-cubic wire cap. The supplier theorems [Theorem 2.1](#) and [Theorem 2.2](#) may be of independent algorithmic interest.

1.2 Related work

Threshold logic originates in the work of Muroga, Toda, and Takasu [43]; standard background includes Muroga’s monograph [42], comparisons between majority and weighted threshold gates [28, 22], bounded-depth threshold circuits [25], and finite integer-weight normalization [27]. At depth two, relevant satisfiability and counting algorithms include the sparse-wire regime [35], dense special cases [3], average-case analysis of small threshold circuits [15], polynomial-representation methods [1], Tamaki’s subquadratic-gate algorithm [54], and Williams’s satisfiability algorithm for $\text{ACC} \circ \text{THR}$ circuits [59], the first contact of the algorithmic method with unbounded-weight threshold gates. These results provide context; the supplier proved here does not use them as black boxes.

Wire lower bounds for threshold circuits begin with Impagliazzo, Paturi, and Saks [34]: at depth d , computing parity requires $n^{1+c^{-d}}$ wires for a fixed $c > 1$, and at depth two it requires $\Omega(n^{3/2})$ wires. The nearest wire theorem is due to Kane and Williams [39, Theorem 1.1]. They proved that the linear-time computable Andreev function [4] cannot have agreement $1/2 + \varepsilon$ with a depth-two linear-threshold circuit using $o(\varepsilon^3 n^{5/2}/\log^{7/2} n)$ wires, in their stated range $\varepsilon \gg \sqrt{\log n/n}$. At advantage $\varepsilon = n^{-c}$ with $c > 1/2$ that theorem is outside its stated range, and inside the range its wire scale decays with ε^3 ; the inverse-polynomial-advantage regime of [Theorem 2.3](#) is not addressed by it. Their target is substantially more explicit than ours. In the gate measure, Chen, Tal, and Wang [13, Theorem 1.1] subsequently crossed the quadratic barrier, proving $n^{5/2-\delta}$ lower bounds

for every fixed $\delta > 0$ by a faster-analysis argument. Kabanets, Kane, and Lu [38, Theorem 1.1] give a second restriction-based comparison: their explicit P-time function has worst-case wire lower bound $n^{3/2+1/d}/\log^{O(d^2)} n$ against degree- d polynomial-threshold gates, whose $d = 1$ specialization again reaches the $n^{5/2}/\text{polylog } n$ scale. For the weight-restricted class $\text{THR} \circ \text{MAJ}$, exponential gate lower bounds are classical: a polynomial-weight bottom gate has a polynomial-rank communication matrix, so sign-rank lower bounds apply, giving exponential bounds for inner product by Forster [18] and for a function in AC^0 by Razborov and Sherstov [50]. No comparable route is known once the bottom weights are unrestricted, which is the regime addressed here.

Our lower-bound conversion belongs instead to the algorithmic method. Williams established the basic principle that nontrivial circuit analysis can force circuit lower bounds [58], and his nonuniform ACC work contains the exact Boolean rectangular-multiplication endpoint used by our batch printer [60, Corollary 4.4 and Appendix C, Corollary C.2]. Later self-improvement results explain why the exact strength and uniformity of an analysis algorithm matter [61], and the scope and oracle-relative limits of the algorithmic method itself are studied by Vyas and Williams [57]. Chen and Williams strengthened the analysis-to-lower-bound connection using PCPs of proximity [14]. The pointwise PCPP formulation later used by Chen, Lyu, and Williams is attributed there jointly to Chen–Williams and Vyas–Williams [56]. Chen, Lyu, and Williams supplied the fixed-machine refuter and almost-everywhere conversion [11], building on the almost-everywhere nondeterministic-time hierarchy of Fortnow and Santhanam [19]; Chen and Ren developed strong average-case consequences [12]. The easy-witness and depth-lower-bound developments of Murray and Williams [44] and Bathie and Williams [5] are adjacent uses of the same broader philosophy, but neither is imported in our proof.

The proof-checking layer also has a longer provenance. Robust PCPs of proximity [6], short PCPs with polylogarithmic query complexity [7], and projection-query PCPs [8] culminate, for our purposes, in the specific restatements and composition proved in [14, Lemmas 25–27] and [11, Lemmas 3.10–3.11 and Theorem 4.6]. We cite those exact latter statements at the load-bearing uses in Appendix C; the earlier papers support the historical narrative and do not silently enlarge the imported contract.

Several standard pseudorandom tools enter the supplier only through sharply delimited roles. Exact-threshold circuits were studied directly by Hansen and Podolskii [26], and Coppersmith’s rectangular multiplication [16] is the historical precursor of the exact Boolean product we actually import from [60]. Universal hashing [9] and limited-independence tail bounds [52], together with bounded-independence fooling of halfspaces [17], provide context for the local pairwise-hash list construction. Its multiscale counting-polynomial design is closest in spirit to probabilistic polynomials for symmetric functions [2], although the deterministic zero-advice construction here is proved locally. The walk originates in Margulis’s explicit expander [41]; the spectral constant is taken from the Gabber–Galil construction [20], while Hoory, Linial, and Wigderson [30, Theorem 8.2] provide a modern exposition; our walk tail is proved locally, rather than imported from Gillman’s more general expander Chernoff bound [21]. Randomness recycling [37], standard Boolean-function analysis [48], and the explicit prime estimates of Rosser and Schoenfeld [51, Theorem 4, eq. (3.14)] likewise locate the tools historically. Appendix A states exactly which of these facts is imported and which is reproved.

Finally, the passage from worst-case hardness to correlation hardness lies in the line from Yao’s XOR principle [62] and its later formulation by Goldreich, Nisan, and Wigderson [24], through hard-core distributions [31] and hard-core predicates [23], hardness versus randomness [45, 36], randomness-efficient generators and extractors [46, 55], and the code-based amplification of Sudan, Trevisan, and Vadhan [53, Theorem 24]. Subsequent work refined amplification within NP [47], with nondeterminism [29], and through uniform direct-product/list-decoding theorems [32, 33] and

a new derandomized XOR lemma [10]. The present proof does not combine these results ad hoc: its only worst-case amplifier import is [11, Lemma 3.9], with [53, Theorem 24] recorded as the primary provenance, and its only approximate-sum XOR import is [11, Lemma 3.8], whose proof adapts Levin’s proof of the XOR lemma [40].

This paper moves the *wire* exponent for otherwise unrestricted circuits in the normalized, strictly layered simple-edge model from the established $5/2$ scale to $3 - o(1)$. The price is explicitness: our hard targets lie in E^{NP} and, for inverse-polynomial advantage, the language may depend on the fixed advantage exponent. Our result therefore does not dominate [39] in every parameter. Its contribution is the combination of a near-cubic wire scale, arbitrary real weights, no auxiliary bottom-gate-count or support promise beyond the physical-wire cap itself, inverse-polynomial correlation hardness, an almost-everywhere length quantifier, and one language shared by symmetric and threshold top gates. The almost-everywhere quantifier is measured against prior algorithmic-method lower bounds, which are infinitely-often statements without the machinery of [11]; the combinatorial theorem of [39] itself holds at every length of its target family. To the best of our knowledge, [39] remains the nearest published unrestricted simple-edge average-case wire lower bound. Our novelty claim concerns the conjunction of the near-cubic scale, arbitrary weights, inverse-polynomial advantage, and the almost-everywhere quantifier; no imported ingredient is claimed as new.

The new resource idea is that a wire budget should be charged as a wire budget. A restriction argument that first upper-bounds the number of bottom gates becomes useless when a circuit has very many thin gates. We instead select $K = \Theta(\log n)$ live variables by the total support incidence. If the circuit has W physical wires, the number of carrier gates touched by the live set is $O(KW/n)$. After residualization, every untouched or constant gate vanishes from the algebraic carrier. The resulting degree depends on the square root of the touched population rather than on the total gate count. That change is exactly what permits a near-cubic wire cap.

The second difficulty is compositional. The almost-everywhere average-case transfer needs estimates for degree-four expressions in componentwise circuit sums, not merely a one-circuit acceptance estimate. The threshold-top route must therefore compose residualization, a zero-advice list, inverse-polynomial walk accuracy, disjoint exact children, prime sketches, radix lookup, and exact batch printing without losing a row, a logarithm, or the pointwise error guarantee. [Section 4](#) and [Appendix A](#) give these fixed-accuracy symmetric, high-accuracy symmetric, and high-accuracy threshold suppliers as one integrated theorem. [Section 5](#) and [Appendix C](#) then restate the exact CLW/CW contracts and replay the weak machine, including its fixed-code order, object types, one-sidedness, common-function contradiction, and target schedule.

1.3 Comparison by resource measure

The two primary measures are incomparable at the relevant scales. In a strictly layered simple-edge circuit with G retained bottom occurrences,

$$G \leq W \leq (n + 1)G.$$

Consequently a lower bound $W \geq n^3 / \text{polylog } n$ implies only $G \geq n^2 / \text{polylog } n$, well below the near- $n^{5/2}$ gate scale of [13]. Conversely, a near- $n^{5/2}$ gate lower bound gives only a wire lower bound of the same order, far below $n^{3-o(1)}$. The wire and gate questions therefore require different resource compilers and neither type of bound implies the other. Every specialization used is proved in this manuscript.

The gate-count consequence is nevertheless at the scale of Tamaki’s algorithmic lower-bound corollary [54], which rules out $O(n^2 / \log^b n)$ depth-two symmetric/threshold gates for an explicit function in E^{NP} for a suitable constant b . The present result is distinguished by its wire measure,

Table 1: Comparison with the nearest unrestricted simple-edge wire lower bound. Neither column dominates the other in all parameters.

Feature	Kane–Williams [39]	This paper
Target	explicit linear-time function	a language in E^{NP}
Wire scale	$\varepsilon^3 n^{5/2} / \log^{7/2} n$	$n^3 / \text{polylog } n$
Advantage	$1/2 + \varepsilon$ for $\varepsilon \gg \sqrt{\log n / n}$	fixed or $1/2 + n^{-c}$
Lengths	every length of its target family	every sufficiently large length
Top classes	threshold top	symmetric and threshold top, one language
Restrictions	arbitrary weights	arbitrary weights; physical-wire cap only

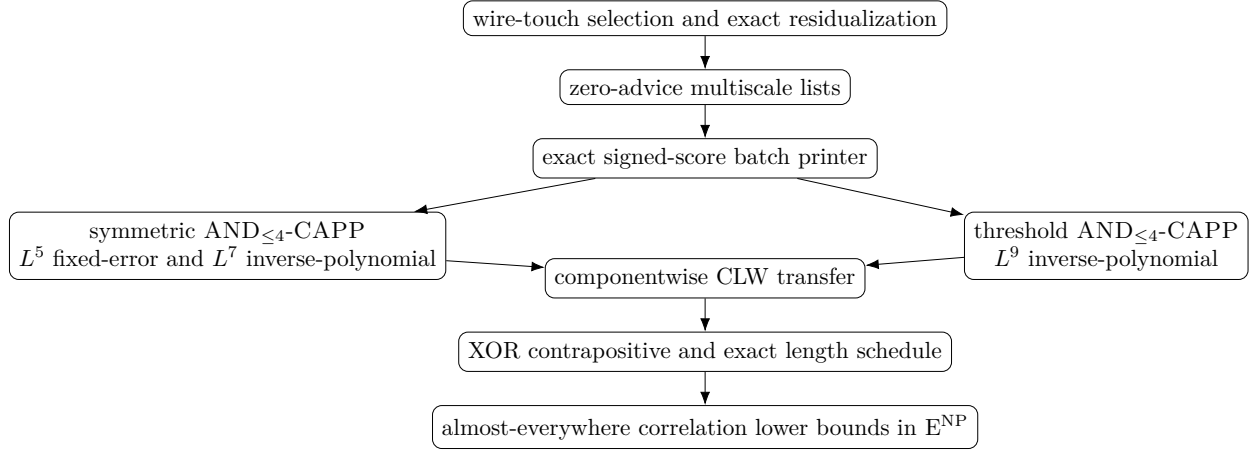


Figure 1: Dependency structure of the proof.

correlation guarantee, arbitrary weights, and almost-everywhere length quantifier; it is not a stronger gate lower bound.

Table 1 records the comparison across the relevant resource and correlation measures.

Figure 1 summarizes the dependency structure of the proof.

The manuscript distinguishes imported facts from local work at every seam. Finite threshold normalization, the qualitative provenance of the disjoint exact-threshold conversion, exact Boolean rectangular multiplication, the Gabber–Galil spectral estimate [20] in the normalization of [30, Theorem 8.2], the Rosser–Schoenfeld prime estimate [51, Theorem 4, eq. (3.14)], and the CLW/CW verifier–refuter interfaces are imported. The local crossing construction and all of its quantitative bounds are replayed locally. Residualization, touching selection, the zero-advice recurrence, walk amplification, exact row construction, signed reduction, all resource ledgers, the componentwise adapters, and the final scheduling are proved here.

1.4 Proof overview

Fix the supplier arity q , let $L_q = \lceil \log_2(q + 2) \rceil$, and, for a fixed positive integer κ , keep $K = \kappa L_q$ variables live. For a bottom halfspace $X_i(y, z)$, define $C_i(z)$ to be the indicator that $X_i(\cdot, z)$ is identically one and put

$$Z_i = X_i \oplus C_i = X_i - C_i.$$

Both constant-zero and constant-one restrictions now give $Z_i = 0$; the removed contribution

survives only in a scalar top offset. Choosing the live set by conditional expectation over support incidences gives, on every column,

$$R = O(KW/q)$$

touched carriers across all four input circuits. Complements and fixed-radix clones reuse those supports. A pairwise nested hash then gives a complete one-hot list of degree

$$D_0 = O(\sqrt{R} + L_q)$$

with constant pointwise error and no advice. When inverse-polynomial error is needed, componentwise majority along one explicit expander walk multiplies degree by $t = \Theta(L_q)$. A threshold top first becomes a disjoint family of exact equations; a small-prime surrogate and $\tau = \Theta(L_q)$ radix digits use the same walk and the same touched-carrier pool.

All list syntax is compiled into total Boolean rows before any column is read. An occurrence-indexed truncated-binomial identity turns the parity of the raw monomial count into one signed family of exact equations. The exact signed-score printer evaluates all columns simultaneously. Its only algebraic import is exact Boolean rectangular multiplication; signed weights, ties, odd arity, and weighted dominance are derived locally. The three resulting load inequalities and wire caps are collected once, in [Table 2](#).

The algorithm may print q^{h_D} external rows. Choosing $\kappa > h_D + \sigma + 10$ makes their column scans cost at most $2^q/q^\sigma$. Radix tuples, trace tuples, and exact-child choices are internal syntax whose canonical expansion is paid once per row in

$$T_{\text{prep}} \leq 2^{O_D(dL_q + L_q^2 + t)} = 2^{o(q)}.$$

The truncated-binomial (BinLift) subsets are accounted for separately: they form the signed exact-threshold occurrence family s_{gen} in [Theorem A.5](#), whose size is absorbed by the $2^{q-K} \text{poly}(q)$ signed-score printing term. They are not included in T_{prep} . The preprocessing term is added to the column time. Every possible scalar residue row is printed first; a column then selects one already printed cell. These two orderings are what rule out an uncharged row family and an adaptive choice of advice.

The lower-bound transfer works componentwise. A guessed real proof coordinate has the form $T = \sum_a \alpha_a C_a$, where each atom C_a obeys the wire cap separately. Expanding the CLW Booleanity, systematic-consistency, second-moment, and clause polynomials produces conjunctions of at most four atoms. Term count controls the number of calls, coefficient mass controls the required accuracy, and neither is added to atom wires. Explicit parity coordinates cost only $O(q)$ wires for a symmetric top and $O(q^2)$ for a threshold top.

One fixed weak machine freezes its data in the following order before it sees a source length: the source transformations and two class presentations; exact width padding and finite syntax envelopes; the target advantage and target-class simulation exponent; the general-oracle exponent and Case-1 amplifier constants; the XOR power, componentwise envelopes, and validity tolerance; and finally the supplier accuracy, saving, cap coefficients, programs, and onsets. Only then is the finite union of the two class modes fixed.

Outer-verifier duplicates are deleted before the PCPP because they form a Boolean conjunction; indexed PCPP duplicates are retained because they carry probability mass. Pointwise PCPP soundness is applied before averaging, so every branch rejects a hierarchy no-instance. The CLW refuter therefore produces, at every sufficiently large source length, a yes-instance on which both modes reject. The deterministic honest PCPP assignment defines one canonical Boolean proof-table

function common to the two modes. A nearby componentwise sum would create an accepting branch, a contradiction.

Finally, the literal CLW XOR contrapositive turns this sum-hard function into correlation hardness. For advantage n^{-c} , take $k = \Theta_c(\log q)$ and retain the stronger reserve $m^{-(c+1)}$ at core arity $m = k(q + O(\log q))$. A correlator becomes only a polynomial family of literal restrictions of itself, so every atom keeps the same absolute wire charge. The cubic arity dilation costs exactly three logarithmic powers, taking the high-accuracy L^7/L^9 supplier denominators to the final L^{10}/L^{12} denominators. At fixed advantage $k = O(1)$, so the fixed-accuracy symmetric L^5 and threshold L^9 rows pass through unchanged. A bounded-jump exact schedule gives $n/3 \leq m \leq n/2$ at every sufficiently large target length; hence $m^{-(c+1)} < n^{-c}$ and a fixed coefficient absorbs the target-to-core wire ratio. Canonical SAT recovery places the resulting total language in E^{NP} .

The delicate claims and their proof locations are therefore visible before the technical proof begins:

Audit map for the main proof seams.

Potential failure	Disposition
thin-gate explosion	charged by support incidences, not gate count
constant-one drift	exact residual $Z_i = X_i - C_i$ and one scalar offset
extra radix or child row	internal syntax versus external-row ledger
column-dependent advice	print every residue row before selecting a cell
pointwise/average mismatch	pointwise list/prime bounds, then Fubini
hidden source-size factor	projection recovery and duplicate outer-clause compaction
illegal aggregate circuit	componentwise caps; sums are never composed for CAPP
length-dependent weak machine	explicit freeze order and fixed program codes
wrong post-padding advantage	$m^{-(c+1)} < n^{-c}$ reserve on $m \geq n/3$

2 Results and model

All logarithms are base two, and

$$L = L(n) = \lceil \log_2(n + 2) \rceil.$$

Physical wires count input-to-bottom incidences and top incidences in a strictly layered simple-edge presentation. Parallel uses are explicit occurrences, zero-weight inputs and zero top coefficients are deleted, and constants are propagated before the resource is measured.

Some depth-two conventions also permit original input variables to feed the top gate directly. Replace each such input-to-top incidence by a fresh identity bottom threshold $I_j(x) = \mathbf{1}[x_j \geq 1]$ and one incidence from I_j to the top, retaining the original top weight in the threshold-top case. The symmetric-top case is the same unweighted replacement. Each direct wire becomes exactly two strictly layered wires, all other wires are unchanged, and the computed function is identical. Consequently every theorem below also holds for this skip-layer convention after shrinking its displayed positive wire coefficient by a factor of two.

