

Smooth slices and the limits of determinantal lower bounds

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Abstract

In an earlier note (arXiv:2606.11090) I showed that the border determinantal complexity of the power sum $x_1^d + \cdots + x_N^d$ is $\Theta(Nd)$. The proof counted the simple solutions of an explicit polynomial system and compared that number with what a small determinant can produce. This left two questions. Does the argument work for other polynomials? And how far can it be pushed?

This article answers both. For the first question, the power sum turns out not to be special. What the argument really counts is the smooth points of the zero set whose tangent hyperplanes pass through some fixed generic points. Every smooth form of degree d in N variables has exactly $d(d-1)^{N-2}$ of these. A determinant of size m has at most about $(4em/N)^{N-1}$. So any polynomial with an affine-linear restriction to a smooth form has border determinantal complexity at least $(N-1)(d-1)/(4e)$, and at least $(N-1)(d-1)/(2e)$ in the symmetric model.

This lets me move from power sums to more natural families. For the homomorphism polynomial of a connected bipartite graph F with q labels, I get $\Omega(q|E(F)|)$. When F is a path with q^2 edges, the bound is $\Theta(q^3)$, and this is tight. These polynomials form a VBP-complete family.

For the second question, the answer is that the method does not go much further. I compute the count exactly for generic matrices, and this shows that the constant $4e$ cannot be improved asymptotically. I then use bounds on the topology of linear sections to show that the comparisons of polar counts in Section 6 cannot prove more than $O(dN)$, even when singular strata are taken into account. For the permanent this means at most $O(n^3)$. For the power sum of degree N it means $(1+o(1))N^2/(4e)$, so the bound from the earlier note is the best this approach can give.

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Keywords: determinantal complexity, border complexity, polar degrees, homomorphism polynomials, multihomogeneous Bézout theorem.

1 Introduction

I study how small a matrix can be if its determinant is a given polynomial f and its entries have degree at most one. Such a determinantal representation has the form

$$f(x) = \det M(x), \quad M(x) = A_0 + x_1 A_1 + \cdots + x_N A_N, \quad (1)$$

where the A_i are constant $m \times m$ matrices over $\mathbb{C}$. The least possible m is the *determinantal complexity* of f , denoted by $\text{dc}(f)$. Determinantal complexity is a central model in algebraic complexity. Families of polynomials with polynomially bounded dc form the class VBP, and the permanent is VNP-complete [16], so a superpolynomial lower bound on $\text{dc}(\text{per}_n)$ would separate VBP from VNP.

The known lower bounds are much weaker than this. For the $n \times n$ permanent, Mignon and Ressayre [8] proved a lower bound of order n^2 , and Landsberg, Manivel and Ressayre [7] extended it to the border model defined below. Note that the permanent has n^2 variables, so a quadratic bound does not go beyond the number of variables. More recently, a September 2026 preprint of OpenAI [9] reports a cubic lower bound in the border model. Its argument uses kernel equations, a Schur-complement reduction and a multihomogeneous count of solutions, which are also the main tools of Section 3 below.

An elementary lower bound is immediately apparent: a determinant of size m has degree at most m , so $\text{dc}(f) \geq \deg f$. Going beyond the degree is difficult for several reasons. A representation need not have any visible structure. Every entry of M may involve all the variables, and most of the terms in the expansion of $\det M$ may cancel. So counting the monomials of f says little about $\text{dc}(f)$, and studying one convenient representation says little about a cleverer one. I need a property of f that every representation must respect.

I also allow approximation. The *border determinantal complexity* $\overline{\text{dc}}(f)$ is the least m such that f is a limit, coefficient by coefficient, of determinants of matrices of the form (1) of size m . Approximation can only make the problem easier, so $\overline{\text{dc}}(f) \leq \text{dc}(f)$. Note that only the determinants have to converge. The matrices themselves need not, and their entries may grow without bound. If in addition every A_i is required to be symmetric, I write $\text{sdc}(f)$ for the corresponding border quantity. Symmetric here means equal to the transpose, not Hermitian.

This paper continues my earlier work on the power sum $x_1^n + \dots + x_n^n$. In [12] I proved a quadratic lower bound on its border determinantal complexity, including the symmetric and exact models, and verified the proof in Lean. That proof counted the points of the zero set picked out by certain conditions on the first derivatives, and their number was the degree of the dual hypersurface. In a short note [14] I then gave a much simpler proof, using the implicit function theorem and an explicit system of polar points with simple solutions. The same note shows that the border determinantal complexity of $x_1^d + \dots + x_N^d$ is $\Theta(Nd)$ for all $N \geq 3$ and $d \geq 2$. In this paper I ask what else this counting argument can prove, and where it stops. An earlier and longer version of these results is [13]. Here the lower bounds use only Bézout's theorem and the implicit function theorem, and the singularity theory needed for the ceiling is kept to Section 6 and Appendix B.

The idea

Fix generic points $c_1, \dots, c_{N-2}$ in $\mathbb{P}^{N-1}$. I call a smooth point of $V(f) \subset \mathbb{P}^{N-1}$ a *polar point* if its tangent hyperplane passes through all of them. There are only finitely many polar points, and they are cut out by f and its first derivatives. The proof counts them in two ways and compares the answers.

Counting on the target side. Suppose $V(f)$ is smooth of degree d . A point is polar when f vanishes there and so does the derivative of f in each direction c_j . This gives one equation of degree d and $N - 2$ equations of degree $d - 1$. By Bézout's theorem there are exactly $d(d - 1)^{N-2}$ polar points.

Counting on the matrix side. Now suppose $f = \det M$. At a smooth zero of f , the matrix M drops rank by exactly one. Its kernel is spanned by a vector v , and its cokernel by a vector u . The first derivatives of f then take a simple form: up to a common factor, $\partial_i f$ equals $u^\top A_i v$. So each

polar point gives a solution of a system of equations that is bilinear in three groups of unknowns, namely x , v and u . The key point is that the degrees of these equations do not depend on m . Only the number of unknowns grows with m . Bézout's theorem for several groups of variables then allows at most $(4em/(N-1))^{N-1}$ solutions.

Comparing the counts. Both counts are roughly $(N-1)$ st powers, so it helps to take $(N-1)$ st roots. Think of each count as the number of lattice points in a cube. The root is then the side length of that cube. The target needs a cube of side at least $d-1$. A determinant of size m supplies a cube of side at most $4em/(N-1)$. For the determinant to have enough polar points, it must have size $m \geq (N-1)(d-1)/(4e)$.

The same picture shows what will not help. Suppose one of the counts is improved by a constant factor C . After taking roots, the side length changes by the factor $C^{1/(N-1)}$, which tends to one as N grows. To change the bound in an essential way, a count would have to change by a factor exponential in N .

Results

To state the results I need one more definition. A polynomial P has an *affine-linear restriction* equal to f if substituting affine-linear forms in N variables for the variables of P gives f . Making the same substitution in a determinantal representation of P gives a representation of f of the same size. So any lower bound for f is also a lower bound for P .

