

A short proof of a quadratic lower bound on the border determinantal complexity of $x_1^n + \cdots + x_n^n$ over $\mathbb{C}$

A simple human exposition

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Abstract

I give a short proof that the border determinantal complexity of $F_n = x_1^n + \cdots + x_n^n$ over $\mathbb{C}$ is at least $(n-1)^2/(4e)$, and at least $(n-1)^2/(2e)$ in the symmetric model. The same argument shows that the border determinantal complexity of $x_1^d + \cdots + x_N^d$ is $\Theta(Nd)$ for all $N \geq 3$ and $d \geq 2$.

The earlier article arXiv:2606.13628, by the same author, introduced the first superlinear lower bounds and includes verification by Lean formalization. However, the AI written exposition was hard to follow. Kumar and Volk (arXiv:2609.34462) give a short proof of the exact ordinary case, but do not address the border or symmetric cases.

In this article, I provide a greatly simplified exposition of the border and symmetric cases, using the implicit function theorem and an explicit section of simple solutions in place of the conormal specialization arguments of arXiv:2606.13628. I choose an explicit polynomial system with $n(n-1)^{n-2}$ simple solutions for F_n . I then use the implicit function theorem to ensure that each point persists under small perturbations. Finally, kernel equations and the multihomogeneous Bézout theorem bound the number of these points for a small determinant.

The exposition is due to the human author. The mathematical simplification was obtained in discussion with ChatGPT Astra and refined in conversation with Claude Opus 5.5.

1 Introduction

Consider the polynomial

$$F_n(x_1, \dots, x_n) = x_1^n + \cdots + x_n^n.$$

which is easy to specify and evaluate. I study the question of whether it can be the determinant of a small matrix whose entries have degree at most one (affine linear entries).

The determinantal representation of a polynomial f has the form

$$f(x) = \det A(x), \quad A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n, \quad (1)$$

where the A_i are constant $m \times m$ matrices over $\mathbb{C}$. The least possible m is the *determinantal complexity* of f , denoted by $\text{dc}(f)$. Constant terms in the entries are allowed.

An elementary lower bound is immediately apparent; a determinant of size m has degree at most m , so $\text{dc}(F_n) \geq n$. There are also representations of size $O(n^2)$. For example, one can build a small

block for each power x_i^n and then combine the blocks to add their contributions. For reference, I show such a construction in Section 8.

The challenge involved in proving a lower bound is to show that a more economical representation cannot exist. This is challenging for various reasons: a representation need not assign a separate block to each variable. Every entry may involve every variable, and the determinant expansion may contain extensive cancellations. The fact that one construction uses roughly n entries for each of n powers is therefore not a lower-bound argument. I need a property of the polynomial that every representation must respect.

Repeated squaring would compute each x_i^n using $O(\log n)$ multiplications. Thus F_n has a small arithmetic circuit. The question here is how much space is needed when all of that computation must fit into one affine-linear determinant.

Determinantal complexity is a central model in algebraic complexity. Valiant's work [9] placed it in a general theory of algebraic computation, and the permanent versus determinant problem has been a main source of lower-bound questions. For an $n \times n$ permanent, Mignon and Ressayre [5] proved a lower bound of order n^2 . That polynomial has n^2 variables. For the power sum above, both the degree and the number of variables are n , so a quadratic lower bound goes beyond both of these basic parameters. The earlier bound of Kumar and Volk [3] for this family was linear in n .

I also allow approximation. Suppose that a sequence of polynomials f_j , each having a size- m determinantal representation, converges coefficient by coefficient to f . The least such m is the *border determinantal complexity*, written $\overline{\text{dc}}(f)$.

This agrees with the Zariski-closure definition because the size- m determinants form a constructible set, whose Euclidean and Zariski closures over $\mathbb{C}$ coincide [2, Theorems 4.12 and 4.15].

Note that approximation can only make the problem easier, so

$$\overline{\text{dc}}(f) \leq \text{dc}(f).$$

The matrices representing the f_j need not converge; it's possible that their entries become arbitrarily large while their determinants still converge.

If all the coefficient matrices in (1) are symmetric, I write $\text{sdc}(f)$ and $\overline{\text{sdc}}(f)$ for the corresponding symmetric quantities. Symmetric here means transpose-symmetric, not Hermitian.

Theorem 1.1. *For every integer $n \geq 3$, over $\mathbb{C}$,*

$$\text{dc}(F_n) \geq \overline{\text{dc}}(F_n) \geq \frac{(n-1)^2}{4e}, \quad \text{sdc}(F_n) \geq \overline{\text{sdc}}(F_n) \geq \frac{(n-1)^2}{2e}.$$

In particular, all four complexities have order n^2 .