A threshold gate is $\mathbf{1}[\sum_i w_i x_i \geq \theta]$, and a symmetric gate is an arbitrary Boolean function of the unweighted number of its accepting inputs. Both circuit classes have bottom threshold gates and one top gate of the displayed type. If bottom occurrence G_i has support $\text{supp}(G_i)$ and a nonzero top incidence, then

$$W(C) = \sum_i (|\text{supp}(G_i)| + 1).$$

Agreement means

$$\text{agr}(C, f) = \Pr_{x \in \{0,1\}^n} [C(x) = f(x)].$$

We use $E^{NP} = \text{DTIME}(2^{O(n)})^{NP}$.

Threshold gates may have arbitrary real weights semantically. For one gate, first delete every zero-weight input and regard the remaining variables as its support. On that finite Boolean cube, standard Muroga normalization, equivalently [13, Lemma 3.2], supplies an equivalent integer threshold description with every parameter in $[-s^s, s^s]$, where s is that gate's retained fan-in. Extend this description by zero on omitted variables. It need not preserve the numerical weights; the point needed here is that it introduces neither a new variable incidence nor another gate occurrence.

Throughout the algorithmic statements, **normalized** means this support-first bounded integer representation, with a fixed canonical binary syntax; a threshold top is normalized after zero top coefficients are deleted, while a symmetric top is stored as its $(M+1)$ -bit count lookup. At the displayed wire caps, the total normalized description length is bounded by one fixed polynomial in the arity. Thus the deterministic algorithms receive only a finite description with a uniform polynomial-bit envelope. An alleged arbitrary-real circuit is replaced existentially, gate by gate, before the transfer guesses it, and literal restrictions, normalization, coefficient collection, and constant propagation never increase physical wires. Fixing inputs preserves both classes and cannot add wires. Output negation changes only the symmetric lookup, or replaces a top inequality $L \geq \theta$ by $-L \geq -\theta + 1$, so it also preserves the class and wire count.

Both target classes contain parity below the displayed caps. For a symmetric top, use identity bottoms and the parity lookup. For a threshold top, use the staircase gates $T_j(x) = \mathbf{1}[\sum_i x_i \geq j]$ and the top gate

$$\mathbf{1} \left[\sum_{j=1}^n (-1)^{j+1} T_j(x) \geq \frac{1}{2} \right].$$

This uses $O(n^2)$ physical wires, which is $o(n^3/L^p)$ for every displayed p .

2.1 Notation and terminology

For a Boolean circuit C on q inputs, CAPP asks for an additive estimate of $\Pr[C(x) = 1]$ under the uniform distribution. An $\text{AND}_{\leq 4}$ -CAPP instance is the conjunction of at most four separately represented circuits. We use *supplier* for the deterministic $\text{AND}_{\leq 4}$ -CAPP algorithm furnished to the lower-bound transfer, and *printer* for the batch subroutine that outputs an exact table of signed scores. These terms name algorithms, not additional circuit classes.

Local terminology and its standard algorithmic meaning.

Term	Meaning
supplier	deterministic $\text{AND}_{\leq 4}$ -CAPP algorithm used by the analysis-to-lower-bound transfer
printer	batched exact score evaluation by rectangular matrix multiplication
carrier	a residual bottom-gate coordinate that can still vary on the live variables
touching	incidence of a retained live variable in a bottom-gate support
residualization	subtracting the column-dependent constant-one indicator so that every constant restricted gate has residual value zero
zero-advice list	an explicit multiscale counting-polynomial list whose correct entry is selected from the input column, without guessed advice

The “zero-advice” qualifier distinguishes the local deterministic list from the sampled probabilistic-polynomial construction of [2]: every seed is enumerated explicitly, and the correct row is selected only from the input column.

Principal parameters.

Symbols reused as local functions or linear forms are qualified in the corresponding section.

Symbol	Meaning
n	final input length in the lower-bound theorems
q	native arity of a supplier call or padded verifier
$L(r)$	$\lceil \log_2(r+2) \rceil$; we write $L_q = L(q)$
$W(C)$	physical-wire count of a normalized depth-two circuit
$K = \kappa L_q$	number of variables retained by the restriction; $\kappa \in \mathbb{N}$ is fixed
R	number of carrier coordinates touched by the live variables
D_0	degree of the unamplified multiscale list
D	requested accuracy exponent
d_{row}	Boolean degree of a final printer row
σ	requested polynomial saving exponent
t, τ	expander-walk length and number of radix digits
Q	modulus exponent in the truncated-binomial lift
h_D	exponent bounding the number of external rows
ρ_{coef}	exponent bounding coefficient mass in the transfer
C_{batch}	absolute constant in the exact batch theorem
C_{clk}	polynomial-clock exponent in the CLW transfer

The circuit-analysis result is a standalone algorithmic contribution. A *normalized* threshold gate is reduced to its nonzero support of size s and represented by integer parameters in $[-s^s, s^s]$, as supplied by [43]; see also [13, Lemma 3.2]. We fix one canonical binary syntax. A threshold top is normalized on its retained bottom occurrences, while a symmetric top is represented by its count lookup.

Theorem 2.1 (Inverse-polynomial fourfold supplier). *For every fixed accuracy exponent $D > 0$ and requested polynomial saving exponent $\sigma > 0$, there are constants $a_S, a_T > 0$ and deterministic algorithms with the following behavior. Given at most four normalized q -input circuits from one of the two named classes, the algorithm estimates their conjunction probability to error at most q^{-D} in total time*

$$\frac{2^q}{q^\sigma},$$

including input processing, whenever every input has, respectively,

$$W(C_i) \leq a_S \frac{q^3}{L_q^7} \quad \text{or} \quad W(C_i) \leq a_T \frac{q^3}{L_q^9}.$$

The constants may depend on D and σ .

Theorem 2.2 (Fixed-accuracy symmetric supplier). *For every fixed $0 < \epsilon < 1/2$ and $\sigma > 0$, there are a constant $a_S^{\text{fix}} > 0$ and a deterministic algorithm that estimates the conjunction probability of at most four normalized q -input SYM $\circ$ THR circuits to additive error at most ϵ in total time $2^q/q^\sigma$, including input processing, whenever every input satisfies*

$$W(C_i) \leq a_S^{\text{fix}} \frac{q^3}{L_q^5}.$$

The constant may depend on ϵ and σ .

The proofs of [Theorem 2.1](#) and [Theorem 2.2](#) occupy Appendices [A](#) and [B](#). The lower-bound transfer begins in [Section 5](#).

Theorem 2.3 (Inverse-polynomial correlation hardness). *For every fixed $c > 0$, there exist a language*

$$F_c \in E^{NP}$$

and constants $b_{S,c}, b_{T,c} > 0$ such that the following holds at every sufficiently large input length n .

If $C \in \text{SYM} \circ \text{THR}$ and

$$W(C) \leq b_{S,c} \frac{n^3}{L^{10}},$$

then

$$\text{agr}(C, (F_c)_n) < \frac{1}{2} + n^{-c}.$$

If $C \in \text{THR} \circ \text{THR}$ and

$$W(C) \leq b_{T,c} \frac{n^3}{L^{12}},$$

then the same correlation inequality holds.

The same language works for both classes. The language and the two positive constants are joint existential witnesses; the construction freezes both class caps before freezing the one finite-mode weak machine.

Corollary 2.4 (Every fixed multiplier). *For the same F_c , both of the following hold for every fixed $A > 0$ and every sufficiently large n :*

$$W(C) \leq A \frac{n^3}{L^{11}} \implies \text{agr}(C, (F_c)_n) < \frac{1}{2} + n^{-c}$$

for $\text{SYM} \circ \text{THR}$, and

$$W(C) \leq A \frac{n^3}{L^{13}} \implies \text{agr}(C, (F_c)_n) < \frac{1}{2} + n^{-c}$$

for $\text{THR} \circ \text{THR}$.

Indeed A/L is eventually smaller than the corresponding fixed positive constant. The same observation proves a more precise consequence: for every function $u(n) \rightarrow \infty$, with no constructibility assumption,

$$A \frac{n^3}{L^{10}u(n)}$$

and

$$A \frac{n^3}{L^{12}u(n)}$$

are defeated by this same language after a u -dependent onset.

Theorem 2.5 (Fixed advantage). *For every fixed $0 < \gamma < 1/2$, there exist a common language $F_\gamma \in E^{NP}$ and positive constants $b_{S,\gamma}, b_{T,\gamma}$ such that, for every sufficiently large n , agreement at least $1/2 + \gamma$ with $(F_\gamma)_n$ implies*

$$W(C) > b_{S,\gamma} \frac{n^3}{L^5}$$

for $C \in \text{SYM} \circ \text{THR}$, and

$$W(C) > b_{T,\gamma} \frac{n^3}{L^9}$$

for $C \in \text{THR} \circ \text{THR}$.

Thus the same fixed-advantage language defeats every fixed multiplier at denominators L^6 and L^{10} . The unequal exponents reflect two real compiler costs: inverse-polynomial amplification costs two logarithmic powers after printer squaring, while exact radix lookup for an arbitrary threshold top costs two more.

3 Why this is not a moving-delta corollary

The fixed- δ theorem says, separately for every fixed $\delta > 0$, that a language is hard at wire scale $n^{3-\delta}$. It gives no uniform relation among the language, algorithm, constants, and onset as $\delta \downarrow 0$. This logical gap is large enough to contain every polylogarithmic endpoint.

For example,

$$h(n) = \frac{n^3}{\exp(\sqrt{\log n})}$$

eventually dominates $n^{3-\delta}$ for each fixed $\delta > 0$, but satisfies

$$h(n) = o\left(\frac{n^3}{(\log n)^K}\right)$$

for every fixed K . Hence all fixed-power statements are compatible with the complete failure of every explicit logarithmic statement.

Nor can one safely schedule row j at length n without quantitative onset control. The theorem permits the onset of row j to occur after the finite interval in which its bound exceeds the desired n^3/L^K scale. For correlation, selector packaging is additionally invalid because all of a correlator's signed bias may concentrate on the ineligible rows. The explicit-log theorem below instead fixes one uniform supplier and one refuter construction.

4 The residualized wire-sensitive supplier

Fix a live coordinate set I . Write a normalized bottom halfspace as

$$X_i(y, z) = \mathbf{1} \left[\sum_{j \in I} w_{ij} y_j + \sum_{j \notin I} w_{ij} z_j \geq \theta_i \right].$$

Let

$$m_i^- = \sum_{j \in I: w_{ij} < 0} w_{ij}$$

and define

$$C_i(z) = \mathbf{1} \left[\sum_{j \notin I} w_{ij} z_j \geq \theta_i - m_i^- \right].$$

Then $C_i(z) = 1$ exactly when $X_i(\cdot, z)$ is identically one. In particular $C_i \leq X_i$, and

$$Z_i = X_i \oplus C_i = X_i - C_i$$

is Boolean, vanishes whenever the restricted gate is constant, and equals X_i whenever it is nonconstant. Over $\mathbb{F}_2$ this is a degree-one substitution. The carrier pool grows by only a constant factor and no residual coordinate has fixed-one drift.

For a symmetric top, the omitted contribution is the scalar

$$F(z) = \sum_i C_i(z).$$

For up to four threshold-top outputs, [Proposition A.6](#) converts each top gate to a disjoint OR of polynomially many exact equations. Expanding the desired conjunction chooses at most one child per input circuit; a domination-base stack turns each child tuple into one exact equation. Averaging its reductions modulo all primes below an explicit polynomial cutoff gives a pointwise inverse-polynomial surrogate in which every modular predicate is one polynomial-range nonnegative score. Residualization moves the constant-one contribution into one scalar residue $f_{g,p}(z)$ for that equation and prime. The algorithm prints every possible residue row before the column scan and then selects the correct row; it never enumerates an independent drift value for every level or digit.

Let q now denote supplier arity and put $L_q = L(q)$. Fix a positive integer κ . TouchingSelect, the conditional-expectation selection procedure of [Appendix A.4](#), finds one live set of size

$$K = \kappa L_q$$

for which every restriction column has at most

$$R = O\left(\frac{KW}{q}\right)$$

nonconstant carriers across the union of the at most four input circuits. Indeed, a support of size f is touched by a random K -set with probability at most Kf/q , and conditional expectation derandomizes the summed support bound. Every gate nonconstant on any column must be touched.

Take $J = \lceil 2L_q \rceil$. [Lemma A.1](#) and [Lemma A.2](#) print the consecutive-window indicators and the exact downward recurrence for a pairwise multiscale list of degree

$$D_0 = O(\sqrt{R} + L_q)$$

and seed entropy $O(L_q)$. Moreover $R2^{-J} = O(L_q^{1-p}) = o(1)$ at every displayed supplier denominator $p \geq 5$, so the lemma's finite terminal premise holds from a sufficiently large arity onward. There is no bad-column term and no drift-advice entropy.

4.1 Direct symmetric top

For constant final error, multiply at most four scalar list outputs. The exact batch theorem in [Appendix A.12](#) first converts bottom thresholds disjointly to exact equations, canonically reduces the squarefree polynomial, and applies the occurrence-indexed truncated-binomial lift. [Theorem B.3](#) derives the required signed table from Williams’s stated exact Boolean rectangular product by local binary slicing and weighted dominance. Its leading closure term is

$$KD_0L_q \lesssim q.$$

After squaring the $\sqrt{R}$ term, the normalized master parameter is

$$\Xi_{\text{S,const}} = \Theta \left(\frac{WK^3L_q^2}{q^3} \right).$$

At $K = \kappa L_q$, a sufficiently small positive coefficient times q^3/L_q^5 makes this parameter smaller than the printer constant.

For error q^{-D} , take the odd walk length

$$t = 2 \left\lceil \frac{C_DL_q}{2} \right\rceil + 1$$

walk on a fixed power of the Margulis expander over the exact pairwise hash-seed space. Componentwise majority reduces error to $2^{-\Omega(t)}$ while multiplying degree by t and adding only $O_D(L_q)$ seed bits. The inverse-polynomial direct master is

$$\Xi_{\text{S,inv}} = \Theta_D \left(\frac{WK^3t^2L_q^2}{q^3} \right).$$

It closes at a sufficiently small positive multiple of

$$\frac{q^3}{L_q^7}.$$

4.2 Threshold top

Binary or fixed-radix reconstruction uses

$$\tau = \Theta_D(L_q)$$

digit lists over the same distinct-carrier pool. Touching is charged once, not once per digit. Walk majority is applied before the exact modular-radix lookup. The conjunction of up to four outputs was already converted to a polynomial family of stacked top equations, and each equation-prime row has exactly one pre-radix score.

Writing d_{row} for the final Boolean row degree (and retaining D only for the requested accuracy exponent), the composed degree is

$$d_{\text{row}} = O_D(t\tau D_0),$$

and the master parameter is

$$\Xi_{\text{T}} = \Theta_D \left(\frac{WK^3t^2\tau^2L_q^2}{q^3} \right).$$

In all three masters, only the $\sqrt{R}$ part of D_0 depends on the wire cap. The additive L_q part and the batch theorem's additive $K + \log(Q + 1) + L_q$ terms contribute only fixed powers of $L_q = o(q)$ after $Q = K + 1$; they impose no additional denominator.

Thus a sufficiently small positive multiple of

$$\frac{q^3}{L_q^9}$$

supports arbitrary fixed inverse-polynomial accuracy. Enlarging the explicit prime cutoff makes each nonzero stacked equation pass on a $q^{-D-O(1)}$ fraction of primes, while increasing the walk constant makes every modular row pointwise reliable to the same scale. The polynomial child-tuple count is paid by these accuracy constants, not by changing the logarithmic exponent.

4.3 Selectable polynomial saving and additive preprocessing

The constants matter. Let D be the requested accuracy and let $\sigma > 0$ be the requested saving exponent. Stacked-equation indices, primes, scalar residues, expanded walk seeds, and constant input-side labels form only q^{h_D} **external rows** for one fixed h_D . Choose

$$\kappa > h_D + \sigma + 10.$$

Then the combined external column phases have time at most

$$\frac{2^q}{q^\sigma}$$

after every polynomial overhead.

The exact radix lookup inside one threshold row has up to

$$(b_0 M + 1)^\tau = 2^{O_D(L_q^2)}$$

internal tuple monomials. These are not external calls. For each fixed external row, the exact batch theorem charges their canonical expansion in

$$T_{\text{prep}} \leq 2^{O_D(dL_q + L_q^2 + t)},$$

where d is the final Boolean row degree. The truncated-binomial counter is used with the minimal modulus exponent $Q = K + 1$: this uniquely recovers a count in $[0, 2^K]$ and makes the leading printer constraint $KdL_q = O(q)$. Since the strict printer inequality gives $dL_q = O(q/K) = O(q/L_q)$,

$$\log_2 T_{\text{prep}} = O_D(q/L_q + L_q^2) = o(q).$$

There are q^{h_D} external rows, so the complete preprocessing charge is

$$q^{h_D} T_{\text{prep}} = 2^{o(q)}.$$

It is therefore eventually below $2^q/q^\sigma$. This charge is added to, and never multiplied by, the 2^{q-K} enumeration of columns. The printer loads grow like a fixed multiple of

$$\sqrt{a} \kappa^{3/2}$$

when $W = aq^3/L_q^p$. Choosing

Table 2: Complete leading supplier ledger.

Mode	Extra	Load	Cap
SYM, fixed	none	$WK^3L_q^2/q^3$	aq^3/L_q^5
SYM, q^{-D}	t	$WK^3t^2L_q^2/q^3$	aq^3/L_q^7
THR, q^{-D}	$t\tau$	$WK^3t^2\tau^2L_q^2/q^3$	aq^3/L_q^9

$$0 < a \leq a_0(D)\kappa^{-3}$$

therefore restores the printer inequality and proves the inverse-polynomial guarantees in [Theorem 2.1](#).

In addition, for every fixed $0 < \epsilon < 1/2$ and $\sigma > 0$, [Theorem 2.2](#) gives a positive constant a_S^{fix} and a deterministic additive- ϵ estimator for symmetric-top AND-four at total time $2^q/q^\sigma$ on

$$W \leq a_S^{\text{fix}} \frac{q^3}{L_q^5}.$$

This is the formal supplier used by the symmetric half of [Theorem 2.5](#). The threshold half continues to use the L_q^9 inverse-polynomial supplier because its polynomial child-tuple sum requires inverse-polynomial primitive accuracy.