Theorem 1.1 (Smooth restrictions). *Suppose a polynomial P has an affine-linear restriction equal to a homogeneous polynomial f of degree $d \geq 2$ in $N \geq 3$ variables, and $V(f) \subset \mathbb{P}^{N-1}$ is smooth. Then*

$$\overline{\text{dc}}(P) \geq \frac{N-1}{4e} (d(d-1)^{N-2})^{1/(N-1)} \geq \frac{(N-1)(d-1)}{4e}, \quad (2)$$

$$\overline{\text{sdc}}(P) \geq \frac{N-1}{2e} (d(d-1)^{N-2})^{1/(N-1)} \geq \frac{(N-1)(d-1)}{2e}. \quad (3)$$

The main application is to graph homomorphism polynomials, which have a particularly simple smooth restriction. Let $F = (L \sqcup R, E)$ be a bipartite multigraph, and let q be the number of labels. The homomorphism polynomial is

$$\text{hom}_{F,q}(X) = \sum_{\alpha: L \rightarrow [q]} \sum_{\beta: R \rightarrow [q]} \prod_{uv \in E} x_{\alpha(u)\beta(v)}. \quad (4)$$

Each labelling of the vertices contributes one monomial. It is the product, over the edges uv , of the variable x_{ab} , where a is the label of the left end u and b is the label of the right end v . Parallel edges are counted with multiplicity. Now set the off-diagonal entries of X to zero. A labelling then survives only if every edge has both ends labelled the same. If F is connected, this forces the whole graph to carry a single label, and

$$\text{hom}_{F,q}(\text{diag}(y_1, \dots, y_q)) = y_1^{|E|} + \dots + y_q^{|E|}. \quad (5)$$

Theorem 1.2 (Graph homomorphism polynomials). *For every connected bipartite multigraph F with $d = |E(F)| \geq 2$ and every $q \geq 3$,*

$$\overline{\text{dc}}(\text{hom}_{F,q}) \geq \frac{(q-1)(d-1)}{4e}, \quad \overline{\text{sdc}}(\text{hom}_{F,q}) \geq \frac{(q-1)(d-1)}{2e}.$$

For the path P_{q^2} with q^2 edges, the family $W_q = \text{hom}_{P_{q^2},q}$ is VBP-complete under constant-depth c -reductions, and $\text{dc}(W_q)$ and $\overline{\text{dc}}(W_q)$ are both $\Theta(q^3)$.

The completeness statement is part of the classification of Dwivedi, Pago and Seppelt [2, Theorem D]. The same diagonal restriction gives cubic bounds for their VP-complete family of binary trees and their VNP-complete family of grids. All of these polynomials have degree of order q^2 in q^2 variables. So the bound beats the trivial degree bound by a factor of order q , which is still far from separating the classes.

It is natural to try to improve these bounds in two ways. One could count the matrix side exactly instead of using Bézout's theorem. Or one could use the singularities of the target to get larger counts. The second half of the paper shows that neither changes the order of the bound. For the first, I compute the exact count $T_N(m)$ for generic matrices (Theorem 5.1). It satisfies

$$T_N(m)^{1/(N-1)} = (4e + o(1)) \frac{m}{N} \quad \text{as } N \rightarrow \infty \text{ and } m/N \rightarrow \infty, \quad (6)$$

which is the same growth as the Bézout estimate. A bound that holds for all matrices must be at least what generic matrices attain, so $4e$ is the best constant this system of equations can give. For the second, Theorem 6.1 shows that comparisons of polar counts, even including the contributions of singular strata, cannot rule out any size $m \geq 4dN$. For the permanent this is $O(n^3)$. For the power sum of degree N the limit can be sharpened to $(1 + o(1))N^2/(4e)$, which is the constant in the lower bound. I should stress that these are limits on what the comparisons can prove. They are not upper bounds on determinantal complexity.

To summarize, the method works well when the target has a large smooth linear section. There it gives a bound of order the degree times the number of variables. It works poorly when singularities make every smooth section small. For the permanent, generic smooth sections give only $O(n^2)$ (Section 4).

The rest of the paper is organized as follows. Section 2 counts the polar points of the target and shows that they can be taken to be simple. Section 3 bounds the number of polar points of a determinant and proves Theorem 1.1. Section 4 treats singular targets through their smooth linear sections and proves Theorem 1.2. Section 5 computes the exact count for generic matrices, Section 6 proves the ceiling, and Section 7 lists some open questions.

Throughout, all varieties and matrices are over $\mathbb{C}$. I write N for the number of variables, d for the degree, m for the size of a determinant and q for the number of labels.

2 Counting polar points on the target

In this section I find the polar points as the solutions of a system of equations. It is not enough to know how many there are. The solutions must also be *simple*, meaning that the Jacobian of the system is invertible at each of them. Simple solutions survive small perturbations of the equations, and this is what will let me pass to limits in the border model.

For $c \in \mathbb{C}^N$ write $D_c f = \sum_i c_i \partial_i f$ for the derivative of f in the direction c . The tangent hyperplane at a smooth point $[x]$ is $\{[y] : \sum_i y_i \partial_i f(x) = 0\}$. So $D_c f(x) = 0$ says exactly that this hyperplane contains $[c]$, and the polar points are the solutions of $f = 0$ and $D_{c_t} f = 0$ for $1 \leq t \leq N - 2$. These equations are homogeneous, so each projective solution is a line of affine solutions. A normalization $\lambda(x) = 1$ picks out one point on each line.

2.1 An explicit choice for a power sum

For the power sum $\Phi_{N,d} = x_1^d + \cdots + x_N^d$ the polar points can be written down by hand. The first thing one might try is to make the first $N - 1$ derivatives equal. The last coordinate is then determined by an equation of the form $x_N^d = R$. When $R \neq 0$ this has d distinct roots, but when

$R = 0$ its only root is $x_N = 0$, repeated d times, and nothing prevents R from vanishing. To rule this out, I fix a small $a \neq 0$ with $(N-2)|a|^d < 1$ and impose

$$x_1 = 1, \quad \partial_i \Phi_{N,d} = a^{d-1} \partial_1 \Phi_{N,d} \quad (2 \leq i \leq N-1), \quad \Phi_{N,d} = 0. \quad (7)$$

Each relation $\partial_i \Phi_{N,d} = a^{d-1} \partial_1 \Phi_{N,d}$ has the form $D_c \Phi_{N,d} = 0$ with $c = e_i - a^{d-1} e_1$, so the solutions are polar points for these particular directions. Up to exchanging x_1 and x_{N-1} , this is the system used in [14].

Lemma 2.1. *The system (7) has exactly $d(d-1)^{N-2}$ solutions, and every solution is simple.*

Proof. Since $x_1 = 1$, the middle equations say that $x_i^{d-1} = a^{d-1}$. So each of $x_2, \dots, x_{N-1}$ is a times a $(d-1)$ st root of unity. The last equation then says $x_N^d = -1 - \sum_{i=2}^{N-1} x_i^d$. The sum has absolute value at most $(N-2)|a|^d < 1$, so the right side is nonzero, and x_N takes d distinct values for each of the $(d-1)^{N-2}$ earlier choices. Use $x_2, \dots, x_N$ as the variables. Then the Jacobian is lower triangular, with determinant $d^{N-1}(d-1)^{N-2} x_N^{d-1} \prod_{i=2}^{N-1} x_i^{d-2}$, which is nonzero. $\square$

The number $d(d-1)^{N-2}$ is the degree of the dual variety of the smooth Fermat hypersurface. This geometric count was the starting point of my original argument [12]. Kumar and Volk [6] later made the points explicit for the exact ordinary model. They report that the outline of their proof came out of an AI-assisted reading of [12]. Compared with making the derivatives equal, the small ratio a^{d-1} in (7) is what makes every solution simple, and this is what lets the points survive perturbation in the border model. As an example, take $N = d = 3$ and $a = 1/2$. Then $x_2 = \pm 1/2$, and correspondingly $x_3^3 = -9/8$ or $-7/8$. This gives six points, with no repeated root. Had I made the derivatives equal instead, I would get $x_2 = \pm 1$, and at $x_2 = -1$ the last equation would become $x_3^3 = 0$, with three coinciding roots.

