Electronic Colloquium on Computational Complexity, Report No. 301 (2026)

Asymptotic non-negative rank can be exponentially smaller than non-negative rank

Pavel Hrubeš*

November 2026

Abstract

We investigate the behavior of the non-negative rank of a matrix, rk_+, on Kronecker product, $\times$. Beasley et al. [2013] constructed matrices with $\mathrm{rk}_+(M \times M)$ strictly smaller than $\mathrm{rk}_+(M)^2$. We will show that

(i). there exists an M with $\mathrm{rk}_+(M \times M) \leq O(\mathrm{rk}_+(M) \log(\mathrm{rk}_+(M))$.

Defining asymptotic non-negative rank as $\overline{\mathrm{rk}}_+(M) := \lim_{k\to\infty} \sqrt[k]{\mathrm{rk}_+(M^{(k)})}$, where $M^{(k)}$ is the k-th Kronecker power, we show that

(ii). there exists an M with $\overline{\mathrm{rk}}_+(M) \leq O(\log \mathrm{rk}_+(M))$,

(iii). there exist M, N with $\overline{\mathrm{rk}}_+(M \times N) < \overline{\mathrm{rk}}_+(M)\,\overline{\mathrm{rk}}_+(N)$.

We also refute a conjecture of Vandaele et al. [2016] that $\mathrm{rk}_+(M \times M) \geq \Omega(\mathrm{rk}_+(M)\,\mathrm{rk}(M))$.

1 Introduction

The non-negative rank of a matrix M, $\mathrm{rk}_+(M)$, can be defined as the smallest k so that M can be written as the sum of k non-negative matrices of rank one. This quantity has numerous applications in linear optimization, communication complexity, and other fields. In [13], Yannakakis formulated a geometric interpretation of non-negative rank in terms of linear projections of polytopes, and also noted a connection between non-negative rank and non-deterministic communication complexity. The geometric interpretation has spawned many applications, e.g., [2, 7]. The communication angle allows to characterize deterministic communication of a function in terms of the non-negative rank of its communication matrix, see [10] or the monograph [11].

In this paper, we address the behavior of non-negative rank under the Kronecker product. Non-negative rank shares many properties with the usual linear-algebraic rank, but multiplicativity on Kronecker product is not one of them. In [1], [12], Beasley et al. and Vandaele et al. present specific 4×4 matrices M with

*Institute of Mathematics of ASCR, Prague, Czech Republic. hrubes@math.cas.cz. This work was supported by Czech Science Foundation GAČR grant 25-16311S.

ISSN 1433-8092

$\mathrm{rk}_+(M \times M)$ strictly smaller than $\mathrm{rk}_+(M)^2$. Our first contribution, Theorem 4, is to make this gap more robust: we present a matrix M with non-negative rank r such that the non-negative rank of $M \times M$ is $O(r \log r)$.

The *asymptotic non-negative rank* of a matrix M is $\lim_{k\to\infty} \sqrt[k]{\mathrm{rk}_+(M^{(k)})}$ where $M^{(k)}$ is the k-th Kronecker power of M. This quantity was investigated by Chee, Le and Ta [6] where the authors study Strassen's asymptotic spectrum of non-negative matrices. Braun et al. [3] (building on [5, 4]) have shown that asymptotic non-negative rank is closely related to Wyner's common information. For Boolean matrices, there is a kinship between asymptotic non-negative rank and amortized communication complexity (cf. Remark 7 below).

Our second contribution, Theorem 6, shows that asymptotic non-negative rank can be exponentially smaller than non-negative rank. In addition, in Theorem 10, we show that $\overline{\mathrm{rk}}_+$ itself is not multiplicative on Kronecker product. In the language of [6], this means that $\overline{\mathrm{rk}}_+$ is not an element of the asymptotic non-negative spectrum.

Our gap between $\overline{\mathrm{rk}}_+$ and rk_+ is based on the fact (proved in [8]) that rk_+ is a highly discontinuous quantity. In contrast, we point out in Section 5 that $\overline{\mathrm{rk}}_+$ is much less so: it does not change a lot under small perturbations.

2 Preliminaries

A real matrix M is non-negative, $M \geq 0$, if all its entries are. If M is an $m \times n$ matrix M, its *non-negative rank*, $\mathrm{rk}_+(M)$, is the smallest s such that M can be written as

$$M = AB\,,$$

where A and B are non-negative matrices of dimensions $m \times s$ and $s \times n$, respectively. Alternatively, it is the smallest s so that M can be written as a sum of non-negative rank-one matrices. If $M \not\geq 0$, we set $\mathrm{rk}_+(M) := \infty$.