4.4 The deterministic estimators

The full estimators are defined once in [Appendix A.13](#). The important ordering is that every seed-and-offset row is printed before a column computes its residual offset and selects an existing cell. For every fixed input and seed, both the selected row and its target are bits, including on malformed one-hot lists. Fubini therefore bounds the symmetric estimator’s error by the joint input-seed list-failure probability; no seed must work on every input. For a threshold top, the disjoint-child identity is pointwise zero or one, and the prime and list errors are chosen below the inverse of the polynomial child-tuple count. Fubini and the triangle inequality then give the claimed q^{-D} error without discarding columns or performing a signed affine recombination.

4.5 Integrated supplier conservation certificate

The supplier is one composition theorem, not a list of independently invoked subroutines. The following table records the complete leading ledger after $K = \kappa L_q$, $R = O(KW/q)$, and $D_0 = O(\sqrt{R} + L_q)$. Here $t = \Theta_D(L_q)$ is the walk length and $\tau = \Theta_D(L_q)$ is the number of radix digits.

At each displayed cap the leading load is $\Theta_D(a\kappa^3)$. The parameters are chosen in this order: first D and σ ; then the conversion, prime, radix, and walk constants; then the external-row exponent h_D ; next $\kappa > h_D + \sigma + 10$; and finally a positive cap coefficient $a \leq a_0(D)\kappa^{-3}$ and one finite onset.

The distinction between one row and all rows is equally important.

- **One external row.** It fixes one child tuple g , prime p , walk seed e , and scalar residue f , and is charged one complete 2^{q-K} -column score table.
- **Internal row syntax.** Radix tuples, trace tuples, exact-child choices, and their canonical polynomial expansion are charged in $T_{\text{prep}} = 2^{O_D(dL_q + L_q^2 + t)}$ before that row’s column scan. Live assignments and BinLift subsets generate the signed occurrence family s_{gen} , which is bounded by $2^{(q-K)/100}$ and paid inside the signed-score printing term of [Theorem A.5](#).

- **All external rows.** There are at most q^{h_D} choices, so both one-row charges are multiplied by q^{h_D} exactly once.
- **Column-dependent data.** The actual residual offset $f_{g,p}(z)$ selects one cell only after every possible residue row has been printed.

The strict printer inequality gives $dL_q = O(q/L_q)$, so

$$q^{h_D} T_{\text{prep}} = 2^{o(q)}.$$

The column tables cost at most

$$q^{h_D} 2^{q-K} \text{poly}(q) \leq \frac{2^q}{q^\sigma}$$

by the choice of κ . The preprocessing term is **added** to this quantity and satisfies the same final bound after the finite onset. No radix tuple or exact-child tuple is also counted as an external row, and no scalar offset is chosen before its complete table exists. This simultaneously closes the row-multiplicity, hidden-logarithm, and adaptive-row seams in the three-mode supplier composition.

5 Componentwise CLW transfer with polynomial saving

The transfer uses the following source/local boundary.

- **CLW outer verifier.** The published contract gives one native random/address width, projection queries, a 3CNF predicate, perfect completeness, and N^{-10} no-instance acceptance. Locally we prove exact width padding, projection-table recovery, duplicate outer-clause deletion, and compact substitution.
- **CLW/CW pointwise PCPP.** The published contract gives a deterministic honest assignment on a true input, soundness against every auxiliary assignment on a false input, and systematic parity coordinates. Locally we prove the componentwise validity polynomials, native parity atoms, occurrence aliases, and the acceptance midpoint.
- **CLW fixed-machine refuter.** The published contract gives a disagreement at every sufficiently large source length against one fixed weak machine. Locally we construct one finite union of class modes and prove one-sidedness, common-function extraction, and canonical recovery.
- **CLW XOR contrapositive.** The published contract gives the literal ε_k -dependent restriction-sum envelope. Locally we prove supported-versus-carried cap retyping, the three-log arity dilation, and the padding-safe advantage reserve.
- **CLW/STV worst-case amplification.** The published contract turns size- $S(q)$ worst-case hardness into an $O(q)$ -variate output hard at size $S(q)^{1/c_{\text{STV}}}$ and advantage $S(q)^{-1/c_{\text{STV}}}$. Locally we prove one branch-independent arity envelope, choose G before k without circularity, and give the exact target schedule.

Nothing in the local column silently strengthens a published theorem. In particular, outer duplicate clauses may be deleted because they form an ordinary conjunction, whereas PCPP indexed-clause multiplicities are retained because they define an acceptance distribution.

The objects passed across the splice have the following literal types.

- **Padded outer verifier.** Its varying input is $u \in \{0, 1\}^q$, where $q = C_{\text{clk}} \log N + O(\log \log N)$, and it has polynomially many projection queries. It defines the ordinary substituted circuit.
- **Guessed general oracle.** It is a q -input fan-in-two circuit of size at most q^G . It is substituted before the PCPP and is never sent to CAPP.
- **PCPP coordinate function T_X .** On $u \in \{0, 1\}^q$ it is one rational componentwise sum with fixed term, mass, and coefficient-bit envelopes. Every occurrence of variable X aliases this same function.
- **Carried atom.** Before address fixing it has $q_0 = q + O(\log q)$ inputs and satisfies $W \leq B_{\mathfrak{C}}(q_0)$. Literal address fixing returns it to native arity without increasing wires.
- **Systematic coordinate.** It is a native q -input explicit parity with $O(q)$ wires for a symmetric top or $O(q^2)$ wires for a threshold top. It enters as a separate cheap atom.
- **CAPP input.** It is a native q -input conjunction of at most four atoms, each separately below $A_{\mathfrak{C}}(q)$. It estimates one monomial of a validity or clause polynomial.
- **Case-2 hard table f .** It is a total Boolean function on q_0 bits, with malformed addresses set to zero, and is the common componentwise-sum target for both class modes.
- **XOR core.** It has $m = kq_0$ inputs, where $k = \Theta_c(\log q)$ or $O_\gamma(1)$, and converts sum-hardness into correlation hardness.

The resource distinction in the CAPP row is crucial: coefficient mass and term count govern estimation accuracy and the number of calls, but the wire cap is imposed **componentwise** on each atom. No linear sum is composed into one larger circuit and no sum of atom wires is charged to a supplier call.

The fixed-machine quantifier order is as follows. Fix the published source transformations and class presentations, then the exact-width padding and parsers, the target advantage and simulation exponent, the general-oracle exponent and amplifier constants, the XOR power and componentwise envelopes, the validity tolerance and supplier parameters, the supported and carried coefficients, and the supplier programs. Fix the finite union-machine code only after these choices. The source length, refuter word, branch guesses, and canonical hard objects are chosen afterward.

In particular, no program, cap coefficient, onset, term envelope, or parser inside the weak machine depends on the current source length or on a guessed circuit.

For a base class $\mathfrak{C}$, define

$$\text{Lin}_q(B, J, \Lambda)$$

to contain rational sums

$$P(x) = \sum_{a=1}^s \alpha_a C_a(x)$$

with

$$s \leq J, \quad \sum_a |\alpha_a| \leq \Lambda, \quad W(C_a) \leq B$$

for every individual atom; all rational coefficients have polynomial bit length. When the transfer writes $P \in [0, 1] \text{Lin}_q$, it additionally imposes $0 \leq P(x) \leq 1$ pointwise. This is a semantic property

of the approximator produced by the XOR contrapositive, not a promise that the weak machine tries to decide: the machine permits every syntactically bounded rational sum, and its one-sided soundness proof handles arbitrary real values through the Booleanity and second-moment tests. A degree-four expression in such sums and constants expands into at most $(J + 1)^4$ AND-four calls. Its propagated additive error is at most

$$\eta(1 + \Lambda)^4,$$

while $(J + 1)^4$ affects only time. No sum of atom wires is ever presented to CAPP.

[Appendix C.9](#) restates the four published CLW/CW/STV semantic interfaces, and [Appendix C.10](#) replays the weak machine. Before applying the pointwise PCPP, the local compaction lemma recovers the promised projection table and deletes repeated outer 3CNF clauses. Since the outer predicate has only $t_{\text{out}} = \text{poly}(q)$ answer variables, at most $8 \sum_{j \leq 3} \binom{t_{\text{out}}}{j} = q^{O(1)}$ distinct clauses remain. Thus the substituted circuit has size $q^{O(1)}$; no source-level $\text{poly}(N)$ representation factor enters the PCPP witness or call count. PCPP clause multiplicities, by contrast, are retained because that theorem measures the fraction of its indexed list.

For an auxiliary proof occurrence the machine tests

$$T_{ij}^2(1 - T_{ij})^2,$$

for a systematic occurrence it tests

$$(\text{Enc}_s - T_{ij})^2,$$

and in both cases it also bounds $\mathbb{E}[T_{ij}^2]$. These are degree-four expressions in the guessed sums. The PCPP clause pool, guessed proof descriptions, term counts, witness length, coefficient masses, and coefficient-bit bounds are a finite list of fixed polynomial envelopes once the target advantage exponent is fixed. They are frozen before the weak machine; no one machine is claimed to cover all polynomial envelopes. Let $q^{p_{\text{call},c}}$ bound the number of analysis calls and their remaining polynomial overhead, write $2^q \leq T(N)q^{d_{\text{pad}}}$, and let the coefficient mass be at most $q^{\rho_{\text{coef}}}$. Request

$$D > 4\rho_{\text{coef}} + 10$$

and

$$\sigma > p_{\text{call},c} + d_{\text{pad}} + 10$$

from [Theorem 2.1](#). Then all ordinary validity calls together cost at most $T(N)/q^{10}$.

A systematic code coordinate is an explicit parity of at most $q/2$ variables. Compile it directly into the selected target class: identity bottoms plus a symmetric lookup use $O(q)$ wires, while cumulative threshold bottoms plus alternating top weights use $O(q^2)$ wires. Thus the parity atom is eventually below every supported cap $a_{\mathfrak{C}}q^3/L(q)^p$. Expanding $(\text{Enc}_s - T)^2$ now uses only native-arity conjunctions of at most two carried/parity atoms, and all systematic tests cost

$$q^{p_{\text{call},c}} \frac{2^q}{q^\sigma} \leq T(N)q^{-10}.$$

The absolute atom charge is unchanged. Distinguish the supplier-supported cap

$$A_{\mathfrak{C}}(q) = a_{\mathfrak{C}} \frac{q^3}{L(q)^p}$$

from the carried cap $B_{\mathfrak{C}}(q_0) = b_{\mathfrak{C}}q_0^3/L(q_0)^p$ on the addressed function, where $q_0 = q + O(\log q)$. Choosing $b_{\mathfrak{C}} \leq a_{\mathfrak{C}}/64$ gives

$$B_{\mathfrak{C}}(q_0) \leq A_{\mathfrak{C}}(q)$$

eventually; before the coefficient factor, this arity ratio is only $1 + o(1)$. Thus address restriction is legal, while systematic parity never changes the native arity. This is why the theorem obtains a positive constant rather than every constant at the boundary.

The native verifier width is padded locally to the exact value

$$q(N) = \lceil \log_2 T(N) \rceil + A_{\text{pad}} \lceil \log_2(\log_2 T(N) + 2) \rceil + A_{\text{pad}}.$$

The added random bits are ignored and native proof addresses receive a zero suffix, so completeness, acceptance fractions, and projection typing are unchanged; moreover $2^q/T(N) = q^{O(1)}$. The complete witness has $q^{O_c(1)} = o(N)$ bits and is eventually below the refuter's $N/10$ bound. Let d_{ver} bound source construction, projection-table recovery, outer-clause compaction, and substitution. Choose the polynomial clock exponent with $C_{\text{clk}} > d_{\text{ver}} + 2$:

$$T(N) = N^{C_{\text{clk}}}$$

The compacting work is $N^{d_{\text{ver}}}q^{O(1)} = o(T(N))$, while choosing the requested CAPP saving beyond the fixed call and padding exponents makes the complete analysis phase at most $T(N)/q^{10}$. Hence the machine is genuinely $o(T(N))$. Freeze a rational validity tolerance $\zeta > 0$, and require every propagated estimate to have error below both ζ and one twentieth of the PCPP completeness-soundness gap. The first bound makes the Booleanity and second-moment thresholds sound; the second protects the acceptance midpoint. The machine estimates the arithmetized clause mean and accepts only above that midpoint. On a hierarchy NO word, outer soundness, pointwise PCPP soundness, and validity rounding keep every branch strictly below it. The fixed-machine refuter consequently produces, at every sufficiently large source length, a YES word on which every class mode rejects.

At that conflict, either Case 1 supplies a general-circuit-hard oracle, or Case 2 chooses the lexicographically first eligible succinct general oracle before the class mode. In Case 2, the PCPP's fixed deterministic honest-assignment algorithm defines one canonical Boolean proof family and hence

$$f(i, j, u) = \hat{T}_{ij}(u),$$

with malformed indices mapped to zero. If an eligible componentwise sum approximated f , occurrence averaging would construct a valid accepting proof in that class mode without increasing any atom's wires, contradicting rejection. Thus the same Case-2 proof-table function is far from componentwise sums of both circuit classes.

6 XOR amplification and the three-log dilation

For target advantage n^{-c} after padding, let

$$\varepsilon_k = (1 - \zeta)^{k-1}(1/2 - \zeta)$$

be the source parameter in CLW's XOR contrapositive and take

$$k = A_{c+1} \lceil \log_2(q+2) \rceil,$$

where the fixed A_{c+1} is large enough that $\varepsilon_k \leq q^{-(c+2)}$. The final Case-2 function has arity

$$m = k(q + O(\log q)) = \Theta_c(q \log q).$$

Because $m = \Theta_c(q \log q)$, eventually

$$\varepsilon_k \leq q^{-(c+2)} \leq m^{-(c+1)}.$$

The term-count and coefficient-mass envelopes are evaluated at the actual source value $\varepsilon_k = q^{-O_c(1)}$; they remain fixed polynomials in q . This avoids silently substituting a different advantage into CLW's quantitative bound. The one-power $m^{-(c+1)}$ reserve is what makes the later padding to target arity n preserve the theorem's stricter threshold n^{-c} .

A correlator for the k -fold XOR yields polynomially many literal restrictions of that correlator as the atoms of one approximate linear sum. No XOR gate is composed on top of an atom.

Suppose first that a symmetric-top correlator has

$$W(C) \leq b_{S,c} \frac{m^3}{L(m)^{10}}.$$

Every restricted atom has the same absolute charge, and

$$W(C_a) \leq O_c(b_{S,c}) \frac{q^3 (\log q)^3}{(\log q)^{10}} = O_c(b_{S,c}) \frac{q^3}{(\log q)^7}.$$

Choose $b_{S,c}$ below the fixed supplier coefficient after all arity constants. The atom is legal. The threshold-top calculation is identical:

$$\frac{m^3}{L(m)^{12}} = O_c \left(\frac{q^3}{L(q)^9} \right).$$

Thus exactly three logarithmic powers pay the cubic arity dilation. No fourth power is needed for the fixed-positive theorem. A fourth power converts the fixed positive constant into every fixed multiplier, proving [Corollary 2.4](#).

For fixed advantage, $k = O_\gamma(1)$ and $m = \Theta_\gamma(q)$. Choose this constant large enough not only to drive the XOR residual advantage below γ , but also to dominate the fixed output-arity constant in the Case-1 amplifier of [\[11, Lemma 3.9\]](#) and [\[53, Theorem 24\]](#). There is no logarithmic arity dilation, so the supplier exponents themselves become the final exponents in [Theorem 2.5](#).

7 Case 1, every length, and E^{NP}

If no small general oracle makes the outer verifier a tautology, apply the exact Case-1 contract stated as [Fact C.4](#). It gives a function on at most $c_{\text{amp}}q$ variables that is hard at both circuit size and advantage exponent G/c_{STV} , and its complete truth table is computable from the hard outer-oracle table in $2^{O(q)}$ time. After finite normalization, bottom and top threshold gates have polynomial-bit descriptions, while a symmetric top has an $M+1$ -entry lookup. Binary addition and comparison therefore simulate a displayed-cap circuit by a fan-in-two circuit of size $q^{G_{\text{sim}}}$ for one fixed G_{sim} , even after the final length is $n = \Theta(q \log q)$. Choose the general-oracle cutoff exponent G so that

$$\frac{G}{c_{\text{STV}}} > \max\{G_{\text{sim}}, c + 2\}.$$

Only after this choice, enlarge the fixed XOR multiplier so that its core arity dominates $c_{\text{amp}}q$. The Case-1 output is then hard at advantage at most $q^{-(c+2)} \leq m^{-(c+1)}$, so it has the same padding-safe reserve as Case 2. This order removes any circular dependence between the amplifier arity, G , and the XOR schedule.

For an exact target schedule, enumerate $s = 1, \dots, n$ and take source length $N_s = 2^s$. Compute $q(N_s)$ from the displayed padding formula and the branch-independent clause-address width

$$r(N_s) = \lceil d_G \log_2(q(N_s) + 2) \rceil, \quad q_0(N_s) = q(N_s) + r(N_s) + 1.$$

Every branch-specific power-of-two clause list is repeated to this fixed envelope. Select the largest s whose amplified core $m_s = k(q(N_s))q_0(N_s)$ has at most $n/2$ variables. The exact padded width changes by $O(1)$ when s increases. In the inverse-polynomial tier, $m_s = \Theta_c(s \log s)$ and the occasional integer jump of the XOR power changes the core by $O_c(s) = O_c(n/\log n) = o(n)$ at the selected scale; in the fixed-advantage tier, $m_s = \Theta_\gamma(s)$ and the jump is $O_\gamma(1)$. Maximality therefore gives

$$m_s = n/2 - o(n)$$

in the inverse-polynomial tier and $m_s = n/2 - O_\gamma(1)$ in the fixed tier. In particular $n/3 \leq m_s \leq n/2$ eventually, with

$$q = \Theta(n/\log n)$$

in the inverse-polynomial tier, while $q = \Theta(n)$ in the fixed-advantage tier. This is a deterministic function of n and the fixed source constructions; it never searches for a hard function. Unused variables are ignored, and fixing an ignored suffix of any alleged correlator preserves at least its average agreement and never increases wires. The direction of the advantage comparison is now explicit:

$$m_s^{-(c+1)} \leq 3^{c+1} n^{-(c+1)} < n^{-c}$$

for all sufficiently large n . Also $n/m_s \leq 3$ and $L(n)/L(m_s) = 1 + o(1)$, so a fixed shrinkage of $b_{\text{S},c}, b_{\text{T},c}$ absorbs the target-to-core wire-cap ratio.

The refuter succeeds at every sufficiently large selected length. Canonical recovery first asks which side of the fixed dichotomy holds. In Case 1, no size- q^G general-oracle circuit makes every unrolled outer-verifier row accept; the construction prefix-recovers the first honest outer-oracle truth table and applies the fixed general-circuit amplifier. Its output has at most $c_{\text{amp}}q$ variables; the choice of k above ensures $c_{\text{amp}}q \leq m_s$, so ignored variables pad it to the same scheduled core. In Case 2, an eligible succinct circuit exists; the construction prefix-recovers the first accepted description and the core already has arity $m_s = kq_0$. In every SAT formula, one shared truth-table or description block is used across all verifier rows, and full Tseitin equivalences prevent a witness from choosing a different circuit on each row. The fixed deterministic PCPP honest-assignment algorithm then computes the Case-2 proof coordinate requested at (i, j, u) directly in $q^{O(1)}$ time; no exponentially long proof table is recovered by SAT. Refuter execution, amplification, XOR evaluation, and recovery therefore cost $2^{O(n)}$ with an NP oracle. Malformed headers and addresses return zero, so the final language is total and lies in E^{NP} .