2.2 Any smooth hypersurface

For a general smooth f the polar points cannot be listed by hand. Generic derivative conditions still give the same count, with all solutions simple.

Lemma 2.2. *Let f be homogeneous of degree $d \geq 2$ with $V(f) \subset \mathbb{P}^{N-1}$ smooth. There are vectors $c_1, \dots, c_{N-2}$ and a linear form λ such that the system*

$$f(x) = 0, \quad D_{c_1} f(x) = \dots = D_{c_{N-2}} f(x) = 0, \quad \lambda(x) = 1 \quad (8)$$

has exactly $d(d-1)^{N-2}$ solutions in $\mathbb{C}^N$, all simple.

Proof. Since $X = V(f)$ is smooth, the derivatives $\partial_i f$ have no common zero on X . So the hypersurfaces $D_c f = 0$, which have degree $d-1 \geq 1$, form a linear system on X without base points. By Bertini's theorem a generic member of this system cuts X transversally. The same holds for each further generic member, cutting the intersection obtained so far. After $N-2$ cuts what remains is a finite set of reduced points, and Bézout's theorem says that there are $d(d-1)^{N-2}$ of them. Now choose λ nonzero at each of these points, and scale each solution so that $\lambda(x) = 1$. By transversality the first $N-1$ equations have a Jacobian of rank $N-1$ on the affine cone. Its kernel is the radial direction, and the equation $\lambda(x) = 1$ removes it. So the full Jacobian is invertible. $\square$

When the hypersurface is reduced but singular, I count only its smooth points. The *top polar degree* $\delta(f)$ is the number of smooth points of $V(f)$ that satisfy $N-2$ generic derivative relations. If the dual variety is a hypersurface, $\delta(f)$ is its degree. Otherwise $\delta(f) = 0$. In either case the refined Bézout theorem still gives

$$\delta(f) \leq d(d-1)^{N-2}. \quad (9)$$

The standard facts about polar degrees can be found in [10, 15].

3 Counting polar points of a determinant

Now let $f = \det M$. If I wrote the polar system (8) directly for $\det M$, I would have to use a polynomial of degree up to m and derivatives of degree up to $m - 1$. That runs into a wall, since Bézout's theorem would then allow $m(m - 1)^{N-2}$ polar points, and comparing this with Section 2 only gives $m \geq d$. To do better, I rewrite the derivative conditions so that their degrees do not depend on m .

3.1 From derivatives to kernel vectors

Differentiating $\det M$ brings in the cofactors of M . Jacobi's formula gives $\partial_i f = \text{tr}(\text{adj } M \cdot A_i)$, and the cofactors can have degree up to $m - 1$. Away from the zero set, the adjugate $\text{adj } M = (\det M) M^{-1}$ is a full $m \times m$ matrix. The hope is to find points where it collapses to something smaller, and the smooth zeros of f are such points. Let p be a smooth zero. If $M(p)$ had corank two or more, its adjugate would vanish, and with it every derivative of f . So $M(p)$ has corank exactly one. Choose nonzero vectors v and u with $M(p)v = 0$ and $u^\top M(p) = 0$. Observe that the adjugate of a corank-one matrix has rank one, with columns in the kernel and rows in the left kernel. So $\text{adj } M(p) = \kappa v u^\top$ for some $\kappa \neq 0$, and

$$\partial_i f(p) = \kappa u^\top A_i v. \quad (10)$$

The cofactors are now controlled by two vectors of length m . The factor κ is unknown, but it does not affect a linear relation among the derivatives, so the condition $D_c f(p) = 0$ becomes $u^\top A_c v = 0$, where $A_c = \sum_i c_i A_i$. This is the main point of the whole argument: the new condition is bilinear in u and v , however large m is.

Adding u and v as unknowns raises a question. To count with Bézout's theorem, each polar point must give a single isolated solution. So I need to know that the new unknowns, taken projectively, cannot turn an isolated polar point into a positive-dimensional family. The next lemma shows that they cannot.

Lemma 3.1 (Corank one). *If $\text{rank } M(p) = m - 1$, then near p the equations $M(x)v = 0$ and $u^\top M(x) = 0$ determine $[v]$ and $[u]$ uniquely on $\det M(x) = 0$, and their solution set is locally isomorphic to that hypersurface.*

Proof. Choose bases, and affine charts for $[v]$ and $[u]$, so that

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \Delta \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ w \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ z \end{pmatrix},$$

where Δ is invertible near p . In these coordinates the equations $Mv = 0$ and $u^\top M = 0$ are equivalent to $w = -\Delta^{-1}\gamma$, $z^\top = -\beta\Delta^{-1}$ and $\alpha - \beta\Delta^{-1}\gamma = 0$. The last expression equals $\det M / \det \Delta$, so it cuts out the hypersurface. The first two express w and z as regular functions of x . $\square$

Now consider the *kernel system*

$$M(x)v = 0, \quad u^\top M(x) = 0, \quad u^\top A_{c_t} v = 0 \quad (1 \leq t \leq N - 2), \quad \lambda(x) = 1. \quad (11)$$

By Lemma 3.1, near each polar point of $\det M$ this system is just the polar system (8), with $[v]$ and $[u]$ attached as functions of x . In particular, an isolated polar point has an isolated lift. The kernel system may have other solutions, some of them on curves or larger sets where the rank of M drops further. These do not matter, since I only count the lifts of polar points.

The system has one equation too many for Bézout's theorem, which wants as many equations as unknowns. One of the kernel equations is redundant, because $u^\top(Mv) = (u^\top M)v$. To remove it, choose a hyperplane $H \subset \mathbb{C}^m$ that contains none of the finitely many kernel lines at the polar points. Keep only the $m - 1$ equations $u^\top Mw = 0$ for w in a basis of H . Near the polar points $v \notin H$, so $H + \mathbb{C}v = \mathbb{C}^m$. Then $Mv = 0$ together with the retained equations implies $u^\top M = 0$. So nothing changes near the points being counted.

3.2 Keeping the variable groups apart

If I ignore the groups of variables and remember only that the equations in (11) are quadratic, ordinary Bézout gives a bound exponential in m . Comparing it with the $d(d - 1)^{N-2}$ polar points of the target would give a lower bound of order only $N \log d$. But the equations have more structure than their degrees show. The equations $Mv = 0$ involve only x and v , the left-kernel equations involve only x and u , and the derivative conditions involve only u and v . The multihomogeneous form of Bézout's theorem takes advantage of this by keeping the groups apart. Give the groups x , v and u the degree symbols h , a and b . An equation that is bilinear in x and v then contributes the factor $h + a$, instead of the number 2, and the number of isolated solutions is at most a certain coefficient of the product of these factors. To see which coefficient, note that each group of variables has a dimension, and the equations have to cut all of these dimensions down to points. Expanding the product amounts to letting each equation spend its condition on one of the groups it involves. The relevant coefficient counts the expansions in which every group receives exactly as many conditions as its dimension.