$M \times N$ will stand for the Kronecker (tensor) product, and

$$M^{(k)} := \underbrace{M \times \cdots \times M}_{k}\,.$$

The *asymptotic non-negative rank* of M is $\overline{\mathrm{rk}}_+(M) = \lim_{k\to\infty} \sqrt[k]{\mathrm{rk}_+(M^{(k)})}$. The limit exists by Fekete's lemma and equals $\inf_k \sqrt[k]{\mathrm{rk}_+(M^{(k)})}$. Observe that

$$\mathrm{rk}(M) \leq \overline{\mathrm{rk}}_+(M) \leq \mathrm{rk}_+(M)\,,$$

where rk is the usual linear-algebraic rank.

Here are some further properties of rk_+ and $\overline{\mathrm{rk}}_+$ that we will use. For rk_+, they follow directly from definitions; for $\overline{\mathrm{rk}}_+$ one can consult [6] if needed.

Fact 1. *Let $M, N \geq 0$.*

(i). $\mathrm{rk}_+(M) \leq \min(m, n)$*, if M has dimension $m \times n$.*

(ii). $\mathrm{rk}_+(aM) = \mathrm{rk}_+(M)$*, if* $0 < a \in \mathbb{R}$.

(iii). $\mathrm{rk}_+(M+N) \le \mathrm{rk}_+(M)+\mathrm{rk}_+(N)$, *provided* M, N *have the same dimension.*

(iv). $\mathrm{rk}_+(M \times N) \le \mathrm{rk}_+(M) \cdot \mathrm{rk}_+(N)$.

The above properties hold also for $\overline{\mathrm{rk}}_+$. *In addition,*

(v). $\overline{\mathrm{rk}}_+(M^{(k)}) = \overline{\mathrm{rk}}_+(M)^k$.

Given real matrices M, N of the same dimension, $M \ge N$ stands for entry-wise inequality (i.e., $M_{i,j} \ge N_{i,j}$ for every i, j). $J_{m,n}$ is the all-ones $m \times n$ matrix and $J_n := J_{n,n}$. The subscripts will be omitted if clear from context.

3 An exponential gap between rk_+ and $\overline{\mathrm{rk}}_+$

Our main construction is based on the following strong discontinuity of non-negative rank.

Fact 2 ([8], Corollary 3.5)**.** *For every* n *there exists an* $n \times n$ *matrix* Q *with positive integer entries such that* $\mathrm{rk}_+(Q) = O(\log(n))$ *but for every* $\epsilon > 0$, $\mathrm{rk}_+(Q - \epsilon J) \ge n^{\Omega(1)}$.

Lemma 3. *Let* S, T *be matrices of the same dimension,* R *an arbitrary matrix and* $z \in \mathbb{R}$ *non-negative. Then* $(S-T) \times R \times (S - zT)$ *is a non-negative linear combination of the matrices* $(S-(z+1)T) \times R \times T$, $S \times R \times (S-(z+1)T))$, *and* $T \times R \times T$.

Proof. The following identity is easily verified:

$$(S-T)\times R \times (S-zT) = \frac{zS \times R \times (S-(z+1)T) + (S-(z+1)T) \times R \times S}{z+1} + zT \times R \times T\,,$$

□

Theorem 4. *For every* r_0 *there exists a non-negative square matrix* M *(of a polynomial dimension in* r_0*) such that* M *has non-negative rank* $r \ge r_0$ *and* $\mathrm{rk}_+(M \times M) \le O(r \log r)$.

Proof. Let Q be the matrix from Fact 2. Let $r := \lim_{\epsilon \to 0_+}(Q - \epsilon J)$. Fix an $\epsilon_0 > 0$ such that $\mathrm{rk}_+(Q - \epsilon J) = r$ for every $\epsilon \in (0, \epsilon_0]$. To see that both the limit and ϵ_0 exist follows from the fact that $\mathrm{rk}_+(Q - \epsilon J)$ is an integer-valued function of ϵ with finitely many discontinuities (cf. Observation 3.8 in [8]). Let M be the matrix

$$M := Q - \frac{\epsilon_0}{2} J\,.$$

The previous lemma, applied with $z = 1$, $S = Q$, $T = \frac{\epsilon_0}{2} J$ and $R = (1)$, shows that $M \times M$ is a non-negative linear combination of

$$Q \times (Q - \epsilon_0 J)\,,\ (Q - \epsilon_0 J) \times Q\,,\ J \times J\,.$$