Proof of Theorem 2.3. Fix $c > 0$. Invoke Theorem 2.1 at the finite accuracy and saving exponents frozen in Appendix C.10. Its two modes supply componentwise AND-four CAPP at carried exponents $p_S = 7$ and $p_T = 9$, with positive supported coefficients; (C.1.1)–(C.1.2) shrink these once to positive carried coefficients before the one union machine is fixed. Lemma C.8 gives one class-independent hard table at every sufficiently large selected source length. Lemma C.9, used at its literal ε_k with $k = \Theta_c(\log q)$, converts that table to correlation hardness and charges exactly three logarithmic powers under $m = \Theta_c(q \log q)$. Fact C.4, together with the target-class simulation bound fixed before G , supplies the corresponding core hardness in the other side of the dichotomy. Lemma C.10 and the bounded-jump schedule then give one total language $F_c \in E^{NP}$ at every sufficiently large target length. The final restriction and cap comparisons in Appendix C.12 preserve advantage n^{-c} and yield positive constants at denominators $L^{7+3} = L^{10}$ and $L^{9+3} = L^{12}$. The union machine was fixed with both class modes, so this is one F_c for both conclusions. Support-first normalization in Section 2 transfers the conclusion from finite integer descriptions to arbitrary real weights without adding a wire. $\square$

Proof of Theorem 2.5. Fix $0 < \gamma < 1/2$. For the symmetric mode use the constant-accuracy supplier of Theorem 2.2, whose carried exponent is 5; for the threshold mode use Theorem 2.1 at exponent 9, since the disjoint child sum still needs inverse-polynomial primitive accuracy. Now the XOR power is a fixed constant, enlarged once to dominate the Case-1 amplifier arity, so $m = \Theta_\gamma(q)$ and no logarithmic arity dilation occurs. Appendix C.10 and Appendix C.12 otherwise apply verbatim, with the general-oracle exponent selected so that the Case-1 inverse-polynomial advantage is eventually below γ . They produce one common $F_\gamma \in E^{NP}$ and positive wire coefficients at denominators L^5 and L^9 at every sufficiently large length. $\square$

8 Exact boundary

The proof establishes a fixed positive coefficient at L^{10}/L^{12} , not all fixed multipliers at those exact denominators. At a fixed coefficient, increasing the requested polynomial saving forces a larger live constant κ , while printer closure forces the eligible wire coefficient to shrink like κ^{-3} . There is no asymptotic resource left to absorb arbitrarily large constants at the exact endpoint.

The one-extra-log corollary is therefore not a cosmetic weakening. Removing it from the every-multiplier statement would require one of:

1. a supplier whose single fixed algorithm handles an unbounded multiplier schedule;
2. a printer inequality with an additional vanishing factor at the exact endpoint; or
3. a correlation-safe universal packaging theorem across infinitely many cap coefficients.

Ordinary selector packaging does not provide Item 3 because correlation bias can concentrate on other selector rows.

The result remains below exact cubic wires by a polylogarithmic factor. It does not yield a gate exponent above $5/2$.

9 Imported results and limitations

Theorem 2.1, Theorem 2.2, Theorem 2.3, and Theorem 2.5 are theorems of this paper. Their proofs use the published interfaces listed in Section 1.3 and restated at first use. In particular,

the external boundary is finite integer normalization; the qualitative CW19 disjoint threshold-to-exact-threshold conversion, used through the self-contained local construction in [Appendix A.12](#); Williams’s stated exact Boolean rectangular product; the Gabber–Galil spectral-radius bound [20] in the normalization of [30, Theorem 8.2]; the Rosser–Schoenfeld Chebyshev-function estimate [51, Theorem 4, eq. (3.14)]; and the CLW/CW XOR, worst-case amplification, verifier, PCPP, and fixed-machine refuter interfaces. Appendices [A–C](#) separate each literal imported contract from the local adapter that follows it. No CTW shifted-product or GapAND theorem is used by the supplier.

[Table 3](#) summarizes the load-bearing imported interfaces and the precise versions used here.

Table 3: Load-bearing external interfaces and the versions used here.

Interface	Source version	Imported conclusion
Integer normalization	Muroga–Toda–Takasu [43]; see also CTW ECCC TR26-039, Lemma 3.2 [13]	Polynomial-bit integer representation on the retained support
Threshold conversion	CCC 2019 full version, Proposition 18(2) and Appendix B [14]	Qualitative disjoint exact-equation decomposition; the quantitative construction is reproved locally
Rectangular product	JACM 61(1), Corollary 4.4 and Appendix C [60]	Exact integer product of the stated Boolean rectangular matrices
Expander spectrum	Gabber–Galil [20] as normalized in HLW, Theorem 8.2 [30]	Spectral-radius bound for the powered walk
Prime estimate	Rosser–Schoenfeld, Theorem 4, equation (3.14) [51]	Lower bound on $\sum_{p \leq x} \log p$
Transfer interfaces	CLW full version, ECCC TR20-150, Lemmas 3.8–3.11 and Theorem 4.6 [11]	The four semantic contracts restated in Appendix C.9 ; the PCPP and hierarchy provenance are [56, 19]

The theorem is deliberately limited in four ways. First, it remains below exact cubic wires by polylogarithmic factors. Second, the hard language lies in E^{NP} rather than linear time, and the inverse-polynomial theorem uses one language for each fixed exponent c . Third, the lower bound concerns strictly layered simple-edge depth-two circuits, not general circuits. Fourth, it does not yield a gate exponent above $5/2$ or a lower bound for NP ; in particular, it makes no claim against the natural-proofs barrier [49], whose candidate pseudorandom functions live at higher depth.

Four open problems follow naturally. First, improve the explicitness of the hard language, from E^{NP} toward E or NE , or make one language work for every advantage exponent c simultaneously. Second, remove the remaining polylogarithmic factors, or show that the exact cubic endpoint requires a different supplier, as [Section 8](#) suggests. Third, raise the gate exponent beyond the $5/2$ scale of [13]; the present wire-charged method does not do so. Fourth, extend wire-sensitive circuit analysis to depth three, where the known wire bounds [39, 38] remain at or below the $n^{5/2}$ scale.

A The residualized AND-four suppliers

A.1 Proof setup for the supplier theorems

Let

$$L_q = \lceil \log_2(q + 2) \rceil.$$

The formal statements are [Theorem 2.1](#) and [Theorem 2.2](#). The normalization convention fixed in [Section 2](#) gives one polynomial bound on the complete input-description length and every numerical bit length used below. The fixed-accuracy algorithm uses the base list directly, with its window constant chosen for error ϵ , and therefore has no walk factor t^2 . Its complete resource proof is the direct fixed-error ledger in Appendices A.7–A.8 together with the estimator in [Appendix A.13](#). The threshold-top denominator remains L_q^9 , because its polynomial family of disjoint child tuples still requires inverse-polynomial pointwise accuracy before summation, even when the outer transfer requests only constant error.

At the displayed wire caps the complete normalized input description has size $q^{c_{\text{desc}}}$ for one absolute constant c_{desc} . To obtain the stated total-time bound, the construction internally requests saving exponent $\sigma + c_{\text{desc}} + 2$ before fixing its positive wire coefficient. Thus no unabsorbed factor $\text{poly}(|C_1| + \dots + |C_j|)$ remains in the theorem statement. The constants may shrink as D or σ increases. This dependence is essential and is retained explicitly in the transfer.

A.2 Exact residualization

Fix the live coordinates $I \subseteq [q]$. For one bottom threshold gate write

$$X_i(y, z) = \mathbf{1}[A_i(y) + B_i(z) \geq \theta_i],$$

where

$$A_i(y) = \sum_{j \in I} w_{ij} y_j, \quad B_i(z) = \sum_{j \notin I} w_{ij} z_j.$$

The minimum of A_i over the live cube is

$$m_i^- = \sum_{j \in I: w_{ij} < 0} w_{ij}.$$

Define

$$C_i(z) = \mathbf{1}[B_i(z) \geq \theta_i - m_i^-].$$

Then

$$C_i(z) = 1 \iff X_i(y, z) = 1 \text{ for every } y.$$

It follows pointwise that $C_i \leq X_i$ and

$$Z_i = X_i - C_i = X_i \oplus C_i \in \{0, 1\}.$$

There are exactly three cases on a column:

restricted X_i	C_i	residual Z_i
constant zero	0	0
constant one	1	0
nonconstant	0	X_i

Thus all constant coordinates become zero and every nonconstant coordinate is preserved. The new gate C_i ignores I , has support contained in the original support, and has a fixed-polynomial-bit integer description after semantic normalization. Here normalization means an equivalent integer threshold over the already present support, not preservation of the original numerical weights. The formal pool grows from M to at most $2M$; this changes $\log(B+1)$ by only an additive constant.

The nested-hash fixed-one counts are now

$$F_j^Z = 0$$

at every level, for every seed and column. Hence all drift discrepancies and the terminal fixed count are identically zero. This is stronger than compressing the old advice: the trajectory no longer exists.

A.3 Recovery of the omitted score

A.3.1 Symmetric output

The original Hamming count is

$$\sum_i X_i = F(z) + \sum_i Z_i, \quad F(z) = \sum_i C_i(z).$$

For every seed and every possible $f \in \{0, \dots, M\}$, print one complete Boolean polynomial implementing the original symmetric lookup with offset f . During the fixed-column scan compute $F(z)$ and select the corresponding printed row.

This order is malformed-seed safe. Even when a residual one-hot vector is wrong, the selected object is one Boolean polynomial and its discrepancy is at most one. Combining estimated one-hot counts as integers after printing would not have this property.

A.3.2 Threshold output

For up to four threshold-top inputs, [Proposition A.6](#) converts every top gate to a disjoint OR of polynomially many exact equations. Expanding the conjunction chooses at most one child from each top gate, and stacking the chosen equations leaves one exact equation $L_g(X) = 0$ over the common literal carrier. A deterministic all-small-primes surrogate replaces that equation by modular predicates

$$g_p(X) = \mathbf{1}[L_g(X) \equiv 0 \pmod{p}].$$

After coefficients are reduced to $\{0, \dots, p-1\}$, each g_p is one modular lookup of one nonnegative weighted score. Write its coefficients as

$$\mu_i = \sum_{d < \tau} v_{i,d} b_0^d, \quad 0 \leq v_{i,d} < b_0.$$

The exact score identity is

$$\sum_i \mu_i X_i = F_{g,p}(z) + \sum_{d < \tau} b_0^d V_{g,p,d}(y, z),$$

where

$$F_{g,p}(z) = \sum_i \mu_i C_i(z), \quad V_{g,p,d} = \sum_i v_{i,d} Z_i.$$

Carries are already represented by this integer equality. Only the residue $f_{g,p}(z) = F_{g,p}(z) \bmod p$ is needed. Print one total Boolean modular-radix row for every $f \in \{0, \dots, p-1\}$, then compute the actual residue and select its row during the column scan. The $\tau = \Theta_D(L_q)$ digit lists do not carry separate offsets.

A.4 Touching and multiscale list

Normalize all four input circuits before selection, and take the union of their original support incidences. A uniformly random K -set touches a given support of size f with probability at most Kf/q . Conditional expectation therefore finds one I for which the number of touched carriers is

$$R \leq c_0 \frac{KW}{q}$$

for a fixed constant c_0 absorbing the four circuits and the source's constant multiplicity. Every bottom gate nonconstant on any I -column must touch I . The inequality holds on every column; no Markov discard or error floor occurs.

The derandomization is explicit. After a partial choice has included S , left undecided coordinates U , and still needs k elements, a support A has conditional touching probability one if $A \cap S \neq \emptyset$, and otherwise

$$1 - \frac{\binom{|U \setminus A|}{k}}{\binom{|U|}{k}}.$$

Compute the summed conditional expectation for including and excluding the next coordinate and keep the no-larger branch. Polynomial-bit binomial arithmetic and the explicit support lists make the search polynomial in the input description.

Every radix digit list is a labelled sublist of the same distinct original carrier pool. Reusing a touched carrier in τ digit positions changes degree and lookup syntax, but not R . Polynomially many stacked equations and primes are evaluated sequentially and are never unioned into one selector instance.

For simultaneous direct inputs and for all digit positions of one modular row, pad every labelled sublist to one common union pool by identically zero coordinates. The common pool has only constant-factor larger size and the same R bound. Consequently all lists use exactly the same field labels, seed space $\mathbb{F}_{2^{r_0}}^2$, and later walk; the phrase “common seed” does not identify unequal hash spaces.

For a residual digit list, the pairwise nested hash and unequal windows of [Lemma A.2](#) give

$$D_0 = O(\sqrt{R} + L_q).$$

The hash seed has $O(L_q)$ bits. There is no advice string, and with $J = \lceil 2L_q \rceil$ the terminal condition is explicit: at every relevant cap,

$$R2^{-J} = O\left(L_q^{1-p}\right) = o(1),$$

where $p \geq 5$ is the displayed denominator exponent. Hence, for every fixed $\rho > 0$, the finite terminal premise $R2^{-J} \leq \rho/2$ holds from a sufficiently large arity onward.

A.5 Powered-walk accuracy

Choose the window constants in [Lemma A.2](#) so that, for each fixed list input, the complete one-hot vector is wrong on at most a constant

$$\rho < 1/128$$

of the exact pairwise hash seeds. The seed set is a power-of-two square, so a fixed power of the Margulis graph acts on it explicitly with nontrivial spectral radius below $1/128$.

Take the odd walk length

$$t = 2 \left\lceil \frac{C_D L_q}{2} \right\rceil + 1.$$

For a fixed input, [Lemma A.3](#) gives

$$\Pr[\text{at least half the vertices are bad}] \leq 2^{-\Omega(t)}.$$

Apply the exact multilinear majority polynomial to each output coordinate. If a strict majority of visited vectors is the correct one-hot vector, componentwise majority recovers that vector exactly. The resulting list has degree at most

$$tD_0$$

and seed entropy $O_D(L_q)$. For one fixed external modular row (g, p) there are $O_D(L_q)$ digit lists, all evaluated on one common walk. Their union failure is below any prescribed $q^{-D'}$ after increasing C_D . The polynomial family of child tuples and primes is handled sequentially as external rows and receives the stronger per-row budget printed in [Appendix A.13](#); no constant-size shortcut is taken.

A.6 Fourfold threshold-top accuracy

The native compiler in [Proposition A.6](#) starts from the desired conjunction itself. Disjointness gives the exact integer identity

$$\bigwedge_{i=1}^j C_i = \sum_{g \in \mathcal{G}} \mathbf{1}[L_g = 0], \quad j \leq 4,$$

where at every input the sum is already a bit and $|\mathcal{G}| = q^{O(1)}$. No Fourier expansion and no CTW pointwise surrogate is used.

For every equation, the all-small-primes average has pointwise error below $q^{-D-c_{\mathcal{G}}-10}$, where $q^{c_{\mathcal{G}}}$ bounds $|\mathcal{G}|$. Every modular row contains only $\tau = O_D(L_q)$ digit lists. Choose the common-walk length so that their joint pointwise failure is below the same budget. Summing over $\mathcal{G}$ then gives total error at most q^{-D} .

Thus fourfold composition changes only the fixed child-count, prime-cutoff, and walk constants. It does not change L_q^9 or the touching charge. The threshold supplier's need for inverse-polynomial internal precision, even in the fixed-advantage transfer, now comes from the polynomial number of stacked child tuples rather than from a normalized CTW coefficient mass.

For direct symmetric tops, multiply the four scalar Boolean approximants. Degree grows by at most four and a constant union bound pays their errors.

A.7 Master inequalities

Take a positive integer κ and put

$$K = \kappa L_q.$$

Apply [Theorem A.5](#) with the minimal legal modulus exponent

$$Q = K + 1.$$

This choice is sufficient because every live-block count lies in $[0, 2^K]$, and it minimizes both the BinLift subset order and the generated occurrence count. Write d_{row} for the final Boolean row degree (to distinguish it from the requested accuracy exponent D in [Theorem 2.1](#)). The leading condition is therefore

$$K d_{\text{row}} L_q \leq c_p q.$$

For direct symmetric fixed error, $d_{\text{row}} = O(\sqrt{R} + L_q)$, so squaring the leading term gives

$$\frac{W K^3 L_q^2}{q^3} = O(1).$$

For inverse-polynomial direct error, $d_{\text{row}} = O_D(t\sqrt{R} + tL_q)$ and the master is

$$\frac{W K^3 t^2 L_q^2}{q^3} = O_D(1).$$

For the threshold top, $d_{\text{row}} = O_D(\tau t \sqrt{R} + \tau t L_q)$, giving

$$\frac{W K^3 t^2 \tau^2 L_q^2}{q^3} = O_D(1).$$

These displays isolate the terms containing $\sqrt{R}$, which are the only cap-dependent constraints. The additive L_q part of $D_0 = O(\sqrt{R} + L_q)$ contributes respectively $K L_q^2$, $K t L_q^2$, and $K t \tau L_q^2$ to the unsquared printer inequality; after $K, t, \tau = O_D(L_q)$ each is a fixed power of L_q and hence $o(q)$. The additive $K + \log(Q + 1) + L_q$ terms in [Theorem A.5](#) are lower order for the same reason.

Substituting $K = L_q$, $t = L_q$, and $\tau = L_q$ yields the exponent identities

$$5 = 3 + 2, \quad 7 = 3 + 2 + 2, \quad 9 = 3 + 2 + 2 + 2.$$

The first 3 is K^3 , the first 2 is the printer coefficient logarithm squared, and the later pairs are walk and radix degree squared.

A.8 Saving versus coefficient and preprocessing

External row indices and internal syntax must be separated. Let $h_D L_q$ bound only the logarithm of the number of **external** rows: stacked-equation labels, primes, scalar residues, expanded walk seeds, and the constant number of input-side labels. The column phase then has exponent

$$q - K + h_D L_q + O(1).$$

The exact radix lookup inside one row has as many as

$$(b_0 M + 1)^\tau = 2^{O_D(L_q^2)}$$

formal tuples. These are internal monomials, not external batch calls. For each fixed external row, [Theorem A.5](#) charges their canonical expansion in the additive preprocessing term

$$T_{\text{prep}} \leq 2^{O_D(dL_q + L_q^2 + t)},$$

where $d = d_{\text{row}}$ is the final Boolean row degree. The strict printer inequality gives $dL_q = O(q/K) = O(q/L_q)$, so

$$\log_2 T_{\text{prep}} = O_D(q/L_q + L_q^2) = o(q).$$

Multiplying by all external rows gives

$$q^{h_D} T_{\text{prep}} = 2^{o(q)}.$$

Consequently, for every fixed σ , the complete preprocessing charge is eventually smaller than $2^q/q^\sigma$, and the external column phase alone determines the polynomial saving. No $2^{O(L_q^2)}$ factor is multiplied into every one of the 2^{q-K} columns.