A small example shows the difference. The equations $xy = 1$ and $xy = x + y$ are two plane conics, so ordinary Bézout allows four solutions. But both are bilinear in x and y , and the coefficient of ha in $(h + a)^2$ is 2. Indeed there are exactly two solutions.

To apply this, homogenize x with a new coordinate x_0 . The unknowns then lie in $\mathbb{P}_x^N \times \mathbb{P}_v^{m-1} \times \mathbb{P}_u^{m-1}$, and the normalization becomes $\lambda(x) = x_0$. The equations and their classes are as follows.

Equations	Number	Degree
$Mv = 0$	m	$h + a$
retained left-kernel equations	$m - 1$	$h + b$
$u^\top A_{c_i} v = 0$	$N - 2$	$a + b$
$\lambda(x) = x_0$	1	h

Multiplying the classes and reading off the right coefficient gives the bound

$$B_N(m) = [h^{N-1} a^{m-1} b^{m-1}] (h + a)^m (h + b)^{m-1} (a + b)^{N-2} = \sum_{j=0}^{N-2} \binom{N-2}{j} \binom{m}{j+1} \binom{m-1}{N-2-j}. \quad (12)$$

Here $[\cdot]$ denotes a coefficient, and I have already taken out the factor h coming from the last equation. This bounds the isolated solutions even if the system has other components of positive dimension [4, Chapter 12]. Using $\binom{N-2}{j} \leq 2^{N-2}$ and Vandermonde's identity,

$$B_N(m) \leq 2^{N-2} \binom{2m-1}{N-1} \leq \left(\frac{4em}{N-1} \right)^{N-1}. \quad (13)$$

The denominator $N - 1$ in this bound comes from the binomial coefficient, and so from keeping the groups of variables apart. It is the source of the factor N in Theorem 1.1.

3.3 Proof of Theorem 1.1

An exact lower bound need not be a border lower bound. A polynomial may have no small representation and still be a limit of polynomials that do. This is where the simplicity of the polar points is used.

Proof of Theorem 1.1. I first treat the case $P = f$. Fix the simple solutions given by Lemma 2.2. Polynomials of degree at most m form a finite-dimensional space, so the implicit function theorem can be applied there. It gives, for each solution, a small ball around it and a neighbourhood of f with the following property. Every g in the neighbourhood has exactly one solution of

$$g = 0, \quad D_{c_1}g = \cdots = D_{c_{N-2}}g = 0, \quad \lambda = 1$$

in the ball, and this solution is simple. Choose the balls to be disjoint, and intersect the neighbourhoods. Note that a simple solution is automatically a smooth zero of g . If the gradient of g vanished there, the Jacobian of the system would be singular.

Now suppose that f is a limit of determinants of size m . Then some determinant $g = \det M'$ of size m lies in the neighbourhood. By the above, g has $d(d-1)^{N-2}$ simple solutions, and these are smooth zeros satisfying the derivative conditions. By Lemma 3.1 each lifts to an isolated solution of the kernel system for M' , so $d(d-1)^{N-2} \leq B_N(m)$. Combined with (13), this gives (2). Note that only the polynomials had to converge here, and nothing was assumed about the matrices M' . The kernel vectors and the hyperplane H are chosen separately for each M' , but the bound $B_N(m)$ depends only on the sizes and degrees of the equations, so it is the same for all of them.

The symmetric case needs one change. When M is symmetric, its left and right kernels coincide, so I only need one of them, and $\text{adj } M = \kappa v v^\top$. The derivative conditions become $v^\top A_{c_t} v = 0$, of degree $2a$. There is no u , and there is no redundant equation to remove. Losing a whole group of m unknowns is what improves the constant. The count is

$$[h^N a^{m-1}] (h+a)^m h (2a)^{N-2} = 2^{N-2} \binom{m}{N-1} \leq \left(\frac{2em}{N-1} \right)^{N-1}.$$

The isolation and perturbation steps work as before, and this gives (3).

Finally, let P be general. An affine-linear substitution preserves the size and the symmetry of a matrix, and it depends continuously on the coefficients. So restricting P to f cannot increase any of these complexities, and the bounds for f also hold for P . $\square$

Remark 3.2. In the exact ordinary model the same argument gives

$$\text{dc}(f) \geq \frac{N-1}{4e} \delta(f)^{1/(N-1)}$$

for every reduced homogeneous f . No perturbation is needed. A generic polar section consists of isolated smooth points, and the kernel count applies to them directly.

4 Restrictions and graph polynomials

Theorem 1.1 needs a smooth target. A singular hypersurface can still have a large smooth linear section, because a generic linear space of small enough dimension misses the singular locus. This section makes that precise and applies it to graph polynomials.

4.1 Avoiding the singular locus

Let P be a squarefree form of degree d , and let $X \subset \mathbb{P}^s$ be its zero set. Write c for the codimension of the singular locus of X in $\mathbb{P}^s$, with $c = s + 1$ if X is smooth.

Corollary 4.1. *If $c \geq 3$, then $\overline{\text{dc}}(P) \geq (c-1)(d-1)/(4e)$ and $\overline{\text{sdc}}(P) \geq (c-1)(d-1)/(2e)$.*

Proof. A generic $(c-1)$ -plane misses $\text{Sing } X$, because $(c-1) + (s-c) < s$. By Bertini's theorem it meets the smooth part of X transversally. So the restriction of P to this plane is a smooth form of degree d in c variables, and Theorem 1.1 applies to it. If X is smooth, apply the theorem to P directly. $\square$

Two examples show what the corollary can and cannot do. For $\det_m$ the singular points are the matrices of corank at least two, which have codimension 4. So the corollary gives a linear bound, as it must, since $\text{dc}(\det_m) \leq m$. For per_n with $n \geq 3$, every matrix with two zero rows is a singular point. Indeed, each first derivative of the permanent is a smaller permanent, and it still has a zero row. These matrices have codimension $2n$, so $c \leq 2n$, and smooth slices give at most a quadratic bound. The permanent has n^2 variables, but its singular locus hides most of them from the method.

4.2 The diagonal restriction

Let F be connected. By (5), $\text{hom}_{F,q}$ restricts to the Fermat form $\Phi_{q,|E|}$. This form is smooth, because its derivatives $|E| y_i^{|E|-1}$ have no common nonzero zero. Theorem 1.1 with $N = q$ and $d = |E|$ then gives the first part of Theorem 1.2. The restriction remembers only that F is connected and how many edges it has. So paths, trees and grids with the same number of edges all get the same bound, even though their homomorphism polynomials can differ greatly in complexity.

4.3 Paths give a matching upper bound

For the upper bound I use the fact that a path can be labelled one vertex at a time. Let the path be $v_0, \dots, v_L$. Once the labels of $v_0, \dots, v_t$ are chosen, only the label of v_t matters for the next edge. This gives an algebraic branching program with $L+1$ layers of q nodes each, one node for each label. Between consecutive layers, the arc from label a to label b has weight x_{ab} or x_{ba} , depending on which side of the bipartition contains v_t . A source is joined to every node of the first layer, and every node of the last layer is joined to a sink, by arcs of weight 1. Paths from the source to the sink then correspond exactly to labellings, and their weights are the terms of (4).