Using the properties from Fact 1, and since J is a rank-one matrix, we have

$$\mathrm{rk}_+(M \times M) \leq 2\,\mathrm{rk}_+(Q)\,\mathrm{rk}_+(Q - \epsilon_0 J) + 1\,.$$

Finally, $\mathrm{rk}_+(M), \mathrm{rk}_+(Q - \epsilon_0 J) = r$ whereas $\mathrm{rk}_+(Q) = O(\log r)$ and hence $\mathrm{rk}_+(M \times M) \leq O(r \log r)$. □

Lemma 5. *Let $S, T \geq 0$ be matrices of the same dimension and and $k \geq 1$ an integer such that $S - kT \geq 0$. Let $s := \mathrm{rk}_+(S)$, $t := \mathrm{rk}_+(T)$, and similarly for $\overline{s}, \overline{t}$ and their respective asymptotic ranks. Then*

$$\begin{aligned} \mathrm{rk}_+((S-T)^{(k)}) &\leq k \cdot \mathrm{rk}_+(S - kT)(s+t)^{k-1} + t^k\,, \\ \overline{\mathrm{rk}}_+(S-T)^k &\leq k \cdot \overline{\mathrm{rk}}_+(S - kT)(\overline{s}+\overline{t})^{k-1} + \overline{t}^k\,. \end{aligned}$$

Proof. For a set of matrices $\mathcal{M}$ of the same dimension, $\mathrm{span}_+(\mathcal{M})$ will denote the set of their non-negative linear combinations. Let $\mathcal{N}_i$ be the set of matrices of the form $A_1 \times \cdots \times A_i$ with $A_1, \ldots, A_i \in \{S, T\}$. ($\mathcal{N}_0$ will consist of the single 1×1 matrix (1).) Let $\mathcal{M}_k$ be the set of matrices

$$\{B_1 \times (S - kT) \times B_2 : B_1 \in \mathcal{N}_i,\, B_2 \in \mathcal{N}_{k-1-i}, 0 \leq i \leq k-1\}\,.$$

We want to show that for every $k \geq 1$

$$(S-T)^{(k)} \in \mathrm{span}_+(\mathcal{M}_k \cup \mathcal{N}_k) \tag{1}$$

This is proved by induction on k. If $k = 1$, $S - T \in \mathcal{M}_1$ and (1) holds. For the inductive step, assume that $(S-T)^{(k)} \in \mathrm{span}_+(\mathcal{M}_k \cup \mathcal{N}_k)$, and we must show $(S-T)^{(k+1)} \in \mathrm{span}_+(\mathcal{M}_{k+1} \cup \mathcal{N}_{k+1})$. Since $(S-T)^{(k+1)} = (S-T) \times (S-T)^{(k)}$, it is enough to show that for every $B \in \mathcal{M}_k \cup \mathcal{N}_k$, $(S-T) \times B \in \mathrm{span}_+(\mathcal{M}_{k+1} \cup \mathcal{N}_{k+1})$. If $B \in \mathcal{N}_k$, we have $(S-T) \times B = (S - (k+1)T) \times B + kT \times B$, where these terms are in $\mathcal{M}_{k+1}$ and $\mathrm{span}_+(\mathcal{N}_{k+1})$, respectively. If $B \in \mathcal{M}_k$ is of the form $B_1 \times (S - kT) \times B_2 : B_1 \in \mathcal{N}_i,\, B_2 \in \mathcal{N}_{k-1-i}$, Lemma 3 expresses $(S-T) \times B$ as a non-negative linear combination of the three matrices

$$S \times B_1 \times (S - (k+1)T) \times B_2,\ (S - (k+1)T) \times B_1 \times S \times B_2,\ T \times B_1 \times T \times B_2\,.$$

The first two are in $\mathcal{M}_{k+1}$ and the third one is in $\mathcal{N}_{k+1}$. This proves (1).

Furthermore, it holds that $\mathcal{N}_k \subseteq \mathrm{span}_+(\mathcal{M}_k \cup \{T^{(k)}\})$. For if $B = A_1 \times \cdots \times A_k \in \mathcal{N}_k$ and there is some i with $A_i = S$, we can replace A_i with $(S - kT) + kT$ to write $B = B' + kB''$ where $B' \in \mathcal{M}_k$ and B'' has one less occurrence of S.