To retain a total $q^{-\sigma}$ saving after all polynomial calls, take

$$\kappa \geq h_D + \sigma + c_1.$$

At cap $W = aq^3/L_q^p$, the leading printer load is

$$O_D(\sqrt{a} \kappa^{3/2}).$$

Choose a below a fixed multiple of κ^{-3} . This closes the printer and leaves a positive coefficient. It also explains why the supplier is not uniform over every coefficient at the exact endpoint.

The transfer may choose a still smaller carried coefficient to absorb its $q + O(\log q)$ addressed-arity restriction. Systematic code coordinates are compiled as native parity atoms, so no half-arity supplier call is required.

A.9 Imported ingredients and local scope

The residual identity, offset reconstruction, touching-set derandomization, pairwise list recurrence, walk-majority wrapper, exact total lookup, explicit CAPP estimators, prime surrogate, preprocessing ledger, exponent algebra, saving/coefficient tradeoff, and AND-four child-tuple expansion are proved in this appendix. The only imported ingredients on this supplier path are:

1. finite integer normalization on the retained support [\[43\]](#); see also [\[13, Lemma 3.2\]](#), with support deletion and zero extension proved locally;
2. the qualitative disjoint threshold-to-exact-threshold conversion of [\[14, Proposition 18\(2\) and Appendix B, Lemmas 48–49\]](#), with the complete construction and safe quantitative bounds replayed locally in [Appendix A.12](#);
3. the exact Boolean rectangular-product theorem stated in [Appendix B](#);
4. the Gabber–Galil spectral-radius estimate [\[20\]](#), in the exact normalization restated as [\[30, Theorem 8.2\]](#), used in [Lemma A.3](#); and

5. the explicit Chebyshev-function lower bound [51, Theorem 4, eq. (3.14)] used in [Lemma A.7](#).

The cited statements are restated at their first uses, and every change of object type after those statements is proved locally. In particular, no CTW shifted-product, GapAND, affine-score, or proof-extracted Williams score-table statement is used by the two supplier theorems above.

A.10 Explicit zero-advice list compiler

Put

$$r_0 = \left\lceil \log_2 \max\{M + 1, 2^J\} \right\rceil.$$

The field representation is uniform and canonical. Enumerate monic binary polynomials of degree r_0 in lexicographic order, use the standard repeated squaring and gcd criterion to select the first irreducible one $p_{r_0}(X)$, and represent $\mathbb{F}_{2^{r_0}}$ as $\mathbb{F}_2[X]/(p_{r_0})$ in the coefficient basis $1, X, \dots, X^{r_0-1}$. Because $2^{r_0} \leq 2 \max\{M + 1, 2^J\}$ and $r_0 = O(L_q)$, even exhaustive candidate enumeration followed by polynomial-time irreducibility tests costs $\text{poly}(M, 2^J)$, within the construction bound below. Take π to be projection onto the first J coefficient bits; $r_0 \geq J$ makes it rank J . Thus field operations, coordinate labels, and the seed encoding are deterministic functions of (M, J) rather than nonuniform choices.

Label the M formal coordinates by the first M distinct coefficient vectors in lexicographic order, denoted $\alpha_i \in \mathbb{F}_{2^{r_0}}$. A seed $(a, b) \in \mathbb{F}_{2^{r_0}}^2$ and a fixed rank- J binary projection π define

$$h_{a,b}(i) = \pi(a\alpha_i + b) \in \{0, 1\}^J.$$

For distinct labels, the two field values are independent and uniform. Hence the projected labels are pairwise independent and uniform, and the seed has exactly $2r_0 = O(L_q)$ bits.

Define the nested chain

$$\mathcal{L}_j(h) = \{i : h(i)_1 = \dots = h(i)_j = 0\}, \quad 0 \leq j \leq J.$$

Lemma A.1 (Consecutive-window indicators). *Let $\ell_1, \dots, \ell_m$ be Boolean literals, let $s = \sum_i \ell_i$, and fix a consecutive integer interval $[u, u + T]$. For every v in that interval there is an explicit multilinear polynomial over $\mathbb{F}_2$ of degree at most T that equals $\mathbf{1}[s = v]$ whenever $s \in [u, u + T]$. All $T + 1$ indicators are constructible in $\text{poly}(M, T)(M + 1)^T$ time when the literals use M formal variables.*

Proof. For $0 \leq j \leq T$, Vandermonde's identity gives

$$\binom{s - u}{j} = \sum_{i=0}^j \binom{s}{i} \binom{-u}{j - i}.$$

Modulo two, $\binom{s}{i}$ is the i th elementary symmetric polynomial in the literals. The evaluation matrix $(\binom{x}{j})_{0 \leq x, j \leq T}$ is lower triangular with diagonal one, so these polynomials span every function on the consecutive values $u, \dots, u + T$. The elementary-symmetric recurrence constructs them, and squarefree reduction leaves at most $(M + 1)^T$ monomials. $\square$

For formal Boolean coordinates $Z = (Z_1, \dots, Z_M)$, put

$$c_j(Z) = \sum_{i \in \mathcal{L}_j} Z_i, \quad \delta_j(Z) = c_j(Z) - 2c_{j+1}(Z).$$

Writing $A_j = \mathcal{L}_j \setminus \mathcal{L}_{j+1}$ and $B_j = \mathcal{L}_{j+1}$ gives

$$\delta_j(Z) = \sum_{i \in A_j} Z_i + \sum_{i \in B_j} (1 - Z_i) - |B_j|.$$

Thus indicators for δ_j on intervals containing negative values are still consecutive-window indicators of a sum of Boolean literals.

Lemma A.2 (Zero-advice pointwise list). *Suppose that on a fixed column at most R coordinates $Z_i(\cdot, z)$ are nonconstant and every other coordinate is identically zero. For every fixed $\rho > 0$, if $R2^{-J} \leq \rho/2$, then the nested hash indexes explicit $(M+1)$ -output multilinear polynomials satisfying, for every fixed (y, z) ,*

$$\Pr_h \left[\text{the output is not the exact one-hot encoding of } \sum_i Z_i(y, z) \right] \leq \rho.$$

Their degree and construction time are

$$D_0 = O_\rho(\sqrt{R} + J), \quad 2^{O_\rho(D_0 \log(M+1))}.$$

Proof. Fix the input and let $A = \{i : Z_i = 1\}$, so $|A| \leq R$. One active coordinate contributes $+1$ or -1 to δ_j with probability 2^{-j-1} each, and otherwise contributes zero. Pairwise independence gives

$$\mathbb{E}_h \delta_j = 0, \quad \mathbb{E}_h \delta_j^2 \leq R2^{-j}.$$

For $j = 3u + v$, $v \in \{0, 1, 2\}$, choose

$$W_j = \lceil C\sqrt{R}2^{-u} + C \rceil$$

with $C = C(\rho)$ chosen so that $4/C^2 \leq \rho/2$. Chebyshev gives

$$\Pr_h[|\delta_j| > W_j] \leq C^{-2}2^{-u-v}.$$

The sum over all levels is below $4/C^2$, while

$$\sum_{j < J} W_j = O(C\sqrt{R} + CJ).$$

At the terminal layer, Markov's inequality gives

$$\Pr_h[c_J > 0] \leq \mathbb{E}_h c_J \leq R2^{-J} \leq \rho/2.$$

Use the constant terminal vector whose zeroth coordinate is one and whose other coordinates are zero. Use [Lemma A.1](#) for every $[\delta_j = d]$, $|d| \leq W_j$, then reconstruct downward through

$$[c_j = s] = \sum_{\substack{0 \leq t \leq M \\ |s-2t| \leq W_j}} [c_{j+1} = t][\delta_j = s - 2t] \pmod{2}.$$

On the simultaneous good event, the true pair is present and unique, so induction produces the complete exact one-hot vector. The degree recurrence

$$d_J = 0, \quad d_j \leq d_{j+1} + 2W_j$$

gives $d_0 = O_\rho(\sqrt{R} + J)$. Constructing throughout in the squarefree quotient $\mathbb{F}_2[Z]/(Z_i^2 - Z_i)$ gives the stated time. $\square$

A.11 Powered-walk proof on the exact seed set

Identify the seed set $\mathbb{F}_{2^{r_0}}^2$ with $\mathbb{Z}_{2^{r_0}}^2$ through a fixed binary labeling; no field-ring isomorphism is claimed. On this vertex set use the eight Margulis neighbors [41], duplicate labelled edges to make degree sixteen, and take the fortieth graph power. The primary Gabber–Galil estimate [20], stated for this eight-neighbor graph as [30, Theorem 8.2], bounds the nontrivial unnormalized eigenvalues by $5\sqrt{2}$. Hence the normalized fortieth power has spectral radius

$$\lambda \leq \left(\frac{5\sqrt{2}}{8} \right)^{40} < \frac{1}{128}.$$

One powered edge label uses exactly 160 bits.

Lemma A.3 (Walk-majority amplification). *Suppose Lemma A.2 has pointwise base failure $\rho \leq 1/128$. Starting from a uniform seed, take an odd length- t walk in the powered graph and apply componentwise majority to the t complete one-hot vectors. The failure probability at every fixed input is at most 2^{-2t-4} ; the amplified list has degree at most tD_0 , uses $2r_0 + 160(t-1)$ seed bits, and is constructible in $2^{O(tD_0L_q+t)}$ time.*

Proof. Fix the input and its bad-seed set B . For specified locations $i_1 < \dots < i_s$, stationarity and the projection-operator bound give

$$\Pr[X_{i_1}, \dots, X_{i_s} \in B] \leq \rho(\rho + \lambda)^{s-1} \leq 2^{-7}(2^{-6})^{s-1}.$$

Majority failure requires at least $s = (t+1)/2$ bad vertices. Union-bounding over at most 2^t location sets gives 2^{-2t-4} . The multilinear majority polynomial has degree at most t and at most 2^t monomials, proving the remaining bounds. $\square$

A.12 Exact-score batch theorem and additive preprocessing

The circuit-analysis algorithm receives polynomial-bit integer threshold descriptions. An alleged arbitrary-real circuit in the final lower bound is first replaced existentially by an equivalent finite integer representation on the Boolean cube; this preserves supports and physical wires. No algorithm manipulates arbitrary real numbers.

Lemma A.4 (Disjoint exact-equation decomposition). *Let*

$$G(x) = \mathbf{1} \left[\sum_{i=1}^n w_i x_i > T \right]$$

be a strict threshold gate with integer parameters, put

$$H = \max\{1, |T|, |w_1|, \dots, |w_n|\}, \quad \bar{L} = \max\{1, \lceil \log_2((n+1)H+1) \rceil\},$$

and allow the literals x_i and $1 - x_i$. Then G is a disjoint OR of at most $\max\{1, n\bar{L}\}$ exact integer equations. In literal form the absolute value of every coefficient is at most $2^{\bar{L}+1} - 1$ and the right-hand side is at most $2^{\bar{L}+1} + 1$. After expanding complemented literals, every coefficient and target has absolute value at most $(n+2)2^{\bar{L}+1}$. The list is constructible deterministically in time polynomial in its explicit output description.

Proof. Put $a_i = |w_i|$, let $z_i = x_i$ when $w_i \geq 0$ and $z_i = 1 - x_i$ when $w_i < 0$, and set

$$A_- = \sum_{i:w_i < 0} a_i, \quad T' = T + A_-.$$

The identity

$$\sum_i w_i x_i = \sum_i a_i z_i - A_-$$

reduces the gate to $\mathbf{1}[\sum_i a_i z_i > T']$. If $T' < 0$, use the single always-true equation $0 = 0$; if $T' \geq \sum_i a_i$, use the single always-false equation $0 = 1$. Hence assume $0 \leq T' < \sum_i a_i$. Let L be the least positive integer for which $T', a_1, \dots, a_n \leq 2^L - 1$. Then $L \leq \bar{L}$, and write

$$a_i = \sum_{b=1}^L 2^{b-1} a_{i,b}, \quad a_{i,b} \in \{0, 1\}.$$

For $(a, b) \in [n] \times [L]$, define the partial score

$$P_{a,b}(z) = \sum_{i=1}^n z_i \left(\sum_{j>b} 2^{j-1} a_{i,j} + \mathbf{1}[i \geq a] 2^{b-1} a_{i,b} \right).$$

Prepend the explicit sentinel 0, and then order these scores as

$$P_{n,L}, P_{n-1,L}, \dots, P_{1,L}, P_{n,L-1}, \dots, P_{1,1}. \quad (\text{A.12.0a})$$

The sequence is nondecreasing and ends at $\sum_i a_i z_i$. Let $P_{a,b}^-$ denote the entry immediately before $P_{a,b}$, with the prepended sentinel used at the first position. Directly from the definition,

$$P_{a,b} - P_{a,b}^- = 2^{b-1} a_{a,b} z_a.$$

Consequently, on every accepting input there is a unique first pair (a, b) whose partial score exceeds T' . At that pair $P_{a,b}^- \leq T' < P_{a,b}$, so its step is nonzero and hence $a_{a,b} = z_a = 1$. Write

$$T' = 2^{b-1} Q_b + R_b, \quad 0 \leq R_b < 2^{b-1}, \quad Q_b \geq 0.$$

Both $P_{a,b}$ and $P_{a,b}^- = P_{a,b} - 2^{b-1}$ are multiples of 2^{b-1} . Therefore the two inequalities $P_{a,b}^- \leq T' < P_{a,b}$ hold if and only if $P_{a,b} = 2^{b-1}(Q_b + 1)$. Thus (a, b) is the first crossing if and only if

$$a_{a,b} = 1, \quad z_a = 1, \quad P_{a,b}(z) = 2^{b-1}(Q_b + 1). \quad (\text{A.12.0b})$$

For a zero bit $a_{a,b}$ omit the child. For a one bit use the single exact equation

$$2P_{a,b}(z) + z_a = 2^b(Q_b + 1) + 1. \quad (\text{A.12.0c})$$

Parity in (A.12.0c) forces $z_a = 1$, after which the equation is precisely (A.12.0b). Thus each accepting input satisfies its unique first-crossing child and every rejecting input satisfies none: the OR is disjoint.

There are at most nL nonconstant children. A coefficient on the left of (A.12.0c) is at most $2(2^L - 1) + 1 = 2^{L+1} - 1$. Since $Q_b \leq 2^{L-b+1} - 1$, its right-hand side is at most $2^{L+1} + 1$. Substitution of $z_i = 1 - x_i$ negates the corresponding coefficient and subtracts it from the target, giving the stated safe bound $(n + 2)2^{\bar{L}+1}$. Finally, in the nonconstant case $T' < \sum_i a_i \leq nH$, so $L \leq \bar{L}$; all loops and arithmetic are polynomial in the displayed output size. $\square$

The preceding proof is a self-contained version of the construction underlying [14, Proposition 18(2) and Appendix B, Lemmas 48–49].¹ All quantitative bounds used here follow from the proof above.

Theorem A.5 (Exact batch evaluator). *There is an absolute C_{batch} with the following property. Let P be an explicitly represented Boolean polynomial over $\mathbb{F}_2$, of degree at most d , in a polynomial-size pool of integer-described threshold gates on $(y, z) \in \{0, 1\}^K \times \{0, 1\}^{q-K}$. Let T_{prep} be the time to substitute disjoint exact-threshold representations and canonically combine the resulting squarefree monomials. Let B be the number of distinct exact-threshold coordinates, and take $Q \geq K + 1$. If*

$$C_{\text{batch}} [K + \log_2(Q + 1) + Qd \log_2(B + 1) + L_q] \leq \frac{q - K}{100},$$

then every exact column count

$$N_P(z) = \sum_{y \in \{0, 1\}^K} P(y, z)$$

is printable in total time

$$T_{\text{prep}} + 2^{q-K} \text{poly}(q).$$

Proof. Rewrite each weak integer threshold coordinate $L \geq \theta$ as $L > \theta - 1$ and apply Lemma A.4. Disjointness makes each OR both its ordinary integer sum and its $\mathbb{F}_2$ sum. Substitution is therefore $\mathbb{F}_2$ -linear and preserves degree. Squarefree reduction and cancellation leave at most

$$N_{\text{mon}} \leq \sum_{j=0}^d \binom{B}{j} \leq (B + 1)^d$$

monomials. A conjunction of exact equations is again one exact equation: write them as $e_j(x) = 0$, let V bound every $|e_j(x)|$ on the Boolean cube, choose an integer $H > 2V + 1$, and use

$$\sum_{j=0}^{\ell-1} H^j e_{j+1}(x) = 0.$$

If some e_j is nonzero, its highest-indexed nonzero term has magnitude at least H^j and strictly dominates the sum of all lower terms. The stacked equation has polynomial bit length because $\ell \leq d$ and every input equation has a polynomial-bit description. The empty monomial is the always-true equation $0 = 0$; an absent monomial contributes nothing.

For an integer $t \geq 0$, define the truncated binomial lift

$$\text{BinLift}_Q(t) = \sum_{j=1}^Q (-2)^{j-1} \binom{t}{j},$$

where $\binom{t}{j} = 0$ for $j > t$. The complete finite sum obeys

$$\sum_{j=1}^t (-2)^{j-1} \binom{t}{j} = \frac{1 - (1 - 2)^t}{2} = t \bmod 2.$$

¹In the CCC 2019 version, the displayed proof of Lemma 48 uses $Q_b > 0$, omits the sentinel from its displayed ordering, and gives $2^{L+1} - 1$ as the right-hand-side endpoint. The local construction uses $Q_b \geq 0$, displays the sentinel, and allows the endpoint $2^{L+1} + 1$. These textual differences do not affect the qualitative decomposition imported from CW19.

Every omitted term with $j > Q$ is divisible by 2^Q , and hence

$$\text{BinLift}_Q(t) \equiv t \bmod 2 \pmod{2^Q}. \quad (\text{A.12.1})$$

Let $G_1, \dots, G_{N_{\text{mon}}}$ be the stacked exact-equation monomials and put $t(y, z) = \sum_u G_u(y, z)$. The elementary-symmetric identity

$$\binom{t(y, z)}{j} = \sum_{\substack{A \subseteq [N_{\text{mon}}] \\ |A|=j}} \prod_{u \in A} G_u(y, z)$$

is indexed by occurrences and remains exact even when two G_u are identical. Stack every selected product into one exact equation. Equation (A.12.1) gives

$$\sum_y \text{BinLift}_Q(t(y, z)) \equiv N_P(z) \pmod{2^Q}.$$

Because $0 \leq N_P(z) \leq 2^K < 2^Q$, the least nonnegative residue determines the exact count. Hardwiring all 2^K live assignments and retaining every subset occurrence produces exactly

$$s_{\text{gen}} = 2^K \sum_{j=1}^{\min\{Q, N_{\text{mon}}\}} \binom{N_{\text{mon}}}{j} \leq 2^K Q \max\{1, N_{\text{mon}}\}^Q$$

signed exact-threshold occurrences, with native coefficient $(-2)^{j-1}$ on a size- j subset. The displayed hypothesis puts this below $2^{(q-K)/100}$ and every coefficient and stacked equation has polynomially many bits.