To turn the program into a determinant, order its $q(L+1) + 2$ nodes so that every arc goes forward, and let A be the weighted adjacency matrix. Then A is strictly upper triangular. So $\det(I - A) = 1$, and $(I - A)^{-1} = I + A + A^2 + \dots$ is a finite sum. Its (s, t) entry is the total weight of the paths from the source s to the sink t . By the Schur complement formula,

$$\det \begin{pmatrix} I - A & -e_t \\ e_s^\top & 0 \end{pmatrix} = e_s^\top (I - A)^{-1} e_t = \text{hom}_{P_L, q}. \quad (14)$$

This is a matrix of size $q(L+1) + 3$ with affine-linear entries. Together with the lower bound, it shows that for $q \geq 3$ and $L \geq 2$ both $\text{dc}(\text{hom}_{P_L, q})$ and $\overline{\text{dc}}(\text{hom}_{P_L, q})$ are $\Theta(qL)$, with absolute constants. Taking $L = q^2$ gives the cubic bound of Theorem 1.2.

Dwivedi, Pago and Seppelt [2, Theorem D] show that paths whose length grows polynomially, of the order used here, give a VBP-complete family under constant-depth c -reductions. Their theorem

also covers two other families. Let B_q be the complete binary tree of height $\lfloor 2 \log_2 q \rfloor$, and let G_q be the $q \times q$ grid. The table lists the three families, with the bounds that follow from their edge counts.

Pattern	Edges	Complete for	Border lower bound
P_{q^2}	q^2	VBP	$(\frac{1}{4e} - o(1))q^3$
B_q	between $q^2 - 1$ and $2q^2 - 2$	VP	$(\frac{1}{4e} - o(1))q^3$
G_q	$2q(q - 1)$	VNP	$(\frac{1}{2e} - o(1))q^3$

The symmetric bounds are twice as large. The completeness results use constant-depth c -reductions rather than p -projections, and these reductions do not carry numerical bounds from one family to another. So the bounds in the table do not follow from one another. Each comes directly from the edge count of its own pattern.

4.4 A larger restriction for a dense pattern?

Now consider the complete bipartite pattern, $H_q = \text{hom}_{K_{q,q,q}}$. Its degree and its number of variables are both q^2 . The diagonal restriction gives $\Omega(q^3)$, while the ceiling of Section 6 is $O(q^4)$. So there is room for a larger bound, and one might hope to find it by keeping more entries of X than the diagonal. The next proposition shows that this does not work.

Proposition 4.2. *Let $E \subseteq [q] \times [q]$ with $|E| \geq 2$, and set the entries outside E to zero in H_q . The resulting projective hypersurface is smooth if and only if no two positions in E share a row or a column. In that case its equation is $\sum_{e \in E} x_e^{q^2}$.*

Proof. Consider a labelling with r_i left vertices labelled i and s_j right vertices labelled j . Its monomial is $\prod_{i,j} x_{ij}^{r_i s_j}$. The variables that occur in it form a rectangle, namely the rows i with $r_i > 0$ times the columns j with $s_j > 0$. After the substitution, a labelling survives only if its rectangle lies inside E .

Suppose first that E is a matching. Then the only rectangles inside E are single positions. A single position (i, j) comes from the one labelling that gives every left vertex the label i and every right vertex the label j , and its monomial is $x_{ij}^{q^2}$. This gives the stated polynomial, which is smooth.

Otherwise, after transposing if necessary, E contains two positions (a, b) and (a, c) in the same row. On these two coordinates the polynomial is $(x_{ab}^q + x_{ac}^q)^q$. Choose nonzero σ, τ with $\sigma^q + \tau^q = 0$, and consider the point with $x_{ab} = \sigma$, $x_{ac} = \tau$ and all other entries zero. Because of the outer power q , the polynomial and its derivatives in these two positions vanish there. Now take the derivative in any other position (i, j) . At this point the only monomials that can contribute are those with exponent one in x_{ij} and no other variables outside the pair. Exponent one means $r_i = s_j = 1$. The r 's and the s 's each sum to $q \geq 2$, so the rectangle of such a monomial has at least two rows and two columns. It therefore contains three positions besides (i, j) , and these cannot all lie in the pair. So every derivative vanishes at this point, and it is a singular point. $\square$

So a smooth restriction of H_q in more than q variables cannot just set entries to zero. It has to mix the entries linearly.

Part of the singular locus can also be found directly. Write $p_q(u) = \sum_i u_i^q$. Evaluating the definition at a rank-one matrix gives

$$H_q(uw^\top) = (p_q(u)p_q(w))^q, \quad \partial_{kl}H_q(uw^\top) = q^2 u_k^{q-1} w_l^{q-1} (p_q(u)p_q(w))^{q-1}. \quad (15)$$

For the derivative, note that differentiating a monomial at $uw^\top$ multiplies it by $r_k s_l / (u_k w_l)$. Summing over the labellings gives the formula when all coordinates are nonzero, and it extends to all u, w as a polynomial identity. It follows that the rank-one matrices $uw^\top$ with $p_q(u) = 0$ are singular points. They form a projective set of dimension $2q - 3$. On the other hand, the diagonal $(q - 1)$ -plane misses the singular locus, because the restriction to it is smooth. The projective dimension theorem then gives

$$q \leq \text{codim Sing } V(H_q) \leq (q - 1)^2 + 1. \quad (16)$$

For $q \geq 3$ the lower bound shows that H_q is irreducible. A repeated factor would make $V(H_q)$ singular in codimension one, and two distinct factors would meet in codimension at most two.

In the earlier version [13] I conjectured that the upper end of (16) is the true codimension. If so, Corollary 4.1 would give $\overline{\text{dc}}(H_q) \geq (q - 1)^2(q^2 - 1)/(4e) = \Omega(q^4)$. The conjecture is still open. The rank-one locus only gives the upper bound, and there could be a larger singular component somewhere else.

5 The exact count for a generic matrix

The bound $B_N(m)$ was obtained after dropping one kernel equation, so it may also count solutions that are not true kernel triples. In [14] I remarked that improving the constant would require an exact understanding of the kernel system, or a new count. This section gives an exact count for generic matrices, and shows that it does not improve the constant $4e$. Let $M(x) = \sum_{i=1}^N x_i A_i$ be a generic homogeneous matrix, with $3 \leq N \leq m^2$, and let $T_N(m)$ be the number of projective triples $([x], [v], [u])$ that satisfy *both* full kernel equations and $N - 2$ generic derivative conditions. Any bound that is valid for all matrices of size m must be at least $T_N(m)$.

The extra solutions really do occur, already in the smallest case. Take $M = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$, and work in the chart $x_3 = v_2 = u_2 = 1$. The full kernel equations give $x = (v_1^{-2}, -v_1^{-1}, 1)$ and $u_1 = v_1$. Take $c = (3, 5, 7)$. The derivative condition is then $3u_1 v_1 + 5(u_1 + v_1) + 7 = 0$, which becomes $3v_1^2 + 10v_1 + 7 = 0$. Its roots -1 and $-7/3$ give the two polar points of the conic $x_1 x_3 = x_2^2$. If the second left-kernel equation is dropped, there is a third solution, $x = (0, 0, 1)$, $v = (1, 0)$, $u = (-5/3, 1)$. It is not a kernel triple, since $u^\top M = (0, 1) \neq 0$. So $B_3(2) = 3$, while $T_3(2) = 2$.