This article gives a direct exposition of the lower bounds in the longer paper [7]. Kumar and Volk [4] give a short proof of the quadratic lower bound in the exact ordinary model.

The main tools I rely on are linear algebra, the implicit function theorem, and a form of Bézout's theorem that keeps track of different groups of variables. I first find the points to count, then show why every nearby polynomial must retain at least that many.

The count in my original proof [7] came from the degree of the dual variety of the Fermat hypersurface. Kumar and Volk's slice [4] made the points explicit. Here I choose the slice so that every point is simple; this is what lets the points persist under perturbation and carries the argument to the border.

2 Overview of the proof

To prove a lower bound, I would like to count something that is large for F_n but limited for a small determinant. The polynomial itself does not suggest such a count: it has only n monomials. Its derivatives are more useful:

$$\frac{\partial F_n}{\partial x_i} = nx_i^{n-1}.$$

An equation such as $x_i^{n-1} = 1$ has $n - 1$ complex solutions. If I can make many coordinates satisfy independent equations of this kind, I obtain roughly n^n points. Even after taking an n th root, that is a substantial count.

This idea is well-known in degree-based circuit lower bounds; see, for example, Baur and Strassen [1]. In that approach, one expresses a computation by polynomial equations, introduces a variable for each intermediate value, and then uses the small degrees of those equations to bound the number of solutions. Note here that a small computation cannot produce arbitrarily many isolated solutions. To apply this to a determinant, I need a description that uses its particular structure.

2.1 Derivatives on the zero set

Differentiating $f = \det A$ brings in the cofactors of A , which can still have degree $m - 1$. Away from $f = 0$, they are encoded by $\text{adj } A = fA^{-1}$, which is a full $m \times m$ matrix. The hope is to find points where that matrix collapses to something smaller.

However, at a smooth zero a of f , the situation is better. The matrix $A(a)$ has rank exactly $m - 1$. Its left kernel and right kernel are therefore lines. Choose nonzero vectors u and v on those lines, so

$$u^\top A(a) = 0, \quad A(a)v = 0.$$

Observe that the adjugate of $A(a)$ has rank one and factors as $\lambda vu^\top$ for some nonzero scalar λ . Jacobi's formula then gives

$$\partial_i f(a) = \lambda u^\top A_i v \quad (1 \leq i \leq n). \quad (2)$$

For a proof of these facts, refer to Section 4.

The cofactors are now controlled by two vectors of length m . Smoothness matters too; if the rank dropped further, all the cofactors would vanish and the derivative relations would say nothing about the point.

A derivative of f can have degree $m - 1$ in x . The expression on the right of (2) is bilinear in two vectors of length m . The scalar λ is unknown, but it disappears from a homogeneous linear relation among the derivatives. For example,

$$\partial_i f(a) - c \partial_j f(a) = 0 \iff u^\top (A_i - cA_j)v = 0.$$

The points to count should therefore lie on $f = 0$, be smooth points of that hypersurface, and satisfy linear relations among the derivatives.

2.2 The number of conditions

The equation $f = 0$ leaves $n - 1$ dimensions. To get isolated points, I need $n - 1$ more conditions. For a homogeneous polynomial, derivative relations alone also respect scaling. Hence, if a nonzero

point satisfies them, then every nonzero scalar multiple does as well. I remove that freedom by fixing one coordinate, say $x_{n-1} = 1$.

That leaves room for $n - 2$ derivative relations. For the power sum, a natural choice is to relate the first $n - 2$ derivatives to the $(n - 1)$ st, because $\partial_i F_n = n x_i^{n-1}$ involves only x_i . Once $x_{n-1} = 1$, each relation reduces to $x_i^{n-1} = c$ for a constant c .

Each of the first $n - 2$ coordinates then has $n - 1$ choices. The last coordinate is determined by $F_n = 0$, which with $x_{n-1} = 1$ reads

$$x_n^n = -\left(1 + \sum_{i=1}^{n-2} x_i^n\right).$$

This predicts the count

$$n(n-1)^{n-2}.$$

The last equation might have a repeated root. An equation $t^n = R$ has n distinct solutions when $R \neq 0$, but when $R = 0$ its only solution is $t = 0$, repeated n times. Nothing so far prevents the right side above from vanishing. With the plain relation $\partial_i F_n = \partial_{n-1} F_n$, each x_i is an $(n - 1)$ st root of unity, so $x_i^n = x_i$, and the right side becomes -1 minus a sum of roots of unity, which can be zero. The smallest case is $n = 3$: taking $x_1 = -1$ gives $x_3^3 = 0$ (Example 3.2).

