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Resolution Is Automatable on Tseitin Formulas

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Abstract

We prove that both regular and unrestricted resolution are automatable on Tseitin formulas, resolving a question of Alekhnovich and Razborov [2]. We use known proof-search algorithms that apply to arbitrary CNFs and are parameterized by the treewidth of the primal graph. Our lower bounds show that these algorithms suffice to find a regular refutation of any Tseitin contradiction in time polynomial in the input length and the length of a shortest unrestricted refutation. We also establish a polynomial equivalence between regular and unrestricted resolution on Tseitin formulas, proving a weaker version of Urquhart's conjecture that their minimum regular and unrestricted resolution refutation lengths are equal.

Our main technical results are unrestricted resolution lower bounds of $2^{\Omega(\mathrm{tw}(G))}$ for Tseitin contradictions on arbitrary connected graphs G, and $2^{\Omega(\mathrm{tw}(L(G)))}$ when G additionally has no bridges, where $L(G)$ is the line graph of G. The first bound combines the Ben-Sasson–Wigderson measure argument with random restrictions based on a distribution on spanning trees. The second follows from a reduction to bounded-degree graphs that replaces vertices with expander gadgets.

Use of AI. The main argument was found with the help of ChatGPT Astra 6.0. The authors simplified the argument and reorganized its presentation to make it easier to follow. The authors take full responsibility for the correctness of the results and their proofs.

1 Introduction

Propositional proof systems provide formal methods for proving the unsatisfiability of Boolean formulas in conjunctive normal form (CNF). Resolution is one of the most extensively studied such systems and provides the proof-theoretic basis of conflict-driven clause learning (CDCL) SAT solvers [23]. Its proof lines are clauses, that is, disjunctions of literals, and its inference rule derives $A \vee B$ from $A \vee x$ and $B \vee \neg x$. A resolution refutation derives the empty clause from the input clauses, certifying their unsatisfiability. Derived clauses may be reused, so a refutation is naturally represented as a directed acyclic graph. A refutation is *regular* if no variable is used as the resolving variable more than once along any path from an input clause to the empty clause.

An execution of a resolution-based SAT solver on an unsatisfiable formula can be translated into a resolution refutation whose length is at most proportional to the running time. Thus lower bounds on refutation length also give running-time lower bounds for such solvers.

Tseitin formulas encode systems of parity equations associated with graphs. Given an undirected graph $G = (V, E)$ and a charge function $c\colon V \to \{0, 1\}$, introduce a variable x_e for each edge and require

$$\bigoplus_{e \ni v} x_e = c(v) \qquad \text{for every } v \in V.$$

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The formula $\mathrm{T}(G, c)$ is the conjunction of the standard CNF encodings of these equations. If G is connected, the formula is unsatisfiable exactly when the total charge is odd. More generally, it is satisfiable if and only if every connected component has even total charge.

In 1968, in one of the founding papers of propositional proof complexity, Tseitin [26] proved the first superpolynomial lower bound for regular resolution using formulas of this form based on square grids. Urquhart [28] subsequently proved exponential lower bounds for unrestricted resolution on bounded-degree expander graphs. Together with formulas expressing the pigeonhole principle, Tseitin formulas have become standard examples for studying the power and limitations of propositional proof systems.

Although unrestricted resolution can be exponentially more succinct than regular resolution [1], the relationship between the two systems on Tseitin formulas has remained open. Urquhart asked whether there is always a regular refutation of minimum size for these formulas [29, Problem 3]; the conjecture also appears explicitly in [1, Conjecture 6.1]. We establish that every Tseitin contradiction with an unrestricted resolution refutation of length S has a regular resolution refutation of length $S^{O(1)}$, thus proving a weaker version of Urquhart's conjecture.

For an unsatisfiable CNF formula F, write $S(F)$ and $S_{\mathrm{reg}}(F)$ for its minimum unrestricted and regular resolution refutation lengths, respectively. We count used input clauses and derived clauses; unused input clauses need not be listed. Write $\ell(F)$ for the encoding length of F and $|F|$ for the number of clauses.