Theorem 5.1. *For a generic $m \times m$ linear matrix in N variables, with $3 \leq N \leq m^2$, the full system has exactly*

$$T_N(m) = \sum_{r=\lceil N/2 \rceil}^{\min(m, N-1)} \binom{m}{r} \binom{r-1}{N-1-r} \binom{2r-2}{r-1} \quad (17)$$

solutions. They are reduced and lie over matrices of rank $m - 1$. Consequently every bound for this system valid for all size- m matrices is at least $T_N(m)$.

The formula comes from the same three-group count on $\mathbb{P}^{N-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$, with one change. Instead of dropping a left-kernel equation, I keep all m of them and account for the single relation among them, which has degree $h + a + b$. This replaces the factor $(h + b)^{m-1}$ by $(h + b)^m / (h + a + b)$, so that

$$T_N(m) = [h^{N-1} a^{m-1} b^{m-1}] \frac{(h + a)^m (h + b)^m (a + b)^{N-2}}{h + a + b}, \quad (18)$$

where the fraction is expanded in descending powers of h . Appendix A justifies this formula and shows that the points it counts are distinct.

At first sight (18) involves alternating signs, but in fact it does not. The identity $(h+a)(h+b) = h(h+a+b) + ab$ gives

$$\frac{(h+a)^m(h+b)^m}{h+a+b} = \sum_{r=1}^m \binom{m}{r} h^r (h+a+b)^{r-1} (ab)^{m-r} + \frac{(ab)^m}{h+a+b}.$$

The last term is divisible by a^m , so it cannot contribute. In the r th term, the factor h^{N-1-r} must come from $(h+a+b)^{r-1}$, which can happen in $\binom{r-1}{N-1-r}$ ways. The remaining factor $(a+b)^{2r-2}$ must then supply $a^{r-1}b^{r-1}$, which it does in $\binom{2r-2}{r-1}$ ways. This proves (17).

For small N the formula gives familiar numbers. It gives $T_3(m) = m(m-1)$ and $T_4(m) = m(m-1)^2$, which are the degrees of the duals of smooth plane curves and smooth surfaces of degree m . In five variables the generic determinantal hypersurface is already singular, and the count drops below the smooth value: $T_5(5) = 120 + 100 = 220$, while a smooth hypersurface of degree 5 in five variables gives $5 \cdot 4^3 = 320$.

Taking roots. To compare with the Bézout bound, put $k = N-1$. Keeping only the term $r = k$ gives, for $m \geq k$,

$$T_N(m) \geq \binom{m}{k} \binom{2k-2}{k-1}. \quad (19)$$

For fixed N this is the leading term in m . Expanding (12) to leading order with Vandermonde's identity, the leading coefficients are $\binom{2k-2}{k-1}/k!$ for $T_N(m)$ and $\binom{2k-1}{k}/k!$ for $B_N(m)$. So $T_N(m)/B_N(m) \rightarrow (N-1)/(2N-3)$ as $m \rightarrow \infty$. In other words, the Bézout estimate overcounts by less than a factor of two. After taking $(N-1)$ st roots this becomes a factor $2^{1/(N-1)}$, which tends to one.

The same is true when N grows as well. Let $N \rightarrow \infty$ and $m/N \rightarrow \infty$. Put the bounds $\binom{m}{k} \geq (m-k+1)^k/k!$ and $\binom{2k-2}{k-1} \geq 4^{k-1}/(2k-1)$ into (19). Stirling's formula then gives $T_N(m)^{1/k} \geq (4e - o(1))m/N$. In the other direction, $T_N(m) \leq B_N(m)$ and (13) give the matching upper bound. Thus

$$T_N(m)^{1/(N-1)} = (1 + o(1)) B_N(m)^{1/(N-1)} = (4e + o(1)) \frac{m}{N}, \quad (20)$$

uniformly in these limits. The omitted factors are controlled by N/m and $\log N/N$.

6 The ceiling of the counting argument

Section 5 shows that counting the matrix side more carefully does not help. The other option is to use a finer invariant of the target. For the top polar degree alone it is easy to see that this fails. By (9), no reduced form of degree d in N variables has more than $d(d-1)^{N-2}$ smooth polar points. The power sum has exactly this many, and yet it is a determinant of size $O(Nd)$. To see this, compute each power x_i^d by a path with d arcs, put the N paths side by side between a common source and sink, and use (14). So any test that uses only the top polar degree, together with a matrix-side bound valid for all determinants of a given size, must stop at size $O(Nd)$.

Singular targets are less simple. Singularities can lower the top polar degree, but they also contribute counts of their own along the singular locus. For example, the dual of a nodal plane cubic has degree 4. Its node adds one to the total polar count defined below, which is therefore 5. For a smooth cubic the count is 6. So I need to bound all such contributions at once. The idea is to compare growth rates. A datum of index j lives on a generic linear section of dimension about j , and the Euler characteristic of such a section is at most about d^j . On the other side, a matrix-side bound of index j must allow about $(m/j)^j$ points, because generic determinants have that many. So the matrix side wins once m/j is a large enough multiple of d . For all $j \leq N$ at once, this happens when m is of order dN .

6.1 The numbers being compared

Let $X \subset \mathbb{P}^s$ be a reduced hypersurface. For $0 \leq j \leq s$ let χ_j be the Euler characteristic of the intersection of X with a generic projective j -plane, and put $\chi_{-1} = \chi_{-2} = 0$. Define

$$Q_j(X) = (-1)^{j-1}(\chi_j - 2\chi_{j-1} + \chi_{j-2}), \quad 1 \leq j \leq s. \quad (21)$$

These are the total polar counts of the constant sheaf. Each one is the ordinary polar count of the smooth part of X , plus nonnegative contributions from the singular strata (see Appendix B). For a smooth hypersurface of degree d , $Q_j(X) = d(d-1)^{j-1}$. For the nodal plane cubic above, $(\chi_0, \chi_1, \chi_2) = (0, 3, 1)$, so $Q_2 = -(1-6) = 5$. In Appendix B I prove that

$$Q_j(X) \leq 8(d-1)^j + 8j + 8 \quad \text{if } \deg X \leq d. \quad (22)$$

It matters that the exponent here is j and not s . A bound in terms of the ambient dimension, used for every j , would be too weak for the comparison.

The data I allow are the ordinary polar degrees, the totals Q_j , and the individual weighted contributions to these totals. Each datum has an index j . For an ordinary polar degree, index j means that one restricts to a generic j -plane and imposes $j-1$ generic tangent-hyperplane conditions.

These data are evaluated on the hypersurface given by a representation. A representation (1) of size m of a homogeneous f of degree d in N variables homogenizes to

$$\det\left(x_0 A_0 + \sum_{i=1}^N x_i A_i\right) = x_0^{m-d} f(x). \quad (23)$$

For $m > d$, its reduced zero set $X_m \subset \mathbb{P}^N$ is the cone over $V(f)$ together with the hyperplane $x_0 = 0$. For $m = d$ only the cone remains. Either way, the reduced degree of X_m is at most $d+1$, however large m is. So by (22) and the positivity proved in Appendix B, every datum of index $j \geq 2$ is at most

$$8d^j + 8j + 8 \leq 16d^j. \quad (24)$$

(For ordinary polar degrees, Bézout's theorem gives this directly, since $(d+1)d^{j-1} \leq 16d^j$.) A datum of index 1 is at most the reduced degree, and so at most m .