Apply [Theorem B.3](#) at residual arity $q - K$. It prints the complete signed integer score table in $2^{q-K} \text{poly}(q)$ time. Reducing each score modulo 2^Q yields $N_P(z)$. Every substitution, squarefree product, and canonical stacked equation is charged in T_{prep} , while every live assignment and BinLift subset is charged through s_{gen} . If $N_{\text{mon}} = 0$, return the all-zero table directly. $\square$

For the present supplier, the complete total radix row has canonical construction time

$$T_{\text{prep}} \leq 2^{O_D(dL_q + L_q^2 + t)}.$$

The L_q^2 term is the internal digit-tuple syntax. It is additive and is never included in the external row entropy.

A.13 Native threshold AND-four compiler and explicit estimators

Proposition A.6 (Disjoint child tuples). *Fix a polynomial integer-description envelope. Given $j \leq 4$ normalized $\text{THR} \circ \text{THR}$ circuits $C_1, \dots, C_j$, there are a common labelled literal-carrier family $\mathcal{H}$ and a family $\mathcal{G}$ of polynomial-bit integer equations over $\mathcal{H}$ such that*

$$\bigwedge_{i=1}^j C_i(x) = \sum_{g \in \mathcal{G}} \mathbf{1}[L_g(x) = 0] \quad (\text{A.13.1})$$

pointwise over the integers. The sum on the right is always a bit, $|\mathcal{H}| = O(W + 1)$, and

$$|\mathcal{G}| \leq q^{c_{\text{top}}}$$

for one fixed constant c_{top} .

Proof. Put every bottom-gate occurrence, its complement, and constants zero and one into $\mathcal{H}$; repeated occurrences retain separate labels. The normalized top linear form and threshold are integral, so first rewrite the paper's weak convention $L \geq \theta$ as the source's strict convention $L > \theta - 1$. This changes no coefficient, bottom occurrence, or wire. By [Lemma A.4](#), the self-contained version of the construction underlying [\[14, Proposition 18\(2\) and Appendix B, Lemmas 48–49\]](#), the top threshold of C_i is a disjoint OR

$$C_i = \text{DOR}(a_{i,1}, \dots, a_{i,d_i}) = \sum_{e=1}^{d_i} a_{i,e},$$

where every child is one polynomial-bit exact equation over literals from $\mathcal{H}$ and $d_i \leq q^{c_0}$ for one fixed c_0 . The last equality holds over the integers because at most one child accepts.

For completeness, the polynomial child bound follows from the local safe endpoint rather than from the shorter printed endpoint. The retained top arity is $m \leq |\mathcal{H}| = O(W + 1) = q^{O(1)}$. Finite normalization gives each top parameter bit length $O(m \log m) = q^{O(1)}$, so the local lemma gives at most $m\bar{L} = q^{O(1)}$ children, each with a $q^{O(1)}$ -bit description. Thus a single fixed exponent c_0 covers every normalized top gate in this supplier.

Expand the product of these j sums. A tuple of chosen children is their conjunction. Stack its at most four equations with the domination construction in [Appendix A.12](#), producing one polynomial-bit equation $L_g = 0$. At any input, either some C_i is zero and no tuple accepts, or every C_i is one and the unique accepting child from each disjoint OR gives exactly one accepting tuple. This proves (A.13.1). Taking $c_{\text{top}} \geq 4c_0 + 1$ bounds the family. $\square$

Lemma A.7 (Explicit prime surrogate). *For every fixed $A > 0$, there is a deterministic polynomial-size prime set $\mathcal{P}_q$ such that, for every $g \in \mathcal{G}$ and every Boolean input x ,*

$$L_g(x) = 0 \implies \frac{1}{|\mathcal{P}_q|} \sum_{p \in \mathcal{P}_q} \mathbf{1}[L_g(x) \equiv 0 \pmod{p}] = 1, \quad (\text{A.13.2})$$

while for $L_g(x) \neq 0$ the same average is smaller than

$$q^{-A - c_{\text{top}} - 10}. \quad (\text{A.13.3})$$

Each modular predicate is one Boolean lookup of one nonnegative polynomial-range weighted score over $\mathcal{H}$.

Proof. Fix β large enough that every equation produced by [Proposition A.6](#) satisfies

$$|L_g(x)| < 2^{q^\beta}.$$

Put $h = A + c_{\text{top}} + 12$, let

$$x_0 = \lceil q^{\beta+h+3} \rceil,$$

and take all primes $p \leq x_0$. Rosser and Schoenfeld [\[51, Theorem 4, eq. \(3.14\)\]](#) prove the stronger inequality

$$\sum_{p \leq x} \log p > x \left(1 - \frac{1}{2 \log x} \right)$$

for $x \geq 563$, with the natural logarithm on both sides. Passing to the base-two convention of this paper only enlarges the left-hand sum, so the exact estimate used here,

$$\sum_{p \leq x} \log p \geq x/3,$$

holds in base two for sufficiently large x , and it implies

$$|\mathcal{P}_q| \geq \frac{x_0}{3 \log x_0}.$$

A nonzero integer of magnitude below 2^{q^β} has fewer than q^β distinct prime divisors. Its accepting-prime fraction is therefore at most

$$\frac{3q^\beta \log x_0}{x_0} < q^{-A - c_{\text{top}} - 10}$$

eventually. A zero integer is divisible by every prime.

For the circuit form, combine repeated labelled literals, absorb constants, reduce every coefficient modulo p into $\{0, \dots, p-1\}$, and move the constant to the target residue. The resulting nonnegative score is at most $|\mathcal{H}|(p-1) = q^{O_\Lambda(1)}$. Its accepted residues are an explicit Boolean lookup. The primes can be enumerated deterministically in polynomial time because $x_0 = q^{O_\Lambda(1)}$. $\square$

Fix g, p . After coefficient reduction, write

$$L_g(X) \equiv a_{g,p,0} + \sum_{i=1}^{|\mathcal{H}|} r_{g,p,i} X_i \pmod{p}, \quad 0 \leq r_{g,p,i} < p. \quad (\text{A.13.4})$$

Apply the residualization of [Appendix A.2](#) separately to the labelled carriers and substitute $X_i = C_i + Z_i$. Define

$$f_{g,p}(z) = \left(a_{g,p,0} + \sum_i r_{g,p,i} C_i(z) \right) \pmod{p}. \quad (\text{A.13.5})$$

Then the modular predicate is exactly

$$\mathbf{1} \left[f_{g,p}(z) + \sum_i r_{g,p,i} Z_i(y, z) \equiv 0 \pmod{p} \right]. \quad (\text{A.13.6})$$

Choose one fixed radix $b_0 \geq 2$ and write

$$r_{g,p,i} = \sum_{d < \tau} v_{g,p,i,d} b_0^d, \quad 0 \leq v_{g,p,i,d} < b_0, \quad \tau = O_A(L_q).$$

Realize digit d as an unweighted population by giving carrier i exactly $v_{g,p,i,d}$ labelled clones. The clone multiplicity is below the fixed radix b_0 . A complement carrier is touched exactly when its underlying bottom occurrence is touched, while constant carriers residualize to zero. Hence every digit list has at most $2b_0R = O(R)$ nonconstant labelled coordinates, without rerunning TouchingSelect, and all digit lists use one common walk seed. Let $P_{g,p,e,d,v}$ be the amplified one-hot polynomial asserting that digit d has population v .

For every possible offset $f \in \{0, \dots, p-1\}$ define the total Boolean modular row

$$Q_{g,p,e,f} = \sum_{\substack{(v_0, \dots, v_{\tau-1}) \\ f + \sum_{d < \tau} b_0^d v_d \equiv 0 \pmod{p}}} \prod_{d < \tau} P_{g,p,e,d,v_d} \pmod{2}. \quad (\text{A.13.7})$$

On legal one-hot lists, exactly one tuple contributes and (A.13.7) equals (A.13.6). On malformed lists it remains one Boolean polynomial; no tuple is decoded as an integer. Its degree is at most τ in the amplified list outputs, and the number of internal radix tuples is at most

$$(b_0|\mathcal{H}| + 1)^\tau = 2^{O_A(L_q^2)}. \quad (\text{A.13.8})$$

These tuples contribute to per-row preprocessing, not to the external row count.

For direct symmetric circuits $C_1, \dots, C_j$, let $P_{e,\mathbf{f}}$ be the product of their total Boolean lookup rows and let

$$T_{e,\mathbf{f}}(z) = \sum_{y \in \{0,1\}^K} P_{e,\mathbf{f}}(y, z)$$

be the exact table printed by [Theorem A.5](#). If $\mathbf{F}(z)$ is the tuple of actual scalar residual offsets, define

$$\hat{p}_S = \frac{1}{|\mathcal{E}|2^q} \sum_{e \in \mathcal{E}} \sum_z T_{e,\mathbf{F}(z)}(z). \quad (\text{A.13.9})$$

For every fixed (y, z) , simultaneous list correctness makes the selected Boolean row equal the desired conjunction. Off that event both values are bits. Fubini therefore bounds the error in (A.13.9) by the joint input-seed list-failure probability; no seed must work on every input.

For threshold-top inputs, let

$$T_{g,p,e,f}(z) = \sum_{y \in \{0,1\}^K} Q_{g,p,e,f}(y, z)$$

be the exact table and set

$$\hat{p}_T = \frac{1}{|\mathcal{P}_q||\mathcal{E}|2^q} \sum_{g \in \mathcal{G}} \sum_{p \in \mathcal{P}_q} \sum_{e \in \mathcal{E}} \sum_z T_{g,p,e,f_{g,p}}(z). \quad (\text{A.13.10})$$

Every average in (A.13.10) is over an explicitly enumerated finite set. By (A.13.1)–(A.13.3), the pointwise prime-surrogate error after summing over g is below q^{-A-9} . For each fixed g, p, y, z , the selected modular row differs from its target only if one of its τ lists fails; both sides are bits off that event. Choose the walk length so that this probability is below $q^{-A-c_{\text{top}}-10}$. Fubini and $|\mathcal{G}| \leq q^{c_{\text{top}}}$ then give

$$\left| \hat{p}_T - \Pr_x \left[\bigwedge_{i=1}^j C_i(x) = 1 \right] \right| \leq q^{-A}.$$

The external labels (g, p, e, f) have total polynomial multiplicity q^{h_A} . Equation tuples, digit tuples, and exact-child tuples are paid by T_{prep} . BinLift subsets instead form the generated signed occurrence family s_{gen} bounded inside [Theorem A.5](#); their printing cost is included in its $2^{q-K} \text{poly}(q)$ term. This completes the native $\text{AND}_{\leq 4}$ estimator without CTW shifted products, GapAND, or affine score recombination.

B Exact signed-score printing

B.1 Imported boundary

The only black box in Appendix B is the following published statement.

Fact B.1 (Williams rectangular-product contract). *For every sufficiently large integer u , Boolean matrices of dimensions*

$$u \times \lceil u^{0.1} \rceil \quad \text{and} \quad \lceil u^{0.1} \rceil \times u$$

can be multiplied exactly over the integers in

$$u^2 \text{poly}(\log u)$$

deterministic bit operations.

This is [60, Corollary 4.4 and Appendix C, Corollary C.2]. We use the stated exact Boolean integer output, not an intermediate threshold-evaluation claim. In particular, the surrounding main-text implementation sentence and Appendix C give different field-cardinality descriptions ($\text{poly}(\log u)$ versus $\text{poly}(u)$). We do not use the smaller-field sentence: the imported interface is only the exact Boolean integer-product corollary, and every signed operation below is a local binary-slicing reduction to that interface.

Everything below is local. The dependency chain is: exact Boolean rectangular multiplication; signed-by-Boolean multiplication by binary slicing; weighted dominance on two half-cubes; and exact signed score tables.

B.2 Signed left matrix, Boolean right matrix

Lemma B.2 (Signed-by-Boolean product). *Fix a constant $\nu \geq 1$. Let*

$$1 \leq d \leq \lceil u^{0.1} \rceil, \quad A \in \mathbb{Z}^{u \times d}, \quad B \in \{0, 1\}^{d \times u},$$

and suppose

$$|A_{ij}| < 2^b, \quad b \leq (\log_2(u + 2))^\nu.$$

Then AB is exactly computable in $u^2 \text{poly}(\log u)$ deterministic bit operations.

Proof. For $0 \leq t < b$, let A_t^+ contain the t th magnitude bit of every positive entry and let A_t^- contain the t th magnitude bit of every negative entry. Both matrices are Boolean, and

$$A = \sum_{t=0}^{b-1} 2^t (A_t^+ - A_t^-).$$

Hence

$$AB = \sum_{t=0}^{b-1} 2^t (A_t^+ B - A_t^- B). \tag{B.1}$$

Zero-pad the common dimension to $\lceil u^{0.1} \rceil$ and invoke the imported contract on the $2b = \text{poly}(\log u)$ Boolean products in (B.1). A bit-plane product entry lies in $[0, d]$, so a recombined answer has at most $b + \lceil \log_2(d + 1) \rceil + 1 = \text{poly}(\log u)$ bits. Constructing the planes costs $udb \text{poly}(\log u)$, and recombination at all u^2 entries costs $u^2 \text{poly}(\log u)$. Finite values below the imported onset are handled directly. $\square$

The signs are arithmetic data in the left matrix. A coefficient of magnitude V is never replaced by V gate occurrences.

B.3 Weighted dominance

Theorem B.3 (Exact signed-score printer). *Fix a constant description exponent. For every sufficiently large r , let $G_1, \dots, G_s$ be integer exact-threshold gates on r variables with polynomial-bit parameters, and let $v_1, \dots, v_s$ be signed polynomial-bit integers. If*

$$s \leq 2^{r/100},$$

then the complete table

$$z \mapsto \sum_{i=1}^s v_i G_i(z)$$

is deterministically printable in $2^r \text{poly}(r)$ bit operations.

Proof. Replace every equality by two successive weak cuts:

$$\mathbf{1}[L = \theta] = \mathbf{1}[L \geq \theta] - \mathbf{1}[L \geq \theta + 1]. \quad (\text{B.2})$$

The requested table is now a signed sum $\sum_{j=1}^m \omega_j H_j(z)$ of $m \leq 2s$ ordinary threshold gates. The coefficients are the native integers $v_i, -v_i$.

First let $r = 2d$, split $z = (a, b)$ into two d -bit halves, and put $U = 2^d$. For gate H_j , split its linear form as

$$\ell_j(a, b) = \ell_j^L(a) + \ell_j^R(b)$$

and define

$$A[a, j] = \ell_j^L(a), \quad B[j, b] = \ell_j^R(b).$$

Then

$$H_j(a, b) = \mathbf{1}[A[a, j] \leq B[j, b]],$$

so the desired output is the weighted dominance matrix

$$P[a, b] = \sum_{j=1}^m \omega_j \mathbf{1}[A[a, j] \leq B[j, b]]. \quad (\text{B.3})$$

Choose $\delta = 0.049$. Since

$$m \leq 2^{r/100+1} = 2U^{0.02} \leq U^\delta$$

eventually, zero-pad the gate index to $\lceil U^\delta \rceil$. For each j , sort together all U values $A[a, j]$ and all U values $B[j, b]$. Break ties by placing every A copy before every B copy. With this order, $A[a, j] \leq B[j, b]$ holds exactly when the rank of the former precedes the rank of the latter.

Partition each sorted list into consecutive buckets of size at most

$$B_{\text{buck}} = \lceil U^{1-\delta} \rceil.$$

For every pair (a, j) , scan the $B[j, b]$ entries in the same bucket and add ω_j precisely for the correctly ordered pairs. The total number of inspected pairs is at most

$$U \cdot U^\delta \cdot U^{1-\delta} = U^2$$

Table 4: Dimension check for the signed-score printer.

Object	Bound	Reason
half-cube size	$U = 2^{r/2}$	complete even split
weak cuts	$m \leq 2s \leq 2^{r/100+1}$	two successor cuts per equality
padded gate dimension	$m \leq U^{0.049}$ eventually	$0.02 < 0.049$
bucket-pair dimension	$O(U^{0.098})$	two factors of $U^{0.049}$
imported allowance	$U^{0.1}$	Williams contract
coefficient and output bits	$\text{poly}(r)$	fixed description exponent
total time	$U^2 \text{poly}(r) = 2^r \text{poly}(r)$	output-linear table

up to polynomial factors in r .

For cross-bucket pairs, let h be a bucket label and form

$$A'[a, (j, h)] = \begin{cases} \omega_j, & A[a, j] \text{ lies in bucket } h, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B'[(j, h), b] = \mathbf{1}[\text{the bucket of } B[j, b] \text{ is later than } h].$$

The common dimension is

$$O\left(U^\delta \frac{U}{U^{1-\delta}}\right) = O(U^{2\delta}) = O(U^{0.098}) \leq \lceil U^{0.1} \rceil$$

for every sufficiently large U . The left matrix has signed polynomial-bit entries and the right matrix is Boolean, so [Lemma B.2](#) computes $A'B'$ exactly. Its (a, b) entry is precisely the contribution from later buckets. Adding the same-bucket contribution gives (B.3) with no thresholding or information loss.

Sorting and matrix construction cost $U^{1+\delta} \text{poly}(r)$. Every score entry is a sum of $O(U^{0.098})$ polynomial-bit coefficients and therefore has $\text{poly}(r)$ bits. The total time is

$$U^2 \text{poly}(r) = 2^r \text{poly}(r).$$

If r is odd, add one ignored variable, run the even-arity algorithm, and retain the zero slice. $\square$

B.4 Dimension certificate

[Table 4](#) records the exact dimensions used in the signed-score reduction.

The strict comparison $0.098 < 0.1$ is the only exponent slack used from rectangular multiplication. Ties, negative coefficients, and odd arity are handled locally.