For the exact lower bound, one can work with a weaker count of distinct points, as in [4]. For the border bound, it is useful to make every selected solution simple from the start. I do this by choosing the first $n - 2$ coordinates to have absolute value $1/2$. Then

$$\left|\sum_{i=1}^{n-2} x_i^n\right| \leq \frac{n-2}{2^n} < 1,$$

so the sum is too small to cancel the 1, the right side is never zero, and all n values of x_n are distinct.

2.3 The quadratic count

The kernel equations involve three groups of variables:

$$x \in \mathbb{C}^n, \quad u \in \mathbb{C}^m, \quad v \in \mathbb{C}^m.$$

Each equation is linear in each group it uses. For example, $u^\top A(x) = 0$ is linear in x and linear in u ; a derivative relation becomes an equation involving u and v but no x .

If I ignore these groups and remember only that the equations are quadratic, ordinary Bézout gives an upper bound exponential in m . Comparing it with roughly n^n points yields a lower bound of order $n \log n$.

The grouped Bézout count assigns each equation to one of the variable groups it uses. Each group must receive as many equations as it has variables. Once $x_{n-1} = 1$ is fixed, there are only $n - 1$ remaining x variables. So exactly $n - 1$ of the kernel equations must be assigned to x ; the others must account for u and v . There are about $2m$ kernel equations to choose from. This gives a binomial coefficient, with the estimate

$$\binom{2m}{n-1} \leq \left(\frac{2em}{n-1}\right)^{n-1}.$$

The remaining choices, from the $n - 2$ derivative relations, contribute at most 2^{n-2} . Section 5 makes this count precise. Together these give an upper bound of the form

$$\left(\frac{Cm}{n}\right)^{n-1}.$$

The power sum supplies at least $(n - 1)^{n-1}$ points. Taking $(n - 1)$ st roots leaves a quantity of order n on that side, and of order m/n on the determinant side. Thus m/n must be at least a constant times n , which in turn forces m to be at least a constant times n^2 .

3 The power-sum system

Fix $n \geq 3$ and put

$$\alpha = 2^{-(n-1)}.$$

For any polynomial f in n variables, define

$$P_i(f) = \partial_i f - \alpha \partial_{n-1} f, \quad 1 \leq i \leq n - 2.$$

I study the square system of n equations

$$f(x) = 0, \quad x_{n-1} = 1, \quad P_i(f)(x) = 0 \quad (1 \leq i \leq n - 2). \quad (3)$$

A solution is called *simple* if the Jacobian matrix of these n equations with respect to the n variables is invertible there. This means that the equations meet independently at the point. In particular, the solution is isolated.

Lemma 3.1. *For $f = F_n$, the system (3) has exactly $N = n(n - 1)^{n-2}$ solutions. Every solution is simple, and every solution is a smooth zero of F_n .*

Proof. On the hyperplane $x_{n-1} = 1$, the derivative equations become

$$x_i^{n-1} = 2^{-(n-1)} \quad (1 \leq i \leq n - 2).$$

Therefore $x_i = \zeta_i/2$, where ζ_i is an $(n - 1)$ st root of unity. There are $(n - 1)^{n-2}$ independent choices for these coordinates. For each choice, the remaining equation is

$$x_n^n = - \left(1 + \sum_{i=1}^{n-2} x_i^n \right). \quad (4)$$

Its right side is nonzero, since

$$\left| \sum_{i=1}^{n-2} x_i^n \right| \leq \frac{n-2}{2^n} < 1.$$

Consequently (4) has exactly n distinct solutions, all nonzero. This proves the required count.

To check simplicity, first use $x_{n-1} = 1$ to eliminate that coordinate. The remaining variables are $x_1, \dots, x_{n-2}, x_n$. The derivative equation indexed by i has derivative

$$n(n - 1)x_i^{n-2}$$

in its own variable and zero in the other remaining variables. These numbers are nonzero. The derivative of F_n with respect to the final variable is nx_n^{n-1} , which is also nonzero. The resulting Jacobian is triangular after ordering the equations appropriately, with nonzero diagonal. The full Jacobian, including $x_{n-1} - 1$, is therefore invertible.

Finally, $\partial_{n-1} F_n = n$ at every solution, so each point is a smooth zero of F_n . $\square$

The number $N = n(n-1)^{n-2}$ is also the degree of the dual of the Fermat hypersurface $\{F_n = 0\} \subset \mathbb{P}^{n-1}$, the variety that records its tangent hyperplanes. The original proof [7] found the count through this geometric interpretation; the explicit system above realizes it with simple solutions.

Example 3.2. For $n = 3$, write $F_n = x^3 + y^3 + z^3$. The system is

$$x^3 + y^3 + z^3 = 0, \quad y = 1, \quad 3x^2 - \frac{1}{4}(3y^2) = 0.$$

Thus $x = 1/2$ or $x = -1/2$. In the first case $z^3 = -9/8$; in the second $z^3 = -7/8$. Each case gives three distinct values of z , for a total of six simple solutions.