On the matrix side, the system of index j imposes $N-j$ generic linear conditions on $[x] \in \mathbb{P}^N$, followed by $j-1$ derivative conditions. The linear conditions turn a generic matrix into a generic matrix in $j+1$ variables. For $j \geq 2$, Theorem 5.1 says that its exact count is $T_{j+1}(m)$. For $j = 1$ the count is $T_2(m) = m$, the number of points where a generic line meets the determinant hypersurface. So any bound $U_j(m)$ for this system that is valid for all matrices of size m satisfies

$$U_j(m) \geq T_{j+1}(m). \quad (25)$$

It is important here that the bound be valid for all matrices. The generic matrices are used only to test a proposed uniform bound, and they need not represent f . An estimate that uses some special property shared by every representation of f would be a different kind of argument, and it is not covered by what follows.

6.2 A precise statement

A comparison of the kind I have in mind works as follows. It takes a finite linear combination of data, $I_m(f) = \left| \sum_{\alpha} \lambda_{\alpha} a_{\alpha}(X_m) \right|$, where a_{α} has index j_{α} . It pairs this with the matched bound

$$U(m) = \sum_{\alpha} |\lambda_{\alpha}| U_{j_{\alpha}}(m), \quad (26)$$

and it excludes size m when $I_m(f) > U(m)$. Of course this is a lower-bound argument only if one has separately proved that $I_m(f) \leq U(m)$ for every representation of f . I do not assume that every datum admits such a proof. The weights on the two sides are matched, so amplifying a datum amplifies its bound as well.

Theorem 6.1 (Ceiling). *Let f be a nonzero homogeneous polynomial of degree $d \geq 2$ in $N \geq 3$ variables. No comparison of the form just described excludes any integer size $m \geq 4dN$. In particular its lower-bound reach is $O(dN)$.*

Proof. Let $j \geq 2$. By (19) with $k = j$,

$$T_{j+1}(m) \geq \binom{m}{j} \binom{2j-2}{j-1} \geq \left(\frac{m}{j}\right)^j,$$

where $\binom{m}{j} \geq (m/j)^j$ follows by multiplying the inequalities $(m-r)/(j-r) \geq m/j$ over $0 \leq r < j$. Since $j \leq N$ and $m \geq 4dN$, this gives $U_j(m) \geq (4d)^j \geq 16d^j$. By (24), this is at least every datum of index j . For $j = 1$, $U_1(m) \geq T_2(m) = m$, which is also at least every datum of index 1. Multiplying by $|\lambda_\alpha|$, summing, and using the triangle inequality gives $I_m(f) \leq U(m)$. $\square$

The theorem still holds if the method first restricts f to fewer variables by a homogeneous linear map, since such a restriction can only decrease the degree and the number of variables.

6.3 The power sum and the permanent

For the power sum, the ceiling can be sharpened so that it matches the constant in the lower bound.

Corollary 6.2. *For $\Phi_{N,N}$, the comparisons covered by Theorem 6.1 cannot certify a size bound larger than $(1 + o(1))N^2/(4e)$.*

Proof. Fix $\varepsilon > 0$ and take $m \geq (1 + \varepsilon)N(N + 1)/(4e)$. Here $d = N$, so by (24) the data of index $j \geq 2$ are at most $16N^j$. I split the indices into two ranges. For $2 \leq j \leq \sqrt{N}$, the bound $T_{j+1}(m) \geq (m/j)^j$ exceeds $16N^j$ once N is large. For $j > \sqrt{N}$, both j and $m/(j + 1)$ tend to infinity uniformly, so (20) applies and gives

$$T_{j+1}(m)^{1/j} \geq (4e - o(1)) \frac{m}{j+1} \geq (1 + \varepsilon)(1 - o(1))N.$$

On the other hand $(16N^j)^{1/j} = (1 + o(1))N$. Index $j = 1$ is handled as in the proof of Theorem 6.1. Summing the matched bounds and letting $\varepsilon \rightarrow 0$ proves the claim. $\square$

So, to leading order, the lower bound $(N - 1)^2/(4e)$ of Theorem 1.1 is the most this method can give for the power sum. The true determinantal complexity may well have a larger constant. For per_n , substituting $d = n$ and $N = n^2$ in Theorem 6.1 gives a ceiling of $O(n^3)$, while the generic smooth slices of Corollary 4.1 reach only $O(n^2)$. A supercubic lower bound for the permanent would need an argument outside the class covered by Theorem 6.1.

7 What remains open

I end with the questions that seem most natural to me.

For homomorphism polynomials, the concrete question is whether better smooth restrictions exist. For H_q , one would like to determine the codimension of the singular locus, or to find a

smooth restriction in more than q variables. Proposition 4.2 rules out restrictions that only keep more entries of X . It does not rule out general linear substitutions.

For the counting method itself, Sections 5 and 6 narrow the options. A better estimate for the same uniform corank-one system cannot change (20), and the counts on singular strata do not change the $O(dN)$ order. To go further, one needs either information that these numbers do not bound, or a property special to representations of f that forces a smaller count than generic matrices do. The ceiling does not apply to arguments based on the rank of the Hessian, to rank methods in general, or to scheme-theoretic information such as the Hilbert function of a dual variety. I have also not proved it for kernel systems of higher corank, or for nearby-cycle invariants, which keep track of the multiplicity of a nonreduced equation. The longer version [13] discusses conditional extensions in these directions. None of the theorems here depend on them.

A The geometric coefficient in the exact count

This appendix proves (18) and shows that it counts distinct points. Let

$$Z = \{([x], [v], [u]) : M(x)v = 0, u^\top M(x) = 0\} \subset \mathbb{P}^{N-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1},$$

and write h, a, b for the hyperplane classes of the three factors. The intersection ring of the product is $\mathbb{Z}[h, a, b]/(h^N, a^m, b^m)$, and integration extracts the coefficient of $h^{N-1}a^{m-1}b^{m-1}$.

Dimension. Consider a generic linear matrix with $N \leq m^2$. Its corank- c locus in $\mathbb{P}^{N-1}$, when nonempty, has codimension c^2 . This follows by taking a generic linear section of the rank strata in the space of all matrices, since the normal space to the stratum at a corank- c matrix B is $\text{Hom}(\ker B, \mathbb{C}^m/\text{im } B)$. Over this locus the pairs of kernel lines form $\mathbb{P}^{c-1} \times \mathbb{P}^{c-1}$. So the part of Z over the corank- c locus has dimension $N - 1 - c^2 + 2c - 2 = N - 2 - (c - 1)^2$. The corank-one part has dimension $N - 2$, and all the other parts are smaller.