A proof system $\mathcal{P}$ is *automatable* on a family of formulas if a deterministic algorithm, given an unsatisfiable formula F from the family, constructs a $\mathcal{P}$-refutation in time polynomial in $\ell(F) + S_{\mathcal{P}}(F)$, where $S_{\mathcal{P}}(F)$ is the minimum refutation size in that system. Whether resolution is automatable on all unsatisfiable CNFs was a long-standing open problem. Alongside this general question, researchers investigated automatability on important special classes. Alekhnovich and Razborov [2] obtained positive results for Tseitin formulas on certain families of graphs and asked whether these results extend to arbitrary underlying graphs. The general question was subsequently settled by the breakthrough result of Atserias and Müller [3]: resolution is not automatable on all unsatisfiable CNFs unless $\mathsf{P} = \mathsf{NP}$. We give an affirmative answer to the polynomial-time automatability question for Tseitin formulas: both unrestricted and regular resolution are automatable on this class. More precisely, we prove that the known proof-construction algorithms can be used to find a regular refutation in time polynomial in the input length and the minimum *unrestricted* refutation length.

1.1 Related work

Constructive upper bounds. The *primal graph* $P(F)$ of a CNF formula has its variables as vertices, with two distinct variables adjacent when they occur together in a clause. Broering and Lokam [8, Theorem 3] showed that, given a tree decomposition of $P(F)$ of width k, a regular resolution refutation of an unsatisfiable F can be constructed in time $\ell(F)^{O(1)}2^{O(k)}$. Combining this with the single-exponential-time constant-factor approximation algorithm for treewidth of Bodlaender et al. [6] gives running time

$$\ell(F)^{O(1)}2^{O(\mathrm{tw}(P(F)))}$$

without requiring a decomposition as part of the input. Alternatively, one can obtain the same running time using the branchwidth-based algorithm of Alekhnovich and Razborov [2, Theorem 2.3] together with the standard comparison between branchwidth and primal treewidth.

The *line graph* $L(G)$ has the edges of G as its vertices, with two distinct vertices adjacent when the corresponding edges share an endpoint in G. For a Tseitin formula $\mathrm{T}(G, c)$, the primal graph is the line graph $L(G)$. Consequently, these algorithms construct regular refutations in time

$$\ell(\mathrm{T}(G, c))^{O(1)}2^{O(\mathrm{tw}(L(G)))}.$$

The same expression therefore also bounds the refutation size.

One way to establish automatability is therefore to prove lower bounds on unrestricted resolution refutation length that match the running time of these algorithms up to polynomial factors.

Lower bounds. Early lower bounds applied to particular graph families. Besides Urquhart's expander lower bound [28], Dantchev and Riis [11] proved a $2^{\Omega(n)}$ lower bound for unrestricted resolution on the $n \times n$ grid. Håstad [18] proved lower bounds of the form $2^{n^{\Omega(1/d)}}$ for depth-d Frege refutations of grid-based Tseitin formulas, ruling out polynomial-size proofs up to depth proportional to $\log n / \log\log n$. Håstad and Risse [19] subsequently strengthened the grid bounds, proving a total-size lower bound of $2^{\widetilde{\Omega}(n^{1/d})}$, where the tilde hides polylogarithmic factors, as well as bounds on the number of proof lines when the size of each line is restricted.

Alekhnovich and Razborov [2] obtained matching size bounds for another class of graphs. They defined the *cyclicity* of a graph as the smallest integer k for which there is a family of cycles covering every edge, such that each cycle has length at most k and each edge belongs to at most k cycles. For graph families of bounded cyclicity, they proved automatability and a lower bound matching the constructive upper bound up to polynomial factors. This includes, for example, bridgeless plane graphs whose faces have bounded boundary length.