If I had instead required $\partial_x F_n = \partial_y F_n$, I would obtain $x = \pm 1$. At $x = -1$, the final equation would be $z^3 = 0$. Three roots would coincide.

4 Kernel equations

Define f as the determinant of an $m \times m$ affine-linear matrix A , as in (1). If I were to write system (3) directly for f , I would have to use f itself, of degree m , and its derivatives, of degree $m-1$. That runs into a wall, since the Bézout bound for equations of that degree is too large for my purpose. Therefore, consider the left and right kernel vectors u and v of $A(x)$. If I add them as extra variables, I replace system (3) with equations that have a degree of at most 2, since each equation is linear in each group of variables it involves: $u^\top A(x) = 0$ is linear in x and in u , $A(x)v = 0$ is linear in x and in v , and the derivative relations become $u^\top (A_i - \alpha A_{n-1})v = 0$, which is linear in u and in v .

4.1 Kernels and the adjugate

Lemma 4.1. *Suppose $\text{rank } A(a) = m-1$. For any nonzero left and right kernel vectors u, v at a , there is a scalar $\lambda \neq 0$ such that*

$$\text{adj } A(a) = \lambda v u^\top, \quad \partial_i f(a) = \lambda u^\top A_i v \quad (1 \leq i \leq n).$$

In particular, these identities apply whenever $f(a) = 0$ and $\nabla f(a) \neq 0$, since those conditions force $\text{rank } A(a) = m-1$.

Proof. At a point of rank $m-1$, the identities

$$A(a) \text{adj } A(a) = \text{adj } A(a) A(a) = 0$$

show that every column of the adjugate lies in the right kernel and every row lies in the left kernel. Both kernels are one-dimensional, so the adjugate is $\lambda v u^\top$. Some $(m-1) \times (m-1)$ minor is nonzero, so $\lambda \neq 0$. Differentiation of a determinant gives

$$\partial_i f(x) = \text{tr}(\text{adj } A(x) A_i). \quad (5)$$

Substituting the factorization of the adjugate gives the claimed derivative identities.

For completeness, formula (5) follows by differentiating the determinant with respect to each matrix entry. The derivative with respect to entry (j, k) is its cofactor, and the derivative of that

entry with respect to x_i is $(A_i)_{jk}$. Summing the products of these two quantities is exactly the trace in (5).

Finally, suppose $f(a) = 0$ and $\nabla f(a) \neq 0$. The rank is at most $m - 1$. If it were at most $m - 2$, all $(m - 1) \times (m - 1)$ minors would vanish, so $\text{adj } A(a) = 0$. Formula (5) would then force every derivative of f to vanish at a . Hence the rank is exactly $m - 1$. $\square$

4.2 Normalizing the kernel vectors

Define S as a finite set of isolated solutions of (3), with $\nabla f(a) \neq 0$ at each point $a \in S$. At each such point, the left and right kernels of $A(a)$ are lines, so u and v are determined only up to a nonzero multiple. To count with Bézout, each point must give a single isolated solution, so I fix the scale of u and v .

Choose vectors $q, p \in \mathbb{C}^m$ such that

$$u^\top q \neq 0, \quad p^\top v \neq 0$$

on every left and right kernel line belonging to a point of S . Such choices exist: each line excludes just one hyperplane of possible choices, and finitely many proper hyperplanes cannot cover $\mathbb{C}^m$. I then normalize the kernel vectors by

$$u^\top q = 1, \quad p^\top v = 1. \tag{6}$$

Each point of S now has exactly one normalized pair (u, v) .

4.3 A redundant equation

The full left and right kernel conditions have a total of $2m$ equations, but contain a redundancy. The scalar $u^\top A(x)v$ can be obtained from either block. In order to apply the square-system Bézout bound, I must remove the redundancy carefully.

Choose a linear map

$$\Lambda : \mathbb{C}^m \longrightarrow \mathbb{C}^{m-1}, \quad \ker \Lambda = \mathbb{C}q.$$

Replace the m equations $A(x)v = 0$ by the $m - 1$ equations $\Lambda A(x)v = 0$. Under the other equations, nothing is lost. The reduced equations imply $A(x)v = cq$ for some scalar c . Multiplying by $u^\top$ gives

$$0 = u^\top A(x)v = c u^\top q = c.$$

Thus $A(x)v = 0$ after all.