Class. The m entries of Mv are sections of $\mathcal{O}(h + a)$. They cut out a subvariety of the expected codimension, whose class is $(h + a)^m$. On this subvariety the m entries of $u^\top M$ satisfy one relation, $\sum_i (u^\top M)_i v_i = 0$. So they form a section of the rank- $(m - 1)$ bundle K defined by

$$0 \longrightarrow K \longrightarrow \mathcal{O}(h + b)^{\oplus m} \xrightarrow{v} \mathcal{O}(h + a + b) \longrightarrow 0.$$

Genericity again gives the expected codimension, and the rule for the top Chern class [4] gives

$$[Z] = (h + a)^m c_{m-1}(K), \quad c_{m-1}(K) = \sum_{r=0}^{m-1} (-1)^r a^r (h + b)^{m-1-r}. \quad (27)$$

The second formula comes from taking the degree- $(m - 1)$ part of $c(K) = (1 + h + b)^m / (1 + h + a + b)$. Alternatively, multiply the claimed polynomial by $h + a + b$, and the sum telescopes to $(h + b)^m - (-a)^m$. Each derivative condition has class $a + b$, so the count is the coefficient of $h^{N-1}a^{m-1}b^{m-1}$ in $[Z](a + b)^{N-2}$. Formula (27) differs from the rational expression in (18) only by a term divisible by a^m , which cannot contribute. This proves (18).

Distinct points. A generic subspace of matrices is transverse to each rank stratum. This means that the induced map $\mathbb{C}^N \rightarrow \text{Hom}(\ker M(x), \mathbb{C}^m/\text{im } M(x))$ is onto. So for any nonzero kernel vector v and cokernel vector u , some $u^\top A_i v$ is nonzero. Hence the derivative conditions form a linear system on Z without base points. By the dimension count, $N - 2$ generic members of this system miss the part of Z of higher corank, and they meet the smooth corank-one part transversally. The intersection therefore consists of reduced points, each counted once. This proves Theorem 5.1, and shows that the count $T_N(m)$ is actually attained.

B Singular contributions and the topological bound

This appendix collects the two standard facts used in Section 6 and proves (22). None of the lower-bound proofs depend on it.

B.1 Splitting the count over strata

I need to split Q_j into a part coming from the smooth locus and parts coming from the singular strata, and to know that each part is nonnegative. Conormal varieties provide this split. For a subvariety $W \subset \mathbb{P}^s$, its conormal variety $\text{Con } W \subset \mathbb{P}^s \times (\mathbb{P}^s)^\vee$ is the closure of the set of pairs (x, H) , where x is a smooth point of W and H is a hyperplane containing the tangent space at x . It has dimension $s - 1$. Its multidegrees

$$\delta_i(\text{Con } W) = \int_{\text{Con } W} h^i \check{h}^{s-1-i}, \quad 0 \leq i \leq s - 1,$$

are nonnegative. For a hypersurface they count the points that satisfy the corresponding point and tangent-hyperplane conditions. The index j of Section 6 corresponds to $j = s - i$.

Let X be a reduced hypersurface. The characteristic cycle of its indicator function has the form

$$(-1)^{s-1} \text{CC}(\mathbf{1}_X) = \sum_S m_S [\text{Con } S], \quad m_S \geq 0. \quad (28)$$

Here S runs through suitable smooth strata, the coefficient on each open component is one, and the normalization is $\text{CC}(\text{Eu}_W) = (-1)^{\dim W} [\text{Con } W]$, where Eu_W is the local Euler obstruction. The coefficients m_S are nonnegative for the following reason. A reduced hypersurface is a local complete intersection, so the shifted constant sheaf $\mathbb{Q}_X[s - 1]$ is perverse, and characteristic cycles of perverse sheaves are effective [1, 5]. This is the only positivity I use. In particular, I make no claim about signed combinations of conormal varieties in general.

The projective index formula [3] gives

$$\sum_S m_S \delta_{s-j}(\text{Con } S) = (-1)^{j-1} (\chi_j - 2\chi_{j-1} + \chi_{j-2}) = Q_j(X). \quad (29)$$

The signs here are easy to get wrong, so I explain how to check them. The index formula is a universal linear relation between two kinds of numbers attached to a constructible function φ . On one side are the signed multidegrees $\Delta_i(\varphi)$ of its characteristic cycle, including the coefficient of the zero section. On the other side are the Euler characteristics of its generic slices. Because the relation is linear and universal, its coefficients and signs can be checked on a basis. Take the indicator function of a linear subspace $\mathbb{P}^r \subset \mathbb{P}^s$. Its slice values are $\chi_j = \max(0, r + j - s + 1)$, whose second difference is 1 at $j = s - r$ and 0 elsewhere. Its conormal multidegrees are a single 1 at $i = r$, with sign $(-1)^r$. These vectors form a basis of the numerical data. Hence $\Delta_i(\varphi) = (-1)^i (\chi_{s-i} - 2\chi_{s-i-1} + \chi_{s-i-2})$ for every φ , and the sign $(-1)^{s-1}$ for hypersurfaces gives (29). Since every summand on the left of (29) is nonnegative, each one is at most the total. These summands are exactly the weighted stratum contributions allowed in Section 6.

B.2 Bounding the Euler characteristics

Let g be a nonzero homogeneous polynomial of degree $d \geq 2$ in $j + 1$ variables. Its projective zero set may be singular, which makes it awkward to work with directly. So I trade it for a smooth affine variety. Let $F = \{z \in \mathbb{C}^{j+1} : g(z) = 1\}$. This is smooth, because Euler's identity

$\sum z_i \partial_i g = dg$ shows that the gradient of g cannot vanish on F . The map $z \mapsto [z]$ is a d -sheeted cover of $U = \mathbb{P}^j \setminus V(g)$, since each line through the origin outside $V(g)$ meets F in exactly d points. Therefore

$$\chi(F) = d\chi(U), \quad \chi(V(g)) = j + 1 - \chi(U). \quad (30)$$

To bound $\chi(F)$, I slice it with generic affine hyperplanes. On a smooth slice of F of dimension r , the critical points of a generic linear function are isolated Morse points. In suitable coordinates they are cut out by $g = 1$, the slice equations, and r first derivatives of g , so Bézout's theorem bounds their number by $d(d-1)^r$. For a generic affine Lefschetz pencil on a smooth closed affine variety Y of dimension r ,

$$\chi(Y) = \chi(Y \cap H) + (-1)^r \#\{\text{critical points}\}.$$

I use this in the version that also controls the hyperplane at infinity, which is the smooth case of the formula of Seade, Tibăr and Verjovsky [11, Lemma 3.1 and formula (2)]. Iterating down to a zero-dimensional section, which has at most d points, gives $|\chi(F)| \leq d \sum_{r=0}^j (d-1)^r$. The sum is at most $2(d-1)^j$ for $d \geq 3$, and equals $j+1$ for $d=2$. So (30) yields

$$|\chi(V(g))| \leq E(d, j) := 2(d-1)^j + 2j + 2. \quad (31)$$

Note that smoothness was needed only for F , where it is automatic. So g itself may be singular, or even nonreduced.

Now apply (31) to the generic linear sections of X . Since $E(d, j)$ is nondecreasing in j , the three terms in (21) give

$$Q_j(X) \leq E(d, j) + 2E(d, j-1) + E(d, j-2) \leq 4E(d, j) = 8(d-1)^j + 8j + 8.$$

For $j=1$ use $\chi_0 = \chi_{-1} = 0$. In fact $Q_1 = \deg X$. This proves (22). By the positivity in (28), each individual contribution is bounded by the same number, however many strata occur.

Finally, the multiplicity of an equation plays no role in any of this. The polynomials g and g^a have the same reduced zero set, and so the same Q_j . Nearby-cycle invariants do keep track of this multiplicity. They would need a separate estimate, and this is why Theorem 6.1 does not cover them.

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