We can now conclude that

$$(S-T)^{(k)} \in \mathrm{span}_+(\mathcal{M}_k \cup \{T^{(k)}\})$$

Applying the properties from Fact 1

$$\begin{aligned} \mathrm{rk}_+((S-T)^{(k)}) &\leq \sum_{M \in \mathcal{M}_k \cup \{T^{(k)}\}} \mathrm{rk}_+(M) \\ &\leq k\,\mathrm{rk}_+(S - kT) \sum_{i=0}^{k-1} \binom{k-1}{i} s^i t^{k-1-i} + t^k \\ &= k\,\mathrm{rk}_+(S - kT)(s+t)^{k-1} + t^k\,. \end{aligned}$$

The same bound holds for $\overline{\mathrm{rk}}_+((S-T)^{(k)})$ in term of $\overline{s}, \overline{t}$, and we note that $\overline{\mathrm{rk}}_+((S-T)^{(k)}) = \overline{\mathrm{rk}}_+(S-T)^k$.

□

Theorem 6. *For every r_0 there exists a non-negative square matrix M (of a polynomial dimension in r_0) such that $\mathrm{rk}_+(M) > r_0$ and $\overline{\mathrm{rk}}_+(M) \leq O(\log(\mathrm{rk}_+(M))$.*

Proof. Let Q be the matrix from Fact 2. Since Q is positive and integer valued, we have $Q - J \geq 0$ and $\mathrm{rk}_+(Q-J) \leq n$. Fix a k large enough so that $(kn)^{\frac{1}{k}} \leq 2$. Let

$$M := (Q - \frac{1}{k}J)\,.$$

Then $\mathrm{rk}_+(M) = n^{\Omega(1)}$. Lemma 5 gives

$$\begin{aligned}\mathrm{rk}_+(M^{(k)}) &\leq k \cdot \mathrm{rk}_+(Q-J)(\mathrm{rk}_+(Q)+1)^{k-1} + 1 \\ &\leq kn \cdot (\mathrm{rk}_+(Q)+1)^k\,.\end{aligned}$$

Hence $\overline{\mathrm{rk}}_+(M) \leq (\mathrm{rk}_+(M^{(k)})^{\frac{1}{k}}) \leq 2(\mathrm{rk}_+(Q)+1) \leq O(\log n)$. □

Remark 7. *Such a gap between $\overline{\mathrm{rk}}_+$ and rk_+ cannot be achieved by Boolean matrices. If M is a matrix with $0/1$-entries then*

$$\log(\mathrm{rk}_+(M)) \leq O(\log(\overline{\mathrm{rk}}_+(M))\log(\mathrm{rk}(M))) \leq O(\log^2(\overline{\mathrm{rk}}_+(M)))\,.$$

Proof sketch. A boolean matrix $M \in \{0,1\}^{n\times n}$ can be viewed as a communication matrix of a Boolean function $f : [n] \times [n] \to \{0,1\}$. Denote $n^1(M)$ the 1-non-deterministic communication complexity of f (i.e., the $\log_2$ of the number of 1-rectangles needed to cover the 1-entries of the matrix). Then we have

$$n^1(M) \leq \log_2 \mathrm{rk}_+(M) \leq O(n^1(M)\log(\mathrm{rk}(M)))\,,$$

where the second inequality follows from Lemma 2.15 in [11]. $M^{(k)}$ corresponds to the conjunction of k independent copies of f. Hence, by the direct sum theorem for n^1 (see Corollary 4.9 in [9])

$$k(n^1(M) - \log\log n - O(1)) \leq n^1(M^{(k)})\,,$$

which implies $n^1(M) \leq \log_2 \overline{\mathrm{rk}}_+(M) + \log\log n + O(1)$, and hence

$$\log(\mathrm{rk}_+(M)) \leq O(\log \overline{\mathrm{rk}}_+(M) + \log\log n)\log(\mathrm{rk}(M))\,.$$

Finally, we can assume that $n \leq 2^{\mathrm{rk}(M)}$, otherwise M would contain duplicate rows or columns. □

4 $\overline{\mathrm{rk}}_+$ is not multiplicative on Kronecker product

The following lemma shows that for *every* positive M, there exists an N such that $\mathrm{rk}_+(M \times N)$ drops to roughly $\mathrm{rk}(M)\,\mathrm{rk}(N)$ – which gives other examples where $\mathrm{rk}_+(M \times N) < \mathrm{rk}_+(M)\,\mathrm{rk}_+(N)$.