Galesi, Itsykson, Riazanov, and Sofronova [15] extended the bounded-depth Frege lower bounds to arbitrary connected graphs by using the polynomial grid-minor theorem and reductions to grid-based Tseitin formulas. They obtained lower bounds $2^{\mathrm{tw}(G)^{\Omega(1/d)}}$ for depth-d Frege and, for sufficiently large depth, upper bounds $\ell(\mathrm{T}(G,c))^{O(1)}2^{\mathrm{tw}(G)^{O(1/d)}}$. For bounded-degree graphs, their results also imply *quasi-automatability* of resolution, regular resolution, and each fixed bounded-depth Frege system extending resolution. Here quasi-automatability allows running time $2^{(\log(\ell(F)+S_{\mathcal{P}}(F)))^{O(1)}}$ in place of a polynomial running time.

For unrestricted resolution, the grid-minor approach gives

$$S(\mathrm{T}(G,c)) \geq 2^{\Omega(\mathrm{tw}(G)^{\lambda})},$$

where $\lambda > 0$ is a constant supplied by the polynomial grid-minor theorem [15, 10]. Before this work, this was the strongest general lower bound expressed solely in terms of treewidth. For example, the bound of Chuzhoy and Tan [10] permits any fixed $\lambda < 1/9$. For general graphs, this grid-minor approach cannot yield an exponent $\lambda > 1/2$. For planar graphs, however, a grid minor of side length $\Omega(\mathrm{tw}(G))$ exists [25]. Combined with the grid resolution lower bound, this yields the matching $2^{\Omega(\mathrm{tw}(G))}$ lower bound for bounded-degree planar graphs.

Stronger general lower bounds were known for *regular* resolution. Itsykson, Riazanov, Sagunov, and Smirnov [20] used the characterization of regular resolution by read-once branching programs solving the clause-search problem. They transformed such a program into a read-once branching program computing a satisfiable Tseitin formula and obtained the lower bound

$$S_{\mathrm{reg}}(\mathrm{T}(G,c)) \geq 2^{\Omega(\mathrm{tw}(G)/\log|V(G)|)}.$$

De Colnet and Mengel [12] used a related reduction to decomposable negation normal form (DNNF) representations of satisfiable Tseitin formulas. Their bound is

$$S_{\mathrm{reg}}(\mathrm{T}(G,c)) \geq \frac{2^{\Omega(\mathrm{tw}(G)/\Delta(G))}}{|V(G)|},$$

where $\Delta(G)$ denotes the maximum degree of G. For bounded-degree graphs, this eliminates the logarithmic loss in the exponent, at the cost of a polynomial factor. Itsykson, Riazanov, and Smirnov [21] subsequently strengthened the DNNF lower bound and obtained

$$S_{\mathrm{reg}}(\mathrm{T}(G,c)) \geq 2^{\Omega(\mathrm{tw}(G))}$$

for every connected graph. These results left open the corresponding exponential treewidth lower bound for unrestricted resolution.

Width and smoothness. The width of a clause is its number of literals, and the width of a resolution refutation is the maximum width of its clauses. Write $w(F)$ for the minimum resolution refutation width

of F. For Tseitin contradictions on connected simple graphs, Galesi, Talebanfard, and Torán [16] showed that $w(\mathrm{T}(G,c)) = \Theta(\mathrm{tw}(L(G))+1)$. Alekhnovich and Razborov [2] called a class of contradictions *smooth* if $w(F) = O(\log S(F))$ uniformly over the class, and asked whether such a bound holds for Tseitin contradictions. This would strengthen the general size–width relation of Ben-Sasson and Wigderson [5], which gives

$$w(F) \le w_0(F) + O\Big(\sqrt{n \log S(F)}\Big),$$

where n is the number of variables and $w_0(F)$ is the maximum width of an input clause. The dependence on n and $w_0(F)$ prevents this general bound from establishing smoothness.

1.2 Our results

An exponential treewidth lower bound.

