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A power saving for pointer chasing

Emanuele Viola*

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Pointer chasing has a long history in communication complexity, beginning with [PS84]; see the book [Vio] for background. Here we focus on the simultaneous number-on-forehead model [CFL83, BGKL03]. This setting is particularly clean: to state the results no model definition is needed. Three-player pointer chasing asks to compute $g(f(i))$ where $i \in [n]$, $f : [n] \to [n]$, and $g : [n] \to [2]$. Each player sees two of these three inputs.

A communication lower bound of $c\sqrt{n}$ is known, see [BHK01, VW09]. Our focus is on upper bounds. A surprising, brilliant protocol in [PRS97] achieves communication $\leq n/\log^c n$. See [Vio] for an overview of their protocol. However the gain over the trivial n is small and raises the question whether it is a reflection of the unbalanced input lengths or a true saving in communication. I show the latter to be the case: a variant of [PRS97] achieves a power saving in communication:

Theorem 1. *There are functions A, B, C with output length n^{1-c}, and a function Dec such that*

$$Dec\big(A(f,g), B(i,g), C(i,f)\big) = g(f(i)).$$

Here A, B, C compute the messages of the players. Their output length is the communication. And *Dec* is the *decoder*, or *referee*, which maps the messages to the output.

To isolate the new key ideas, I first give a *randomized* protocol where the middle function is a permutation π. We then make the protocol deterministic and finally remove the permutation restriction.

Randomized protocols for permutations. Consider a bipartite graph H with vertex set $[n]$ on the left layer and $[n]$ on the right. Given H and π we define another graph $G = G(H, \pi)$ on the n vertices of the last layer: $y, z \in [n]$ are adjacent iff both

$$(\pi^{-1}(y), z) \quad \text{and} \quad (\pi^{-1}(z), y)$$

are edges of H. We define

$$\Gamma(G)$$

to be the minimum size of a partition of the vertices $[n]$ of G into cliques.

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Lemma 2. *[PRS97] Let H be a graph with left degree $\leq D$. There is a protocol that uses $\leq c(D + \Gamma(G(H, \pi)))$ communication on any input (i, π, g).*

Proof. $A(\pi, g)$ is the parity of g on each clique. $B(i, g)$ is the bits of all the $\leq D$ the neighbors of i in H. $C(i, \pi)$ consists of the index of the clique containing $\pi(i)$ and a D-bit mask selecting its other vertices from B's output. This is sufficient to compute pointer chasing because every other vertex z of the clique of $\pi(i)$ is a neighbor of i in H, by the definition of G. Hence the referee can XOR the parity corresponding to the clique, given by A, with the bits corresponding to the other vertices in the clique, which can be obtained by selecting the output of B according to the mask given by C. □

[PRS97] picks a uniform graph H of a certain density. By contrast, I use an ad hoc construction of graphs which is inspired by [AMS13], though it doesn't quite appear there, so everything is proved from scratch. Let $m := \frac{1}{2} \log_2 n$ and independently assign to each vertex v in H a uniform *label* $L(v) \in [2]^m$. Edge $u - v$ is in H exactly when $L(u)$ and $L(v)$ agree $\geq 5m/8$ coordinates. (Any threshold between $m/2$ and $3m/4$ would do; $5/8$ is the midpoint.) A standard tail bound will bound the degree of H by n^{1-c}. The main property to show is that for every π:

$$\mathbb{E}[\Gamma(G)] \leq n^{1-c}. \tag{1}$$

The remainder of this section is to prove this. We introduce a *color* of each vertex y of G, denote $\mathrm{col}(y)$. This color is the coordinate-wise sum of the labels of y and of $\pi^{-1}(y)$, an element in $[3]^m$. Let S be a random color distributed as the color of a vertex, and denote $p_s := \mathbb{P}[S = s]$ where $s \in [3]^m$ is a color. The key idea is to show that each color class is a nearly complete graph; one can then use a standard graph-theoretic argument to conclude that it can be covered by few cliques. Fix two vertices $y \neq z$ in G, and condition on the event that they have the same color s. Define q_s as the prob. that they are not connected in G, which recall means that at least one crossed pair agrees in $< 5m/8$ coordinates, that is, either $L(\pi^{-1}(y))$ and $L(z)$ agree in $< 5m/8$ coordinates or $L(\pi^{-1}(z))$ and $L(y)$ agree in $< 5m/8$ coordinates.

Claim 3. $\mathbb{E}_S q_S \leq n^{-c}$.