Lemma 8. *Let M be an $m \times m$ matrix with positive entries. Let $r_-(M) := \min_{\epsilon>0} \mathrm{rk}_+(M - \epsilon J_m)$. Then for every n, there exists a non-singular $n \times n$ matrix $N \geq 0$ such that*

$$\mathrm{rk}_+(M \times N) \leq n(\mathrm{rk}(M) + 2) + r_-(M) \leq n(\mathrm{rk}(M) + 2) + m\,.$$

Proof. The following identity is easy to verify

$$M \times N = (M + tJ_m) \times (N - J_n) + (M - \epsilon J_m) \times J_n + J_m \times ((t+\epsilon)J_n - tN)\,. \tag{2}$$

By Lemma 3.9 of [8], we can fix $t > 0$ such that $\mathrm{rk}_+(M + tJ_m) \leq \mathrm{rk}(M) + 1$. Let $\epsilon > 0$ be such that $\mathrm{rk}_+(M - \epsilon J_m) = r_-(M)$. . Let $N := J_n + \frac{\epsilon}{t} I_n$. Then N is non-singular, and $N - J_n$, $(t+\epsilon)J_n - tN$ are non-negative with rk_+ at most n. The identity 2 gives

$$\begin{aligned}\mathrm{rk}_+(M \times N) \leq& \mathrm{rk}_+(M + tJ_m)\,\mathrm{rk}_+(N - J_n) + \mathrm{rk}_+(M - \epsilon J_m) \\ &+ \mathrm{rk}_+((t+\epsilon)J_n - tN)) \\ \leq& (\mathrm{rk}(M) + 1)n + r_-(M) + n = (\mathrm{rk}_+(M) + 2)n + r_-(M)\,.\end{aligned}$$

□

The proof of the following lemma is delegated to Section 4.1.

Lemma 9. *There exists a positive matrix Q with $\overline{\mathrm{rk}}_+(Q) > \mathrm{rk}(Q)$.*

Theorem 10. *There exist matrices $M, N \geq 0$ with $\overline{\mathrm{rk}}_+(M \times N) < \overline{\mathrm{rk}}_+(M)\,\overline{\mathrm{rk}}_+(N)$.*

Proof. Let Q be the matrix from the previous lemma. Since $\overline{\mathrm{rk}}_+(Q^{(k)}) = \overline{\mathrm{rk}}_+(Q)^k$ and $\mathrm{rk}(Q^{(k)}) = \mathrm{rk}(Q)^k$, we can find a k such that for $M := Q^{(k)}$, $\overline{\mathrm{rk}}_+(M) > \mathrm{rk}(M) + 3$. If M is $m \times m$, let N be the non-singular $m \times m$ matrix from Lemma 8. Then

$$\overline{\mathrm{rk}}_+(M \times N) \leq (\mathrm{rk}(M) + 3)m < \overline{\mathrm{rk}}_+(M)m = \overline{\mathrm{rk}}_+(M)\,\overline{\mathrm{rk}}_+(N)\,,$$

where $\overline{\mathrm{rk}}_+(N) = m$ because N is non-singular.. □

Let

$$M \oplus N := \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

Observe that for M, N as in Theorem 10 we also have $\overline{\mathrm{rk}}_+(M \oplus N) < \overline{\mathrm{rk}}_+(M) + \overline{\mathrm{rk}}_+(N)$.

In [12], Vandalae et al. conjectured that $\mathrm{rk}_+(A \times A) \geq \mathrm{rk}_+(A)\,\mathrm{rk}(A)$ for every $A \geq 0$. We show that even a weaker form of this conjecture fails.

Proposition 11. *Let c be a constant such that $\mathrm{rk}_+(A\times A) \geq \Omega((\mathrm{rk}_+(A)\,\mathrm{rk}(A))^c)$ for every $A \geq 0$. Then $c \leq 2/3$.*

Proof. Let M be the matrix constructed in the proof of Theorem 4 with $\mathrm{rk}_+(M) = r$ and $\mathrm{rk}_+(M \times M) \leq O(r \log r)$. Inspecting the proof of Theorem 4, one can see that $r_-(M) \leq r$ and $\mathrm{rk}(M) = O(\log r)$. Let $k := \lceil \sqrt{r} \rceil$ and let N be the $k \times k$ non-singular matrix from Lemma 8 such that $\mathrm{rk}_+(M \times N) \leq (\mathrm{rk}(M)+2)k + r_-(M) \leq O(r)$. Let $A := M \oplus N$. Then $\mathrm{rk}_+(A) \geq r$, $\mathrm{rk}(A) \geq \sqrt{r}$ and hence

$$\mathrm{rk}_+(A)\,\mathrm{rk}(A) \geq r^{3/2}\,.$$

On the other hand,

$$\mathrm{rk}_+(A \times A) \leq \mathrm{rk}_+(M \times M) + 2\,\mathrm{rk}_+(M \times N) + \mathrm{rk}_+(N \times N) \leq O(r \log r)\,.$$