4.4 The incidence system

Write $B_i = A_i - \alpha A_{n-1}$ for $1 \leq i \leq n - 2$. Consider the following equations in (x, u, v) :

$$\begin{aligned} u^\top A(x) &= 0, \\ \Lambda A(x)v &= 0, \\ u^\top B_i v &= 0 & (1 \leq i \leq n - 2), \\ x_{n-1} &= 1, \\ u^\top q &= 1, \quad p^\top v = 1. \end{aligned} \tag{7}$$

There are $n + 2m$ variables and

$$m + (m - 1) + (n - 2) + 3 = n + 2m$$

scalar equations. Every point of S gives a solution: the kernel equations and normalizations hold by construction, and Lemma 4.1 converts each derivative relation into the corresponding equation involving B_i .

The system may have other solutions. Some may lie on curves or larger sets, especially where the rank of $A(x)$ drops further. I single out solutions from S , and only count these in the larger system.

Lemma 4.2. *Every point $a \in S$ gives a distinct isolated solution of (7).*

Proof. Let (a, u_a, v_a) be the normalized solution above a . Some $(m - 1) \times (m - 1)$ minor of $A(a)$ is nonzero. It remains nonzero in a neighborhood of a , so throughout that neighborhood the rank is at least $m - 1$.

At any solution of (7), the normalization makes u nonzero. The left kernel equation therefore implies $\det A(x) = 0$. For solutions near (a, u_a, v_a) , the rank is consequently exactly $m - 1$. The reduced right equations imply the full right kernel condition, as shown above. Lemma 4.1 now applies, so

$$P_i(f)(x) = \lambda u^\top B_i v = 0.$$

Together with $f(x) = 0$ and $x_{n-1} = 1$, these are precisely the equations of (3).

Since a is an isolated solution of that system, the only possible x sufficiently near a is a itself. The normalized kernel vectors at a are unique. Hence the only solution sufficiently near (a, u_a, v_a) is that triple. Different points of S have different first coordinates, so their lifts are distinct. $\square$

5 The multihomogeneous count

5.1 The multihomogeneous Bézout theorem

Suppose the variables are divided into groups of sizes $r_1, \dots, r_k$, with $r = r_1 + \dots + r_k$ in total. For each of r equations, record its degree in each group separately. If those degrees are at most $(d_{j1}, \dots, d_{jk})$ for equation j , associate to it the formal linear expression

$$d_{j1}Y_1 + \dots + d_{jk}Y_k.$$

The symbols Y_i are only used to calculate the bound.

Theorem 5.1 (Multihomogeneous Bézout bound). *The number of isolated complex solutions of the system is at most*

$$[Y_1^{r_1} \dots Y_k^{r_k}] \prod_{j=1}^r (d_{j1}Y_1 + \dots + d_{jk}Y_k). \quad (8)$$

The bound remains valid when the system also has positive-dimensional sets of solutions.

This is a standard form of the multihomogeneous Bézout theorem; see Morgan and Sommese [6] and Sommese and Wampler [8, Chapter 8].

Each group of variables has as many dimensions as variables, and the equations have to cut all of them down to points. Expanding the product implies letting each equation spend its condition

on one of its groups. The coefficient keeps only those expansions where every group receives exactly as many conditions as variables.

For example, consider two equations in variables s, t , each of degree at most one in s and at most one in t . The ordinary degree bound is four, since both equations can have total degree two. The grouped bound is

$$[ST](S + T)^2 = 2.$$

Indeed, solving one equation for t and substituting into the other gives an equation of degree at most two in s . The coefficient rule is the higher-dimensional version of this improvement.

5.2 The Bézout coefficient

Apply Theorem 5.1 to the groups x, u, v , of sizes n, m, m . Use formal symbols X, U, V for their degrees. The complete degree calculation is:

Equations	Number	Degree expression
$u^\top A(x) = 0$	m	$X + U$
$\Lambda A(x)v = 0$	$m - 1$	$X + V$
$u^\top B_i v = 0$	$n - 2$	$U + V$
$x_{n-1} - 1 = 0$	1	X
$u^\top q - 1 = 0$	1	U
$p^\top v - 1 = 0$	1	V

It follows that

$$\begin{aligned} |S| &\leq [X^n U^m V^m] XUV (X + U)^m (X + V)^{m-1} (U + V)^{n-2} \\ &= [X^{n-1} U^{m-1} V^{m-1}] (X + U)^m (X + V)^{m-1} (U + V)^{n-2}. \end{aligned} \quad (9)$$

Call the last coefficient $B(m, n)$.

To expand it, suppose I choose $X^k U^{m-k}$ from the first factor. The power of X then forces the choice of $X^{n-1-k} V^{m-n+k}$ from the second factor. The required power of U forces the choice $U^{k-1} V^{n-1-k}$ from the third factor. Therefore

$$B(m, n) = \sum_{k=1}^{n-1} \binom{m}{k} \binom{m-1}{n-1-k} \binom{n-2}{k-1}. \quad (10)$$

As usual, a binomial coefficient is zero when its lower index is negative or exceeds its upper index.