Proof. Let us first fix a color s and suppose $\mathrm{col}(y) = \mathrm{col}(z) = s$. This means

$$L(y) + L(\pi^{-1}(y)) = L(z) + L(\pi^{-1}(z)).$$

Rearranging,

$$L(y) - L(\pi^{-1}(z)) = L(z) - L(\pi^{-1}(y)).$$

This shows that the crossed edges $y - \pi^{-1}(z)$ and $z - \pi^{-1}(y)$ are either both in H or both absent from H. So we just need to analyze one crossed edge. Consider a coordinate in the color. Suppose it is 2. Then all four coordinates of the corresponding labels are 1. Similar, if it is 0 all four coordinates are zero. So for these coordinates, the crossed labels always agree. Now suppose a color coordinate is 1. This can be obtained in two ways, $0 + 1$ or $1 + 0$ in the corresponding sum of labels in the matching pairs. Correspondingly, each crossed pair has two uniform independent bits. So the crossed labels disagree with probability $1/2$. If s has k ones then $q_s = \mathbb{P}[\mathrm{Bin}(k, 1/2) > 3m/8]$, when Bin is the binomial distribution. The threshold $3m/8$ corresponds to *disagreements*, equivalent to $< 5m/8$ agreements.

To bound $\mathbb{E}_S q_S$, at each of the m coordinates flip one fair coin to decide whether the color is 1. If it is, flip another fair coin to decide whether the crossed labels disagree. Hence a disagreement has probability 1/4, and consequently:

$$\mathbb{E}_S q_S = \mathbb{P}[\mathrm{Bin}(m, 1/4) > 3m/8] \leq n^{-c}.$$

□

We use the claim to show that the subgraphs of G induced by the color classes are almost cliques, i.e., they miss few edges. And then we use a general, simple graph-theoretic fact to obtain a cover using a small number of cliques.

Let us fix a color $s \in [3]^m$. Let M_s be the number of missing edges among the vertices with color s. Note that for any two vertices in G, they both have color s with probability p_s^2. And conditional on that, they are nonadjacent with probability q_s. Hence

$$\mathbb{E}M_s = \binom{n}{2} p_s^2 q_s. \tag{2}$$

We shall use the following basic claim:

Claim 4. A graph (on any number of nodes) with M missing edges can be partitioned into $\leq 1 + \sqrt{2M}$ cliques.

Proof. Let r be the minimum number of cliques in a partition. By minimality, there's a missing edge between any two cliques. Hence we miss $\binom{r}{2}$ edges. So $M \geq r(r-1)/2$. So $2M \geq r^2 - r \geq (r-1)^2$. □

Fix a color s and define $\Gamma_s(G)$ like $\Gamma(G)$, but restricted to nodes with color s (i.e., the minimum number of cliques needed to partition the nodes with color s). By Claim 4, convexity, and equation (2) we have

$$\mathbb{E}\Gamma_s \leq 1 + \mathbb{E}\sqrt{2M_s} \leq 1 + \sqrt{2\mathbb{E}M_s} \leq 1 + np_s\sqrt{q_s}.$$

Now sum over all 3^m color classes, use convexity again, and Claim 3:

$$\mathbb{E}\Gamma(G) \leq 3^m + n\mathbb{E}_S\sqrt{q_S} \leq 3^m + n\sqrt{\mathbb{E}_S q_S} \leq n^{1-c} + n \cdot n^{-c} \leq n^{1-c}.$$

This proves equation (1).

Making the protocol deterministic. For this we want a fixed graph H that works for *every* permutation. The idea is simply to pick many independent copies H_i of the graph in the previous section, and let H be their union. We want to drive the probability of not having a suitable clique cover below $1/n!$. Using equation (1) with a first-moment analysis would require about n copies, which is too much. However, a sharper concentration inequality shows that the expectation in equation (1) is concentrated, and so a smaller number of copies suffice. Specifically, if we want $\Gamma(G) \leq n^{1-\epsilon}$ then it is enough to take $n^{c\epsilon}$ copies; for ϵ small enough we can afford the degree blow up.

Specifically, use the method of bounded differences. For a fixed π, consider choosing the labels of the vertices by choosing them for one pair $i, \pi(i)$ at the time. We seek concentration

for $\Gamma(G)$. For that, we note that changing the value of one pair of labels can only change $\Gamma(G)$ by ≤ 1. This is because removing a node y does not increase Γ, and putting it back (with any adjacent edges) increases Γ by ≤ 1. By e.g. Corollary 5.2 in [DP09] to prove

$$\mathbb{P}[\Gamma(G) > \mathbb{E}\Gamma(G) + t] \leq c^{-t^2/n}.$$

The choice $t = n^{1-\epsilon/2}$ suffices.

Extension to arbitrary functions. This is from [BC08]. Specifically their Lemma 2.2 gives a one-way protocol given a protocol for permutations. Their techniques extend to simultaneous.

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