C Componentwise almost-everywhere transfer

C.1 Abstract interface

For each class mode i , after fixing the requested accuracy and saving, let the supplier support the monotone cap

$$A_i(q) = \left\lfloor a_i \frac{q^3}{L(q)^{p_i}} \right\rfloor.$$

The transfer deliberately uses the smaller **carried cap**

$$B_i(q) = \left\lfloor b_i \frac{q^3}{L(q)^{p_i}} \right\rfloor, \quad 0 < b_i \leq a_i/64, \quad (\text{C.1.1})$$

with harmless linear floors at small arities. For every fixed C_{addr} , regular variation gives, for all sufficiently large q ,

$$B_i(q_0) \leq A_i(q) \quad \text{whenever} \quad q \leq q_0 \leq q + C_{\text{addr}} \log q. \quad (\text{C.1.2})$$

Indeed the unscaled ratio is $1 + o(1)$; the factor 64 leaves ample uniform finite slack. This inequality pays the addressed-arity restriction $q_0 \downarrow q$. Systematic code coordinates are handled instead by native parity atoms, whose $O(q^2)$ wires are eventually below every supported near-cubic cap. Assume:

1. literal restrictions and output negation preserve the class and do not increase absolute resource;
2. canonical circuit descriptions and rational coefficients at the cap have polynomial bit length;
3. for every fixed $D, \sigma > 0$, AND-four CAPP at A_i has error q^{-D} and time $2^q/q^\sigma$ after choosing $a_i = a_i(D, \sigma) > 0$ sufficiently small; and
4. cheap parity and the standard Case-1 general-circuit simulation are available.

The inverse-polynomial symmetric/threshold supplier proves this interface with

$$p_S = 7, \quad p_T = 9.$$

C.2 Componentwise sums

Write

$$\|h\|_1 = \mathbb{E}_{x \in \{0,1\}^q} |h(x)|.$$

Let $\text{Lin}_q(B, J, \Lambda)$ contain rational sums

$$P = \sum_{a=1}^s \alpha_a C_a$$

with $s \leq J$, coefficient mass at most Λ , polynomial-bit coefficients, and every atom individually inside B . The notation

$$P \in [0, 1] \text{Lin}_q(B, J, \Lambda)$$

means additionally that $0 \leq P(x) \leq 1$ at every Boolean input; it is not a scalar multiple of the class. Put

$$\overline{\Lambda} = 1 + \Lambda.$$

Expanding a degree-four polynomial in such sums and constants gives at most $(J + 1)^4$ supplier calls. If each has error η , the propagated error is at most

$$\eta \bar{\Lambda}^4.$$

Term count controls time; augmented coefficient mass controls accuracy; neither is added to atom wires.

[Lemma C.7](#) proves that this is exactly the resource split used by the CLW Booleanity and validity tests. Their degree-four polynomial $T^2(1 - T)^2$ explains why AND-four, rather than XOR-two alone, is required.

C.3 Why polynomial saving suffices

First freeze the uniform source constructions. Their stated $\text{poly}(N, \log T)$ construction has one fixed polynomial degree, uniform over polynomial clocks; changing the eventual constant clock exponent changes $\log T$ only by a fixed factor. Let d_{ver} be a fixed exponent bounding that N -dependence and the local verifier-compaction work in [Lemma C.6](#). Now choose the polynomial hierarchy clock

$$T(N) = N^{C_{\text{clk}}}$$

with $C_{\text{clk}} > d_{\text{ver}} + 2$, and use the exact padded PCP width from [Lemma C.5](#),

$$q = C_{\text{clk}} \log N + O(\log \log N).$$

For a fixed target advantage exponent, [Appendix C.10](#) shows that the compacted verifier, PCPP clause pool, guessed proof descriptions, componentwise term count, witness length, and coefficient mass have fixed polynomial exponents. Write

$$2^q \leq T(N)q^{d_{\text{pad}}}$$

for the fixed padding exponent supplied by [Lemma C.5](#), let p_{call} bound the number of CAPP calls and their remaining polynomial overhead, and let ρ_{coef} bound the coefficient-mass exponent. The finite list of required term-count, coefficient-mass, and coefficient-bit envelopes is frozen before the weak machine; the refuter is never asked to diagonalize against all polynomial envelopes simultaneously.

Request

$$D > 4\rho_{\text{coef}} + 10$$

and

$$\sigma > p_{\text{call}} + d_{\text{pad}} + 10.$$

Then ordinary tests cost

$$q^{p_{\text{call}}} \frac{2^q}{q^\sigma} \leq T(N)q^{-10} = o(T(N)).$$

Every systematic code coordinate is a parity of at most $q/2$ inputs. Realize it directly as a native q -input atom in the selected target class: identity bottoms and a symmetric lookup use $O(q)$ wires, while cumulative threshold bottoms and alternating top weights use $O(q^2)$ wires. Thus

$$O(q^2) = o(A_i(q)).$$

After expanding $(\text{Enc}_s - T)^2$, every nonconstant term is either a product of at most two carried atoms or a product of one carried atom with this parity atom. These are native AND-four calls, so their total cost is again

$$q^{p_{\text{call}}} \frac{2^q}{q^\sigma} \leq T(N)q^{-10} = o(T(N)).$$

By (C.1.2), every carried atom obtained from a $q_0 = q + O(\log q)$ input lies inside the supported native cap $A_i(q)$. No half-arity call or same-coefficient comparison $B(q) \leq B(q/2)$ is used.

The non-CAPP construction and compaction cost is

$$N^{d_{\text{ver}}} q^{O(1)} + q^{O(1)} = o(T(N)).$$

Thus the resulting nondeterministic machine runs in time $o(T(N))$. What matters is the ability to request an arbitrarily large **fixed polynomial** saving while retaining some positive boundary coefficient.

C.4 Conflict and common function

The machine constructed in [Appendix C.10](#) has one nondeterministic mode per circuit class. On a hierarchy NO word, validity rounding and PCPP soundness reject every branch. At the refuter's conflicting word, select the canonical general oracle before selecting a class mode. In Case 2, the fixed deterministic honest-assignment algorithm from the pointwise PCPP produces one canonical Boolean proof family from that oracle. The Boolean proof-table function

$$f(i, j, u) = \widehat{T}_{ij}(u)$$

is therefore common to all modes.

If a componentwise linear sum approximates f , average its fixed- (i, j) restrictions over occurrences of each proof variable. Every atom is a literal restriction, so resource does not increase. Occurrence averaging multiplies term count polynomially, preserves coefficient mass, and preserves the $[0, 1]$ semantic promise. [Lemma C.8](#) then constructs an accepting branch, contradicting the conflict.

C.5 Inverse-polynomial advantage

Apply [Lemma C.9](#) contrapositively. To obtain the theorem's advantage n^{-c} after padding, run the core XOR lemma with the slightly stronger exponent $c + 1$ and take

$$k = \Theta_{c+1}(\log q).$$

The output function has

$$m = k(q + O(\log q)) = \Theta_c(q \log q)$$

variables. A correlator yields polynomially many literal restrictions of itself as the atoms of one approximate sum. Their absolute wire charge is the charge of the original correlator.

If

$$W(C) \leq b'_i \frac{m^3}{L(m)^{p_i+3}},$$

then

$$W(C_a) \leq O_c(b'_i) \frac{q^3 L(q)^3}{L(q)^{p_i+3}} = O_c(b'_i) \frac{q^3}{L(q)^{p_i}}.$$

Choose b'_i below the carried coefficient after every fixed comparison. This proves a fixed-positive core endpoint at exponent $p_i + 3$. [Appendix C.12](#) selects a core arity m satisfying $n/3 \leq m \leq n/2$ eventually. It follows that $m^{-(c+1)} < n^{-c}$ eventually. Moreover

$$\frac{n^3/L(n)^{p_i+3}}{m^3/L(m)^{p_i+3}} = O_c(1),$$

so shrinking the final coefficient by this fixed comparison transfers the core resource cap to every padded target length. This is the precise padding step used below.

For every fixed $A > 0$,

$$A \frac{n^3}{L(n)^{p_i+4}} < b'_i \frac{n^3}{L(n)^{p_i+3}}$$

eventually. The same language therefore defeats every multiplier one logarithm below the fixed-positive endpoint. More generally the extra L may be replaced by any divergent $u(n)$.

With $(p_S, p_T) = (7, 9)$, the fixed-positive exponents are $(10, 12)$ and the clean every-multiplier exponents are $(11, 13)$.

C.6 Fixed advantage

For fixed advantage γ , the XOR power is constant and

$$m = \Theta_\gamma(q).$$

No logarithmic arity factor appears. The direct symmetric supplier needs only constant error and has $p_S = 5$; the threshold supplier remains at $p_T = 9$. Hence the fixed-positive endpoints are $(5, 9)$ and the clean every-multiplier endpoints are $(6, 10)$.

C.7 Every target length and E^{NP}

[Appendix C.12](#) gives an exact computable source schedule for every desired output length n . It has $q = \Theta(n/\log n)$ in the inverse-polynomial tier and $q = \Theta(n)$ in the fixed-advantage tier. The chosen XOR power dominates the fixed Case-1 amplifier arity constant, so either branch first pads to the common core m_s and then to n . If a circuit correlates with the padded function, one fixing of the ignored suffix preserves at least the same agreement and cannot increase wires.

Case 1 is excluded by selecting the general-oracle cutoff as in [Fact C.4](#) above the fixed fan-in-two simulation exponent of normalized depth-two circuits at these polynomial resource caps and far enough that the amplified output advantage is at most $q^{-(c+2)} \leq m^{-(c+1)}$. Thus both sides of the dichotomy meet the same padding-safe core threshold.

[Lemma C.10](#) gives canonical recovery of the Case-1 outer truth table or the Case-2 succinct general-oracle circuit through SAT formulas obtained by unrolling all 2^q rows. In Case 2, the PCPP's fixed deterministic honest-assignment algorithm supplies proof coordinates directly; no exponentially large proof is guessed by the weak machine or recovered by SAT. The complete construction runs in $2^{O(n)}$ with an NP oracle and defines a total E^{NP} language at every sufficiently large length.

C.8 Quantifier boundary

The transfer produces

$$\forall c \exists F_c \exists b_{S,c}, b_{T,c} > 0 \forall^* n.$$

It does not produce one language for all c , nor every coefficient at the fixed-positive exponents. Selector packaging does not preserve correlation hardness because signed bias can concentrate off the selected row.

The every-multiplier L^{p+4} consequence is nevertheless on the same language and is purely pointwise: $A/L \rightarrow 0$. No diagonal language construction or algorithm depending on A is used.

C.9 Published transfer interfaces

Fix a time-constructible polynomial clock

$$T(N) = N^{C_{\text{clk}}}$$

with C_{clk} chosen above every fixed source preprocessing exponent.

Fact C.1 (Native outer verifier). [11, Lemma 3.10]; see also [8]. In time $\text{poly}(N, \log T(N))$, the CLW outer construction prints one native width r for both randomness and each proof address, $t_{\text{out}} = \text{poly}(r)$ nonadaptive projection queries, a 3CNF predicate on the answers, perfect completeness, and no-instance acceptance on at most an N^{-10} fraction of native random strings. Each output bit of the query map is a random bit, its negation, or a constant, and

$$2^r \leq T(N) \text{polylog } T(N).$$

Fact C.2 (Pointwise two-query PCPP). [11, Lemma 3.11]; see also [14, Lemmas 25–27] and [56]. Given an ordinary q -input circuit H of size at least q , the transformation prints a polynomial-size 2-SAT instance on systematic variables $Y = \text{Enc}(u)$ and auxiliary variables Z , with a power-of-two indexed clause list and constants $c_p > s_p$. If $H(u) = 1$, a deterministic polynomial-time honest-assignment algorithm outputs Z_u satisfying at least a c_p fraction. If $H(u) = 0$, every Z assignment satisfies at most an s_p fraction. Each systematic coordinate is an explicit parity of at most $q/2$ input variables, and the polynomially many auxiliary coordinates have addresses of $O(\log q)$ bits.

Fact C.3 (Fixed-machine refuter). [11, Theorem 4.6]; see also [19]. For each fixed weak nondeterministic machine $\mathcal{M}$ running in $o(T(N))$ time and guessing fewer than $N/10$ bits, a deterministic $\text{poly}(T(N))$ -time SAT-oracle procedure produces, at every sufficiently large source length N , a word on which $\mathcal{M}$ disagrees with the hierarchy language.

Fact C.4 (Case-1 hardness amplifier). [11, Lemma 3.9]; see also [53, Theorem 24]. There are absolute constants $c_{\text{STV}} \geq 1$ and $c_{\text{amp}} \geq 1$ with the following property. Let $S(q)$ be time constructible and let $h : \{0, 1\}^q \rightarrow \{0, 1\}$ have no general Boolean circuit of size $S(q)$. From the full truth table of h , one can construct in $2^{O(q)}$ time the full truth table of a Boolean function $\text{Amp}(h)$ on at most $c_{\text{amp}}q$ variables such that no circuit of size $S(q)^{1/c_{\text{STV}}}$ has agreement greater than

$$\frac{1}{2} + S(q)^{-1/c_{\text{STV}}}$$

with $\text{Amp}(h)$. We import the literal packaged statement from [11, Lemma 3.9]; [53, Theorem 24] is its code-based primary amplification source. We use the fact only with the fixed polynomial $S(q) = q^G$.

These are semantic source interfaces. Padding random strings to one exact width, padding the indexed clause list to a power of two, adding isolated gates to satisfy size premises, constructing the finite union of class modes, and choosing canonical descriptions are local operations proved below.

The composition pattern itself is also present in the proof of [11, Theorem 1.8]: substitute a guessed succinct oracle into the outer verifier, apply the pointwise PCPP, guess succinct proof-coordinate functions, analyze the indexed clauses, invoke the fixed-machine refuter, and then split between a hard correct outer oracle and hard honest proof coordinates. The proof below changes the analysis class and makes the resource bookkeeping componentwise; it does not import any stronger PCP, refuter, or amplification conclusion than Facts C.1–C.4.

Lemma C.5 (Exact common width). *Fix a constant A_{pad} larger than the polylogarithmic exponent in Fact C.1 and define*

$$q(N) = \lceil \log_2 T(N) \rceil + A_{\text{pad}} \lceil \log_2(\log_2 T(N) + 2) \rceil + A_{\text{pad}}. \quad (\text{C.9.1})$$

The native verifier is equivalent to one with exactly $q(N)$ random bits and exactly $q(N)$ bits per proof address. Perfect completeness and acceptance fractions are unchanged, projection queries remain projections, and

$$\frac{2^{q(N)}}{T(N)} = q(N)^{O(1)}. \quad (\text{C.9.2})$$

Proof. The choice of A_{pad} gives $r \leq q(N)$ eventually. Ignore the final $q - r$ random bits. Every native random string then has exactly 2^{q-r} extensions, so its probability mass is unchanged. Map a native proof address $a \in \{0, 1\}^r$ to $a0^{q-r}$. An honest proof is extended arbitrarily off this zero-suffix face; conversely every extended proof restricts to a native proof on that face. Appending constants preserves the projection form. Finally (C.9.2) follows directly from (C.9.1). Finite lengths are hardwired. $\square$

Lemma C.6 (Compact substituted circuit). *After the source preprocessing, the padded projection verifier may be replaced, without changing its Boolean value or acceptance fractions, by an explicit projection table and a duplicate-free outer 3CNF of size $q^{O(1)}$. Consequently, substituting any size- q^G general-oracle circuit produces an ordinary q -input circuit H_{sub} of size $q^{O_G(1)}$. The construction and compaction cost is $N^{d_{\text{ver}}} q^{O_G(1)}$ for one fixed d_{ver} independent of N .*

Proof. The query map has only qt_{out} output coordinates. If the source prints it as a circuit, evaluate every output on the zero vector and on the q unit vectors. The projection promise then identifies that output as a constant, a random bit, or its negation, giving an explicit $q^{O(1)}$ table after the allowed $\text{poly}(N, \log T)$ preprocessing.

Simplify every outer clause by deleting repeated literals, delete tautological clauses, delete duplicate clauses, and sort the survivors in one fixed lexicographic order. A 3CNF on t_{out} answer variables has at most

$$8 \sum_{j=0}^3 \binom{t_{\text{out}}}{j} = t_{\text{out}}^{O(1)} \quad (\text{C.9.3})$$

distinct non-tautological clauses of width at most three. These operations preserve the conjunction pointwise; unlike the later PCPP clause list, the outer 3CNF carries no multiplicity-weighted acceptance measure. Finally, copy the size- q^G universal evaluator at each of the $t_{\text{out}} = \text{poly}(q)$ projected queries and feed the results to the compact 3CNF. This has size $q^{O_G(1)}$. The uniform construction time of Fact C.1, the $q + 1$ projection-table probes, sorting the clause list, and the polynomial-size substitution are bounded by $N^{d_{\text{ver}}} q^{O_G(1)}$ for one fixed source-construction exponent. $\square$

C.10 Full componentwise weak-machine replay

Apply [Lemma C.6](#) before invoking the PCPP. Substitute a guessed size- q^G general-oracle circuit into the compact projection table and duplicate-free outer 3CNF. The resulting ordinary q -input circuit H_{sub} has size $q^{O_G(1)}$ with no hidden $\text{poly}(N)$ clause factor. Add isolated gates if necessary to meet [Fact C.2](#)'s size-at-least-input premise; this does not change the computed function.

For a given guessed general-oracle branch, let M_0 be the source's power-of-two clause count after substitution. Fix one branch-independent exponent d_G that bounds every such output by $M_0 \leq (q+2)^{d_G}$, and put

$$M = 2^{\lceil d_G \log_2(q+2) \rceil}, \quad r = \log_2 M = O_G(\log q).$$

Repeat the entire indexed list exactly M/M_0 times. This integer is a power of two, so the clause distribution and every satisfaction fraction are unchanged. The machine guesses one real proof function $D_X : \{0, 1\}^q \rightarrow \mathbb{R}$ for every distinct PCPP variable $X \in Y \cup Z$. If position j of indexed clause i names X , define

$$T_{ij} = D_X.$$

Thus repeated occurrences are aliases of one guessed function, not independent guesses. For an auxiliary occurrence define

$$P_{ij}(u) = T_{ij}(u)^2(1 - T_{ij}(u))^2.$$

For a systematic occurrence copied from code coordinate Y_s , define

$$P_{ij}(u) = (\text{Enc}_s(u) - T_{ij}(u))^2.$$

In both cases put

$$Q_{ij}(u) = T_{ij}(u)^2.$$

The weak machine estimates $\mathbb{E}_{i,j,u} P_{ij}$ and every $\mathbb{E}_u Q_{ij}$. All expressions have degree at most four in the guessed sums. For a systematic row, realize Enc_s by the explicit native parity circuit from [Section 2](#). Expanding $(\text{Enc}_s - T_{ij})^2$ then uses only constants, carried atoms, and conjunctions of one parity atom with at most one carried atom; no parity-support enumeration occurs.

Fix a sufficiently small rational $\zeta > 0$ such that

$$6\sqrt{12\zeta} < \frac{1}{10}(c_p - s_p).$$

The machine passes validity only if the first estimated average is at most 2ζ and all estimated second moments are at most $1 + \zeta$.