Proposition 5.2. *Let $f = \det A(x)$ for an $m \times m$ affine-linear matrix. If S is a finite set of isolated solutions of (3) at which $\nabla f \neq 0$, then $|S| \leq B(m, n)$.*

5.3 Comparison of the counts

The exact expression (10) is not needed to read off the order of growth. Bound $\binom{n-2}{k-1} \leq 2^{n-2}$ and use Vandermonde's identity:

$$B(m, n) \leq 2^{n-2} \sum_k \binom{m}{k} \binom{m-1}{n-1-k} = 2^{n-2} \binom{2m-1}{n-1}.$$

For $m \geq n$, the elementary estimate $\binom{a}{b} \leq (ea/b)^b$ gives

$$B(m, n) \leq 2^{n-2} \left(\frac{e(2m-1)}{n-1} \right)^{n-1} \leq \left(\frac{4em}{n-1} \right)^{n-1}. \quad (11)$$

For an exact representation of F_n , $m \geq n$ by degree. Lemma 3.1 and Proposition 5.2 now give

$$n(n-1)^{n-2} \leq B(m, n) \leq \left(\frac{4em}{n-1} \right)^{n-1}.$$

Since $n(n-1)^{n-2} \geq (n-1)^{n-1}$, taking $(n-1)$ st roots yields

$$n-1 \leq \frac{4em}{n-1}, \quad \text{and hence} \quad m \geq \frac{(n-1)^2}{4e}.$$

This proves the ordinary exact lower bound.

6 Border approximation

An exact lower bound need not be a border lower bound. A polynomial could fail to have a small representation and still be a limit of polynomials that do have one.

6.1 Persistence of simple solutions

For a polynomial f define

$$\Phi_f(x) = (f(x), x_{n-1} - 1, P_1(f)(x), \dots, P_{n-2}(f)(x)).$$

Fix a degree bound $d \geq n$ and regard polynomials of degree at most d as vectors of coefficients. The map $(f, x) \mapsto \Phi_f(x)$ is polynomial in those coefficients and in x .

Lemma 6.1. *There is a neighborhood $\mathcal{U}$ of F_n in the coefficient space of polynomials of degree at most d such that, for every $f \in \mathcal{U}$, the system $\Phi_f(x) = 0$ has at least $N = n(n-1)^{n-2}$ distinct simple solutions. Each is a smooth zero of f .*

Proof. Put a small ball around each solution $a_1, \dots, a_N$ from Lemma 3.1, with the balls chosen disjoint. At each (F_n, a_j) , the Jacobian of Φ_f with respect to x is invertible. The implicit function theorem therefore gives a solution $a_j(f)$ in that ball whenever the coefficients of f are sufficiently close to those of F_n .

There are only finitely many balls, so I can choose one coefficient neighborhood $\mathcal{U}$ that works for all of them. The balls keep the solutions distinct. By continuity, shrinking $\mathcal{U}$ if necessary keeps every Jacobian invertible, so the solutions remain simple.

The first row of that Jacobian is $\nabla f(a_j(f))$. An invertible matrix cannot have a zero row. Hence $\nabla f(a_j(f)) \neq 0$ for every j . $\square$

6.2 Proof of the border bound

Suppose $\overline{\text{dc}}(F_n) \leq m$. Every size- m determinant has degree at most m , and this degree condition is preserved under coefficientwise limits. Therefore $m \geq n$. Choose a sequence

$$f_j(x) = \det \left(A_{j,0} + \sum_{i=1}^n x_i A_{j,i} \right) \longrightarrow F_n$$

in the coefficient space of polynomials of degree at most m .

Use Lemma 6.1 with $d = m$. For all sufficiently large j , the polynomial f_j belongs to $\mathcal{U}$ and has at least N simple solutions of (3). At those points its gradient is nonzero. Proposition 5.2 therefore applies to the size- m representation of f_j and gives

$$n(n-1)^{n-2} \leq B(m, n).$$

The calculation in (11) is unchanged, and yields

$$m \geq \frac{(n-1)^2}{4e}.$$

This proves the ordinary border lower bound.

Only the polynomials f_j had to converge. For each fixed j , I first find the N nearby points, and then choose the kernel vectors and normalizations. Note here that the upper bound $B(m, n)$ only depends on the dimensions and degrees of the equations, so it is the same for every member of the sequence.

7 Symmetric representations

For a symmetric matrix, its left and right kernels are the same, so I only need one of them. This lowers the count of auxiliary variables, and is the source of the improved constant.