□

4.1 $\overline{\mathrm{rk}}_+$ of approximate fooling matrices

Here we prove Lemma 9. Let us remark that the lemma is almost obvious if M is required to be *non-negative* instead of *positive*. It is also likely that an *exponential* gap between $\overline{\mathrm{rk}}_+$ and rk for Q positive can be obtained by the information theoretic machinery presented by Braun et al. in [3], but the author prefers to give a self-contained proof of a weaker statement.

Let $M \geq 0$ be a square matrix. Define $\tau(M) = \max_V \mathrm{tr}(V)$, where the maximum is taken over all rank-one matrices with $0 \leq V \leq M$. If $\mathrm{tr}(M) > 0$, we have

$$\mathrm{rk}_+(M) \geq \frac{\mathrm{tr}(M)}{\tau(M)}\,. \tag{3}$$

For if $\mathrm{rk}_+(M) = r$, we can write $M = \sum_{i=1}^r V_i$ where V_i are non-negative rank-one matrices and so $\mathrm{tr}(M) = \sum_{i=1}^r \mathrm{tr}(V_i)$. Non-negativity implies $0 \leq V_i \leq M$ and so $\mathrm{tr}(V_i) \leq \tau(M)$ which means $\mathrm{tr}(M) \leq r \cdot \tau(M)$.

$M \geq 0$ will be called a *z-fooling matrix* if $M_{i,i} = 1$ for every i and $M_{i,j}M_{j,i} \leq z$ for every $i \neq j$.

Lemma 12. *If M is an $n \times n$ z-fooling matrix then $\tau(M^{(k)}) \leq (1 + \sqrt{z}n)^k$.*

Proof. We will assume $z \in [0, 1)$ otherwise the statement is obvious. We will index rows and columns of M by $[n] := \{1, \ldots, n\}$, so that rows and columns of $M^{(k)}$ are indexed by strings $\sigma \in [n]^k$. Let $0 \leq V \leq M^{(k)}$ be a rank-one matrix. Then for every σ, σ', we have

$$V_{\sigma,\sigma}V_{\sigma',\sigma'} \leq z^{h(\sigma,\sigma')}\,,$$

where $h(\sigma, \sigma')$ is the Hamming distance (the number of coordinates where the strings differ). This can be seen as follows. Since V is rank-one and $V \leq M^{(k)}$, we have

$$V_{\sigma,\sigma}V_{\sigma',\sigma'} = V_{\sigma,\sigma'}V_{\sigma',\sigma} \leq M^{(k)}_{\sigma,\sigma'}M^{(k)}_{\sigma',\sigma} = \prod_{i=1}^k M_{\sigma(i),\sigma'(i)}M_{\sigma(i),\sigma'(i)}\,.$$

Since M is z-fooling, we have $M_{\sigma(i),\sigma'(i)}M_{\sigma'(i),\sigma(i)} = 1$ if $\sigma(i) = \sigma'(i)$ and $M_{\sigma(i),\sigma'(i)}M_{\sigma'(i),\sigma(i)} \le z$ otherwise, and so $\prod_{i=1}^{k} M_{\sigma(i),\sigma'(i)}M_{\sigma(i),\sigma'(i)} \le z^{h(\sigma,\sigma')}$.

Let $v_\sigma := V_{\sigma,\sigma}$. Then $\mathrm{tr}(V) = \sum_{\sigma\in[n]^k} v_\sigma$, and we want to bound the sum under the assumptions $v_\sigma \ge 0$ and $v_\sigma v_{\sigma'} \le z^{h(\sigma,\sigma')}$ for every[1] $\sigma,\sigma' \in [n]^k$. Given $\ell \in \{0,\dots,k\}$, let

$$S_\ell := \{\sigma : z^{\frac{\ell+1}{2}} < v_\sigma \le z^{\frac{\ell}{2}}\}.$$

Hence

$$\sum_\sigma v_\sigma \le \sum_{\ell=0}^{k} z^{\frac{\ell}{2}}|S_\ell| + z^{\frac{k+1}{2}}(n^k - \sum_{\ell=0}^{k}|S_\ell|). \tag{4}$$