Let

$$F_i(u) = \widehat{\text{Cons}_i}(T_{i1}(u), T_{i2}(u))$$

be the standard degree-two arithmetization of indexed clause i , including its two source-specified signs. In addition to validity, the machine estimates

$$\mu = \mathbb{E}_{i,u} F_i(u)$$

by expanding it into AND-four supplier calls, and accepts a branch only if

$$\tilde{\mu} \geq \theta_{\text{acc}} := \frac{c_p + s_p}{2}. \quad (\text{C.10.1})$$

Choose every propagated estimation error below

$$\varepsilon_{\text{est}} := \min \left\{ \zeta, \frac{c_p - s_p}{20} \right\}. \quad (\text{C.10.2})$$

The ζ term is needed by the validity thresholds; the gap term is needed by the acceptance midpoint. Since both are fixed positive constants after the PCPP is frozen, the supplier's inverse-polynomial guarantee is eventually smaller than ε_{est} .

Lemma C.7 (Componentwise validity). *Suppose all degree-four estimates have error smaller than a sufficiently small fixed fraction of $c_p - s_p$.*

1. *If the T_{ij} are $[0, 1]$ -valued and are within average ℓ_1 distance ζ of a Boolean proof whose systematic coordinates equal $\text{Enc}(u)$, the real proof passes validity.*
2. *If a real proof passes validity, rounding supplies a Boolean proof $\hat{T}$ with*

$$\mathbb{E}_{i,j} \|T_{ij} - \hat{T}_{ij}\|_2 \leq \sqrt{12\zeta},$$

and every arithmetized two-variable clause obeys

$$\|F_i - \hat{F}_i\|_1 \leq 6 \mathbb{E}_{j \in [2]} \|T_{ij} - \hat{T}_{ij}\|_2.$$

Proof. For completeness, fix the nearby Boolean proof $\hat{T}$. If $T \in [0, 1]$ and $\hat{T} \in \{0, 1\}$, then

$$T^2(1 - T)^2 \leq |T - \hat{T}|$$

for an auxiliary coordinate, while for a systematic coordinate

$$(\text{Enc}_s - T)^2 \leq |\text{Enc}_s - T|.$$

Thus the true average of P_{ij} is at most ζ , every true second moment is at most one, and estimation error at most ζ gives the two passing thresholds.

For soundness, set every systematic rounded value to Enc_s and round an auxiliary value at $1/2$. Pointwise,

$$(T_{ij} - \hat{T}_{ij})^2 \leq 4P_{ij}. \quad (\text{C.10.3})$$

Passing and estimation error at most ζ imply that the true average of P_{ij} is at most 3ζ . Jensen's inequality applied to (C.10.3) gives

$$\mathbb{E}_{i,j} \|T_{ij} - \hat{T}_{ij}\|_2 \leq \sqrt{12\zeta}.$$

The second-moment tests and (C.10.2) give $\mathbb{E}T_{ij}^2 \leq 1 + 2\zeta < 4$, after choosing $\zeta < 3/2$, and hence $\|T_{ij}\|_2 < 2$. Therefore

$$\|T_{i1}T_{i2} - \hat{T}_{i1}\hat{T}_{i2}\|_1 \leq 2 \sum_{j=1}^2 \|T_{ij} - \hat{T}_{ij}\|_2.$$

Every arithmetized two-literal OR has constant, linear, and bilinear coefficients of magnitude at most one. Adding its two linear differences to the product bound yields

$$\|F_i - \hat{F}_i\|_1 \leq 3 \sum_{j=1}^2 \|T_{ij} - \hat{T}_{ij}\|_2 = 6 \mathbb{E}_{j \in [2]} \|T_{ij} - \hat{T}_{ij}\|_2.$$

All preceding inequalities are pointwise real algebra, rounding, and Cauchy–Schwarz. The fourfold expansion in [Appendix C.2](#) computes every expectation using the individual carried atom cap B , term count J , and augmented coefficient mass $\bar{\Lambda}$ separately. Inequality (C.1.2) places every carried native-arity atom inside the supported cap A , and the explicit $O(q^2)$ parity atom is independently below that cap. The expansion never composes a circuit above an atom or adds atom wires. $\square$

The code is frozen in the following order. First fix the source verifier, exact-width padding, PCPP transformation, class presentations, advantage target, and the simulation exponent of the two target classes. Next choose the general-oracle exponent G , apply [Fact C.4](#) to freeze its amplifier constants, choose the XOR power, and freeze the resulting term and coefficient envelopes and the finite set of class tags. Then choose D, σ , the two supported carrier coefficients, and the smaller carried coefficients from (C.1.1), and freeze the deterministic supplier programs, parsers, and finite onsets. Only after all of these choices is the one union machine presented to the refuter. None depends on the current source length or on a guessed circuit.

On input length N , that fixed machine guesses one size- q^G general-oracle circuit and one rational-sum description for every distinct PCPP proof variable in one selected class mode. It deterministically rejects a branch if any description is malformed or noncanonical, has the wrong arity, exceeds its declared term, coefficient-bit, coefficient-mass, or per-atom wire envelope, or fails the fixed general-circuit size bound. It does **not** try to decide the semantic condition $0 \leq P(x) \leq 1$ for every x : all syntactically legal real sums are permitted, and one-sidedness below uses Booleanity and second moments for arbitrary real values. The $[0, 1]$ condition is needed only for the particular contradiction witness produced by [Lemma C.9](#). Every rational coefficient has the already fixed polynomial bit bound. Systematic parities are source-explicit and are compiled deterministically into the native parity circuits of [Section 2](#), not guessed as advice. The total description length is $q^{O_{G,c}(1)} = o(N)$ because $N = 2^{\Theta(q)}$, so it is below $N/10$ eventually.

All ordinary, systematic, second-moment, and clause branches together cost

$$q^{p_{\text{call}}} \frac{2^q}{q^\sigma}.$$

The carried-versus-supported comparison (C.1.2) makes every addressed restriction legal, and the parity atoms have only $O(q^2)$ wires. With $\sigma > p_{\text{call}} + d_{\text{pad}} + 10$, the displayed cost is at most $T(N)q^{-10}$. [Lemma C.6](#) and the initial choice $C_{\text{clk}} > d_{\text{ver}} + 2$ put outer construction, projection-table recovery, duplicate-clause deletion, substitution, PCPP construction, and all syntactic validation in $o(T(N))$. Thus no $\text{poly}(N)$ source-preprocessing factor is silently called polynomial in q .

On a hierarchy NO word, outer soundness makes the substituted circuit H_{sub} accept on at most an N^{-10} fraction of u . For every other u , pointwise PCPP soundness bounds every rounded Boolean proof by s_p . Hence its mean clause value is at most $s_p + N^{-10}$. If validity passes, [Lemma C.7](#) changes this mean by at most $6\sqrt{12\zeta}$, and estimation changes it by at most $(c_p - s_p)/20$. By the choice of ζ , the result is strictly below θ_{acc} for every sufficiently large N . Thus every branch in every class mode rejects, so the union machine is one-sided.

[Fact C.3](#) now supplies a conflicting word at every sufficiently large source length. One-sidedness forces that word into the hierarchy YES side while the entire finite-mode weak machine rejects.

If no size- q^G general-oracle circuit makes the exact-width outer verifier a tautology, the construction enters Case 1 and uses the general-circuit hard oracle. Otherwise, before selecting a class mode, choose the lexicographically first such circuit. Apply the deterministic honest-assignment algorithm in [Fact C.2](#) to its substituted circuit. Let $\hat{T}_{ij}(u)$ be the named systematic encoding bit or auxiliary honest bit at occurrence (i, j) . Define the total Boolean proof-table function

$$f(i, j, u) = \hat{T}_{ij}(u)$$

on $q_0 = q + r + 1 = q + O_G(\log q)$ bits; malformed indices return zero.

Lemma C.8 (Common approximate-sum hardness). *Fix before constructing the weak machine one finite list of polynomial term and coefficient envelopes $J_{0,i}(q), \Lambda_{0,i}(q)$, one for each class mode. The resulting function f has ℓ_1 distance greater than ζ from every*

$$D \in [0, 1] \text{Lin}_{q_0}(B_i(q_0), J_{0,i}, \Lambda_{0,i})$$

in mode i .

Proof. Suppose $\|D - f\|_1 \leq \zeta$. For each PCPP proof variable X with at least one occurrence, average $D(i, j, \cdot)$ over all occurrences mapped to X ; set unused variables to zero. Fixing (i, j) is a literal restriction of every atom, so its class and absolute wires are preserved. Since $q_0 = q + O_G(\log q)$, (C.1.2) places those restricted atoms inside the supplier's supported cap at arity q . Occurrence averaging multiplies term count by at most $2M$, preserves coefficient mass, and adds only $O(\log M)$ coefficient bits. The envelopes frozen before the machine include precisely this polynomial blowup.

For values $v_1, \dots, v_d \in [0, 1]$ and one Boolean bit b ,

$$\left| \frac{1}{d} \sum_a v_a - b \right| = \frac{1}{d} \sum_a |v_a - b|.$$

Averaging this identity with the uniform clause-position occurrence weights and then over u gives

$$\mathbb{E}_{i,j} \|T_{ij} - \hat{T}_{ij}\|_1 \leq \zeta,$$

which is the hypothesis of validity completeness. Because the proof is pointwise $[0, 1]$ -valued,

$$\mathbb{E}_{i,j} \|T_{ij} - \hat{T}_{ij}\|_2 \leq \sqrt{\zeta}$$

by Cauchy–Schwarz and Jensen. The canonical Boolean proof has mean clause value at least c_p , so [Lemma C.7](#) puts the real proof's mean at least $c_p - 6\sqrt{\zeta}$. The fixed estimation error reduces this by at most $(c_p - s_p)/20$, which remains strictly above θ_{acc} by the original choice of ζ . This creates an accepting branch in the selected mode, contradicting the conflict. Since the oracle and honest proof family were chosen before the mode, the same f is hard for every listed class. $\square$

C.11 Precise XOR contrapositive

Lemma C.9 (Componentwise XOR contrapositive). *Let f be ζ -far from every legal componentwise sum in [Lemma C.8](#), and put*

$$\varepsilon_k = (1 - \zeta)^{k-1} (1/2 - \zeta).$$

If a class circuit C on kq_0 inputs has advantage at least ε_k against $f^{\oplus k}$, then fixing $k - 1$ blocks in

$$J_{\text{xor}} = O_\zeta(q_0/\varepsilon_k^2)$$

sampled ways produces a $[0, 1]$ rational sum approximating f within ζ . Apart from one constant atom, every atom is a restriction or negated restriction of C , and the coefficient mass is $O_\zeta(1/\varepsilon_k)$.

This is [11, Lemma 3.8] in the only direction used here; CLW's argument adapts Levin's proof of the XOR lemma [40]. Its proof gives the displayed ε_k -based sample and coefficient bounds; these are separate componentwise envelopes here. The atoms may be restrictions or negated restrictions of C , and both operations preserve the two target classes and their wires. We do not replace ε_k by a different advantage parameter inside the source bound.

Before choosing k , fix the target-class simulation exponent G_{sim} . In the inverse-polynomial tier choose the general-oracle cutoff G so that

$$\frac{G}{c_{\text{STV}}} > \max\{G_{\text{sim}}, c + 2\}.$$

In the fixed-advantage tier it is enough to choose $G/c_{\text{STV}} > G_{\text{sim}} + 1$, since the resulting inverse-polynomial advantage is eventually below every fixed $\gamma > 0$. [Fact C.4](#) then gives a Case-1 output on at most $c_{\text{amp}}q$ variables, hard against size $q^{G/c_{\text{STV}}}$ at advantage $q^{-G/c_{\text{STV}}}$. For fixed advantage, choose the constant k large enough both that ε_k is below the requested advantage and that $kq_0 \geq c_{\text{amp}}q$ eventually. For the final theorem exponent c , take

$$k = A_{c+1} \lceil \log_2(q + 2) \rceil.$$

Choose the fixed A_{c+1} large enough that $kq_0 \geq c_{\text{amp}}q$ from the common onset and

$$\varepsilon_k \leq q^{-(c+2)}. \tag{C.11.1}$$

Since $m = kq_0 = \Theta_c(q \log q)$, (C.11.1) implies, eventually,

$$\varepsilon_k \leq q^{-(c+2)} \leq m^{-(c+1)}. \tag{C.11.2}$$

Substituting the actual $\varepsilon_k = q^{-O_c(1)}$ into [Lemma C.9](#) leaves J_{xor} and the coefficient mass fixed polynomials in q . These, rather than bounds obtained by silently replacing ε_k with $m^{-(c+1)}$, are the finite envelopes frozen in [Lemma C.8](#). No XOR of circuit outputs is submitted to CAPP.

C.12 Exact target schedule and canonical E^{NP} recovery

The source constructions determine the exact padded verifier width $q(N)$. With the fixed envelope exponent d_G from [Appendix C.10](#), use the branch-independent padded clause width

$$r(N) = \lceil d_G \log_2(q(N) + 2) \rceil,$$

which is exactly $\log_2 M$ for $M = 2^{\lceil d_G \log_2(q+2) \rceil}$. Every branch-specific power-of-two list $M_0 \leq M$ is repeated as in [Appendix C.10](#). Hence the exact base arity is

$$q_0(N) = q(N) + r(N) + 1.$$

For a target length n , enumerate integers $s = 1, \dots, n$ and let $N_s = 2^s$. In the inverse-polynomial tier define

$$m_s = k(q(N_s))q_0(N_s),$$

where k is the exponent- $(c + 1)$ integer function in [Lemma C.9](#), and select the largest s with $m_s \leq n/2$. In the fixed-advantage tier use its constant k in the same definition. The functions

q , r , q_0 , k , and m_s are nondecreasing. Here $q(N_s) = \Theta(s)$ and $q(N_{s+1}) - q(N_s) = O(1)$. In the inverse-polynomial tier, the occasional unit jump of $\lceil \log_2(q+2) \rceil$ makes

$$m_{s+1} - m_s = O_c(s),$$

while in the fixed-advantage tier the difference is $O_\gamma(1)$. Since $m_s = \Theta_c(s \log s)$ in the inverse-polynomial tier and $m_s = \Theta_\gamma(s)$ in the fixed tier, the selected boundary has respectively $s = \Theta_c(n/\log n)$ and $s = \Theta_\gamma(n)$. The former jump is therefore $O_c(n/\log n) = o(n)$, and the latter is $O_\gamma(1)$. Maximality now gives $m_s = n/2 - o(n)$ and $m_s = n/2 - O_\gamma(1)$, respectively. In particular, for every sufficiently large n , both selections satisfy

$$\frac{n}{3} \leq m_s \leq \frac{n}{2},$$

and respectively $q = \Theta_c(n/\log n)$ or $q = \Theta_\gamma(n)$. The finite search is deterministic, uses only the fixed source constructions, and never asks whether a function is hard.

Pad the selected hard core by ignoring the final $n - m_s$ variables. If an alleged correlator exists on n variables, averaging over the ignored suffix gives one literal restriction with at least the same agreement and no additional wires. In the inverse-polynomial tier the core was made hard below $1/2 + m_s^{-(c+1)}$, and

$$m_s^{-(c+1)} \leq 3^{c+1} n^{-(c+1)} < n^{-c}$$

eventually. Thus target agreement $1/2 + n^{-c}$ really contradicts core hardness; padding does not reverse the advantage inequality. Likewise, because $n/m_s \leq 3$ and $L(n)/L(m_s) = 1 + o(1)$, a fixed shrinkage of the final carrier coefficient makes the target-length wire cap legal at core arity.

The same m_s is a genuine envelope for both sides of the dichotomy. In Case 2 it is exactly the XOR arity kq_0 . In Case 1 the output of [Fact C.4](#) has at most $c_{\text{amp}}q \leq kq_0 = m_s$ variables by the choice of k in [Lemma C.9](#); pad that shorter amplifier output with ignored variables up to m_s . Thus the schedule never assumes that the hidden $O(q)$ constant in the Case-1 amplifier is one.

Lemma C.10 (Canonical recovery). *The hard slice selected by the Case-1/Case-2 dichotomy is a total function computable in $2^{O(n)}$ time with adaptive SAT access.*

Proof. Run the fixed refuter at the selected source length and construct the exact-width outer verifier. First ask whether one valid canonical size- q^G circuit makes every one of its 2^q verifier rows accept. The SAT formula shares one description-variable block across all rows, enforces the fixed syntax and size bound, and uses complete Tseitin equivalences for every evaluation trace. Satisfiability therefore means that one circuit works universally, not that each row chooses a different circuit.

If this formula is unsatisfiable, enter Case 1. Introduce one variable for every outer-oracle address, conjoin all verifier rows, and use prefix self-reduction to recover the lexicographically first accepted truth table. Perfect completeness guarantees existence. There are at most 2^q prefix queries, each to a formula of length $2^{O(q)}$, after which the fixed worst-case-to-average-case amplifier is evaluated.

If the circuit formula is satisfiable, enter Case 2. Polynomially many prefix queries recover the lexicographically first accepted circuit description. Given that circuit and a point u , the deterministic algorithm in [Fact C.2](#) computes the canonical auxiliary assignment Z_u in $q^{O_G(1)}$ time; systematic coordinates are explicit encoding parities. Thus a requested value of $f(i, j, u)$ is computed directly, and malformed headers or addresses return zero. No SAT recovery of an exponentially long PCPP proof is needed.

In the inverse-polynomial Case-1 branch the core has only $O(q) = O(n/\log n)$ variables. Normalize an alleged real-weight target circuit before fixing its ignored suffix. Each bottom threshold and the threshold top have polynomial-bit integer descriptions; a symmetric top has an explicit lookup on the $M + 1$ possible counts. Standard binary adders and comparators therefore give a fan-in-two simulation polynomial in q , M , and the description length. Since $M \leq W \leq n^3 = q^{O(1)}$, the exponent G_{sim} frozen before the weak machine bounds this simulation. The earlier choice $G/c_{\text{STV}} > \max\{G_{\text{sim}}, c + 2\}$ and [Fact C.4](#) make the Case-1 function hard against every such simulation at advantage at most $q^{-(c+2)} \leq m^{-(c+1)}$, matching (C.11.2). No parameter is chosen inside the recovery procedure.

The refuter and all canonical searches cost

$$\text{poly}(T(N_s)) + 2^{O(q(N_s))} \leq 2^{O(n)},$$

and truth-table evaluation of the final padded function also costs $2^{O(n)}$. Thus the resulting total language lies in E^{NP} . $\square$

The fixed weak union machine is frozen only after the requested advantage exponent and the two positive carrier coefficients are fixed. The resulting language and coefficients are therefore joint existential witnesses. What is common is the language across the two class modes, not a constructional independence of the language from its cap constants.

AI Disclosure

Generative-AI models provided by OpenAI, Anthropic, and Google were used substantively for exploratory proof search, source navigation, drafting and revision, and adversarial proof review. These uses materially affected the proof development and exposition. AI outputs were treated as proposed mathematics and review material, not as authority. AI systems are not authors; Dev Nag assumes responsibility for the theorem, proof, citations, and originality.

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