Suppose $f = \det A(x)$ with each A_i symmetric. At a smooth zero a of f , choose a nonzero vector u in the kernel. The adjugate is a nonzero symmetric rank-one matrix with image $\mathbb{C}u$, so

$$\text{adj } A(a) = \lambda uu^\top \quad \text{for some } \lambda \neq 0.$$

Hence

$$\partial_i f(a) = \lambda u^\top A_i u.$$

Let S again be a finite set of smooth isolated solutions of (3). Choose q nonorthogonal to the finitely many kernel lines and normalize by $q^\top u = 1$. The equations are now

$$\begin{aligned} A(x)u &= 0, \\ u^\top (A_i - \alpha A_{n-1})u &= 0 & (1 \leq i \leq n-2), \\ x_{n-1} &= 1, & q^\top u = 1. \end{aligned} \tag{12}$$

There are $n + m$ variables and $m + (n-2) + 2 = n + m$ equations. No kernel equation needs to be removed.

Isolation follows from this as in Lemma 4.2.

There are now only two variable groups, of sizes n and m . The kernel equations have degree expression $X + U$, the $n - 2$ derivative equations have degree expression $2U$, and the normalizations contribute X and U . Thus

$$\begin{aligned} |S| &\leq [X^n U^m] XU(X + U)^m (2U)^{n-2} \\ &= 2^{n-2} \binom{m}{n-1}. \end{aligned}$$

The choice of $n - 1$ factors of X from $(X + U)^m$ accounts for the binomial coefficient.

For $f = F_n$, or for a sufficiently close symmetric determinant by Lemma 6.1, this gives

$$n(n-1)^{n-2} \leq 2^{n-2} \binom{m}{n-1} \leq \left(\frac{2em}{n-1} \right)^{n-1}.$$

Taking $(n - 1)$ st roots as before proves

$$\overline{\text{sd}}(F_n) \geq \frac{(n-1)^2}{2e}.$$

The exact symmetric lower bound follows as well.

8 Upper bounds

Put $r = n - 1$, and let J be the $r \times r$ matrix with ones immediately above the diagonal and zeros elsewhere. For one variable t , set

$$T(t) = I_r - tJ.$$

This is an upper triangular matrix with determinant one. Since $J^r = 0$,

$$T(t)^{-1} = I_r + tJ + \cdots + t^{r-1} J^{r-1}.$$

In particular, if e_1, e_r are the first and last coordinate vectors,

$$e_1^\top T(t)^{-1} e_r = t^{r-1} = t^{n-2}.$$

8.1 An ordinary representation

Let $T_i = T(x_i)$, and form

$$D = \text{diag}(T_1, \dots, T_n), \quad b^\top = (x_1 e_1^\top, \dots, x_n e_1^\top), \quad c = \begin{pmatrix} x_1 e_r \\ \vdots \\ x_n e_r \end{pmatrix}.$$

The matrix

$$M(x) = \begin{pmatrix} 0 & b^\top \\ -c & D \end{pmatrix}$$

has affine-linear entries and size $1 + n(n - 1)$. The block determinant formula gives

$$\det M = \det D (b^\top D^{-1} c) = \sum_{i=1}^n x_i^2 e_1^\top T_i^{-1} e_r = \sum_{i=1}^n x_i^n.$$

Thus $\text{dc}(F_n) \leq 1 + n(n - 1)$.

I sum the contributions of the n blocks by adding the extra row and column. Each block's inverse contributes x_i^{n-2} . The two connecting entries from b and c add the remaining factor x_i^2 .

8.2 A symmetric representation

The same construction can be made symmetric by pairing each block with its transpose. Define

$$C_i = \begin{pmatrix} 0 & T_i^\top \\ T_i & 0 \end{pmatrix}, \quad w_i = \begin{pmatrix} -\frac{1}{2}x_i e_1 \\ x_i e_r \end{pmatrix}.$$

Then C_i is symmetric and

$$C_i^{-1} = \begin{pmatrix} 0 & T_i^{-1} \\ T_i^{-\top} & 0 \end{pmatrix}.$$

The block order can be checked directly:

$$C_i C_i^{-1} = \begin{pmatrix} T_i^\top T_i^{-\top} & 0 \\ 0 & T_i T_i^{-1} \end{pmatrix} = I_{2r}.$$

The two cross terms in the quadratic expression are equal, so

$$w_i^\top C_i^{-1} w_i = 2 \left(-\frac{1}{2}x_i e_1 \right)^\top T_i^{-1} (x_i e_r) = -x_i^2 e_1^\top T_i^{-1} e_r = -x_i^n.$$

Let $C = \text{diag}(C_1, \dots, C_n)$ and stack the w_i into a column w . Since $\det C_i = (-1)^r$ and $nr = n(n-1)$ is even, $\det C = 1$. Therefore the symmetric affine-linear matrix

$$M_{\text{sym}}(x) = \begin{pmatrix} 0 & w^\top \\ w & C \end{pmatrix}$$

satisfies

$$\det M_{\text{sym}} = -\det C w^\top C^{-1} w = F_n.$$

Its size is $1 + 2n(n-1)$. These two constructions, together with the lower bounds, complete Theorem 1.1.