For every $\sigma,\sigma' \in S_\ell$ we have $z^{\ell+1} < v_\sigma v_{\sigma'} \le z^{h(\sigma,\sigma')}$ and so $h(\sigma,\sigma') \le \ell$. Hence $|S_\ell| \le s_\ell := \binom{k}{\ell} n^\ell$. Since $\sum_{\ell=0}^{k} s_\ell = (1+n)^k \ge n^k$, we can bound (4) as

$$\sum_\sigma v_\sigma \le \sum_{\ell=0}^{k} z^{\frac{\ell}{2}} s_\ell = \sum_{\ell=0}^{k} z^{\frac{\ell}{2}} \binom{k}{\ell} n^\ell = (1+\sqrt{z}n)^k .$$

□

The previous lemma and (3) directly imply:

Proposition 13. *Let M be an $n\times n$ z-fooling matrix. Then $\overline{\mathrm{rk}}_+(M) \ge \frac{n}{1+\sqrt{z}n}$.*

Proof of Lemma 9. Let Q_z be the matrix

$$Q_z = \begin{pmatrix} 1 & 1 & z & z \\ z & 1 & 1 & z \\ z & z & 1 & 1 \\ 1 & z & z & 1 \end{pmatrix}.$$

For $z \in [0,1)$, Q_z is a z-fooling matrix and $\mathrm{rk}(Q_z) = 3$. On the other hand, $\overline{\mathrm{rk}}_+(Q_z) \ge 4/(1+4\sqrt{z}) > 3$ for every $z > 0$ sufficiently small. □

5 Continuity of $\overline{\mathrm{rk}}_+$

Observe that Lemma 3 and 5 express an approximate form of *log-convexity* of $\overline{\mathrm{rk}}_+$. Suppose $M \ge N \ge 0$ with N being a rank-one matrix. Then Lemma 3 gives

$$\overline{\mathrm{rk}}_+(M)^2 \le 2\,\overline{\mathrm{rk}}_+(M-N)\,\overline{\mathrm{rk}}_+(M+N) + 1 .$$

A similar bound on $\overline{\mathrm{rk}}_+(M)^k$ in terms of $\overline{\mathrm{rk}}_+(M-(k-1)N)$ and $\overline{\mathrm{rk}}_+(M)$ follows from Lemma 5 for a general k. These properties give global constraints on the asymptotic non-negative rank function. Locally, this gives the following bound on how much can $\overline{\mathrm{rk}}_+$ change under perturbations.

[1] Under the convention $0^0 = 1$, this entails $v_\sigma \le 1$.

Theorem 14. *Let $M \geq N \geq 0$ be matrices of the same dimension. Then*

$$\lim_{\epsilon \to 0_+} \sup_{\lambda \in [-\epsilon, \epsilon]} |\overline{\mathrm{rk}}_+(M + \lambda N) - \overline{\mathrm{rk}}_+(M)| \leq \overline{\mathrm{rk}}_+(N)\,.$$

Proof. Let $M_\lambda := M + \lambda N$ and $r(\lambda) := \overline{\mathrm{rk}}_+(M_\lambda)$. Then M_λ is non-negative for every $\lambda \geq -1$. Given $\epsilon_k := \frac{1}{2k+1}$ with k sufficiently large, we want to bound $|r(\lambda_2) - r(\lambda_1)|$ for $\lambda_1, \lambda_2 \in [-\epsilon_k, \epsilon_k]$.

Without loss of generality we can assume $\lambda_1 \leq \lambda_2$. Since

$$M_{\lambda_2} = M_{\lambda_1} + (\lambda_2 - \lambda_1)N\,,$$

we have

$$r(\lambda_2) - r(\lambda_1) \leq \overline{\mathrm{rk}}_+(N)\,. \tag{5}$$

If the matrices have dimension $m_1 \times m_2$, let $m := \min(m_1, m_2)$ so that $\overline{\mathrm{rk}}_+(N), r(\lambda) \leq m$ whenever $\lambda \geq -1$. Lemma 5, applied to the matrices $S := M_{\lambda_2}$ and $T := (\lambda_2 - \lambda_1)N$, shows that

$$\begin{aligned} r(\lambda_1)^k \leq& k \cdot r(\lambda_1 - k(\lambda_2 - \lambda_1))(r(\lambda_2) + \overline{\mathrm{rk}}_+(N))^{k-1} + \overline{\mathrm{rk}}_+(N)^k \\ \leq& 2km(r(\lambda_2) + \overline{\mathrm{rk}}_+(N))^k\,, \end{aligned}$$