8.3 Power sums of any degree

Nothing in the argument used that the degree equals the number of variables. For $N \geq 3$ and $d \geq 2$, let

$$F_{N,d} = x_1^d + \dots + x_N^d.$$

Corollary 8.1. *For all $N \geq 3$ and $d \geq 2$, over $\mathbb{C}$,*

$$\frac{(N-1)(d-1)}{4e} \leq \overline{\text{dc}}(F_{N,d}) \leq \text{dc}(F_{N,d}) \leq 1 + N(d-1),$$

$$\frac{(N-1)(d-1)}{2e} \leq \overline{\text{sdc}}(F_{N,d}) \leq \text{sdc}(F_{N,d}) \leq 1 + 2N(d-1).$$

In particular, all four complexities of $F_{N,d}$ are $\Theta(Nd)$.

The lower bound in Corollary 8.1 also follows from the general hypersurface bound in [10]; the argument here gives a short direct proof.

Proof. Choose $\rho > 0$ with $(N-2)\rho^d < 1$ and use the system (3) with N in place of n and $\alpha = \rho^{d-1}$. On $x_{N-1} = 1$ the derivative relations become $x_i^{d-1} = \alpha$, so $x_i = \rho\zeta_i$ with $\zeta_i^{d-1} = 1$, giving $(d-1)^{N-2}$ choices. The last coordinate satisfies $x_N^d = -(1 + \sum_{i \leq N-2} x_i^d)$, and the right side is nonzero because $|\sum_{i \leq N-2} x_i^d| \leq (N-2)\rho^d < 1$. The proof of Lemma 3.1 then gives exactly $d(d-1)^{N-2}$ solutions, all simple and all smooth zeros. The proof of Lemma 6.1 uses only simplicity, and Proposition 5.2 holds with N in place of n , so

$$(d-1)^{N-1} \leq d(d-1)^{N-2} \leq B(m, N) \leq \left(\frac{4em}{N-1} \right)^{N-1}.$$

Taking $(N-1)$ st roots gives $m \geq (N-1)(d-1)/(4e)$. The symmetric bound follows in the same way from $d(d-1)^{N-2} \leq 2^{N-2} \binom{m}{N-1} \leq (2em/(N-1))^{N-1}$.

For the upper bounds, repeat Section 8 with blocks of size $r = d-1$, so that $e_1^T T(t)^{-1} e_r = t^{d-2}$. This gives an ordinary representation of size $1 + N(d-1)$ and a symmetric one of size $1 + 2N(d-1)$. In the symmetric case, replace $-\frac{1}{2}$ by $+\frac{1}{2}$ in w_i when $N(d-1)$ is odd, since then $\det C = -1$. $\square$

For $d = N = n$ this is Theorem 1.1. The bound is superlinear in $\max(N, d)$ whenever both grow, which is the regime singled out as open in [3].

9 Discussion

Note that a high degree alone does not imply many usable points. For example, $x_1 \cdots x_n$ is the determinant of an $n \times n$ diagonal matrix. Taking a smooth zero of the polynomial, exactly one coordinate is zero, implying that the gradient has only one nonzero entry. So its tangent directions barely vary. At singular points, many derivative equations automatically vanish, and this results in larger sets rather than isolated points.

Similarly, the permanent is highly singular, and the same issue arises. The permanent would hence need a fresh count of usable points on its smooth part. However, multiplying the degrees of its derivative equations only yields an upper bound on the count, without saying anything about the existence of usable points.

In [10, Theorem 7.7 and Corollary 7.11], I study how far this kind of counting argument can go. For the polar method and the class of refinements covered there, the lower bounds it can prove are at most $O(dN)$ for a homogeneous polynomial of degree d in N variables. The $n \times n$ permanent has degree n and n^2 variables, so the ceiling is $O(n^3)$. A supercubic lower bound would need something beyond that class of arguments.

For F_n too, this argument gives $\Theta(n^2)$, not the optimal constant. Whereas Bézout counts all isolated solutions of the kernel system, the selected points may be only some of them. Additionally, the binomial estimates loosen the bound further. Hence, improving the constant would require an exact understanding of the kernel system, or a new count.

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