where we used the fact that $\lambda_1 - k(\lambda_2 - \lambda_1) \geq -1$ for $\lambda_1, \lambda_2 \in [-\epsilon_k, \epsilon_k]$. Hence $r(\lambda_1) \leq (2km)^{\frac{1}{k}}(r(\lambda_2) + \overline{\mathrm{rk}}_+(N))$ and

$$\begin{aligned} r(\lambda_1) - r(\lambda_2) - \overline{\mathrm{rk}}_+(N) \leq& ((2km)^{\frac{1}{k}} - 1)(\overline{\mathrm{rk}}_+(N) + r(\lambda_2) \\ \leq& ((2km)^{\frac{1}{k}} - 1)2m\,. \end{aligned}$$

Since $\lim_{k\to\infty}(2km)^{\frac{1}{k}} = 1$, this shows that

$$\lim_{k \to \infty} \sup_{-\epsilon_k \leq \lambda_1 \leq \lambda_2 \leq \epsilon_k} (r(\lambda_1) - r(\lambda_2)) \leq \overline{\mathrm{rk}}_+(N)\,.$$

Together with inequality (5), this implies $\lim_{k\to\infty} \sup_{\lambda_1, \lambda_2 \in [-\epsilon_k, \epsilon_k]} |r(\lambda_1) - r(\lambda_2))| \leq \overline{\mathrm{rk}}_+(N)$. □

Remark 15. *If M is positive and N arbitrary, the bound can be improved to*

$$\lim_{\epsilon \to 0_+} \sup_{\lambda \in [-\epsilon, \epsilon]} |\overline{\mathrm{rk}}_+(M + \lambda N) - \overline{\mathrm{rk}}_+(M)| \leq \mathrm{rk}(N) + 2\,.$$

Acknowledgment We thank Anup Rao, Siddharth Iyer and Mika Göös for inspiring discussions.

References

[1] LeRoy B. Beasley, Hartmut Klauck, Troy Lee, and Dirk Oliver Theis. Communication complexity, linear optimization, and lower bounds for the nonnegative rank of matrices (Dagstuhl Seminar 13082). *Dagstuhl Reports*, 3(2):127–143, 2013.

[2] G. Braun, S. Fiorini, S. Pokutta, and D. Steuer. Approximation limits of linear programs (beyond hierarchies). *Math. of Operations Research*, 40(3), 2015.

[3] Gábor Braun, Rahul Jain, Troy Lee, and Sebastian Pokutta. Information-theoretic approximations of the nonnegative rank. *Computational Complexity*, 26(1):147–197, 2017.

[4] Gábor Braun and Sebastian Pokutta. Common information and unique disjointness. *Algorithmica*, 76(3):597–629, 2016.

[5] Mark Braverman and Ankur Moitra. An information complexity approach to extended formulations. In *Symposium on the Theory of Computing*, pages 161–170, 2013.

[6] Yeow Meng Chee, Quoc-Tung Le, and Duy Hoang Ta. On the asymptotic nonnegative rank of matrices and its applications in information theory. In *International Symposium on Information Theory (ISIT)*, pages 1955–1960, 2024.

[7] Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary, and Ronald de Wolf. Linear vs. semidefinite extended formulations: exponential separation and strong lower bounds. In *Symposium on the Theory of Computing*, 2011.

[8] P. Hrubeš. On ϵ-sensitive monotone computations. *Computational Complexity*, 29(2), 2020.

[9] Eyal Kushilevitz and Noam Nisan. *Communication Complexity*. Cambridge University Press, 1996.

[10] László Lovász. Communication complexity: A survey. In *Paths, Flows, and VLSI-Layout*, pages 235–265. Springer-Verlag, 1990.

[11] Anup Rao and Amir Yehudayoff. *Communication Complexity and Applications*. Cambridge University Press, 2020.

[12] Arnaud Vandaele, Nicolas Gillis, François Glineur, and Daniel Tuyttens. Heuristics for exact nonnegative matrix factorization. *Journal of Global Optimization*, 65(2):369–400, 2016.

[13] Mihalis Yannakakis. Expressing combinatorial optimization problems by linear programs. *Journal of Computer and System Sciences*, 43(3):441–466, 1991.

ECCC ISSN 1433-8092

https://eccc.weizmann.ac.il