Theorem 1.1 (Corollary 5.4)**.** *Let G be a connected graph and c a charge function with odd total charge. Then*

$$S(\mathrm{T}(G,c)) \ge 2^{\Omega(\mathrm{tw}(G))}.$$

Thus the exponential treewidth lower bound previously known for regular resolution also holds for unrestricted resolution. For graphs of bounded degree, this matches the constructive regular resolution upper bound up to polynomial factors in $|V(G)|$ and a constant factor in the exponent.

A line-graph treewidth lower bound. For arbitrary degrees, the relevant parameter in the general proof-construction algorithm is $\mathrm{tw}(L(G))$.

Theorem 1.2 (Theorem 6.1)**.** *For every connected graph G and every odd charge function c,*

$$S(\mathrm{T}(G,c)) + |\,\mathrm{T}(G,c)| \ge 2^{\Omega(\mathrm{tw}(L(G)))}.$$

A connected graph with at least two vertices is *2-edge-connected* if it has no bridges. For such graphs we get a stronger bound:

Corollary 1.3 (Corollary 8.1)**.** *For every 2-edge-connected graph G and every odd charge function c,*

$$S(\mathrm{T}(G,c)) \ge 2^{\Omega(\mathrm{tw}(L(G)))}.$$

For 2-edge-connected graphs, this bound matches the constructive upper bound $S_{\mathrm{reg}}(\mathrm{T}(G,c)) \le |V(G)|^{O(1)} 2^{O(\mathrm{tw}(L(G)))}$. Consequently, Tseitin formulas on 2-edge-connected graphs, together with those on connected bounded-degree graphs, give families on which the general primal-treewidth upper bound for regular resolution is tight even for unrestricted resolution up to polynomial factors.

Automatability and polynomial equivalence. The lower bound on $S(F)+|F|$ in Theorem 1.2, combined with the known proof-construction algorithms, implies the following result.

Theorem 1.4 (Theorem 7.1)**.** *Both unrestricted and regular resolution are automatable on the class of Tseitin formulas. More precisely, there is a deterministic algorithm that, given a Tseitin contradiction F, constructs a regular resolution refutation in time*

$$\big(\ell(F) + S(F)\big)^{O(1)}.$$

This resolves the automatability question posed by Alekhnovich and Razborov [2].

We also prove the following bound, which depends only on the minimum refutation length.

Theorem 1.5 (Theorem 8.2)**.** *There is an absolute constant C such that every Tseitin contradiction F satisfies*

$$S(F) \le S_{\mathrm{reg}}(F) \le S(F)^C.$$

Thus regular and unrestricted resolution are polynomially equivalent on all Tseitin contradictions, proving a weaker version of Urquhart's conjecture.

For 2-edge-connected graphs, Theorem 1.5 follows from the known upper bound and Corollary 1.3. For connected graphs with bridges, we combine regular derivations along the bridge tree. For disconnected graphs, we pass to an unsatisfiable component of minimum resolution complexity.

Smoothness. Our lower bounds also establish smoothness for Tseitin contradictions on 2-edge-connected graphs, and on connected graphs of any fixed maximum degree. Indeed,

$$w(\mathrm{T}(G,c)) \le O(\mathrm{tw}(L(G))) + 1,$$

so Corollary 1.3 gives smoothness in the first case. For graphs of maximum degree at most Δ, the inequality

$$\mathrm{tw}(L(G)) + 1 \le \Delta(\mathrm{tw}(G) + 1)$$

and Theorem 1.1 give $w(\mathrm{T}(G,c)) = O_\Delta(\log S(\mathrm{T}(G,c)))$.

Notice, however, that smoothness fails even for the family of Tseitin contradictions on stars. Indeed, let F_m be the formula on a star with m edges, charge one at the center, and zero charges at the leaves. The leaf clauses $\neg x_i$ and the center clause $x_1 \vee \cdots \vee x_m$ yield a refutation of length $2m+1$. Every refutation must nevertheless use a center clause, all of which have width m. Hence

$$w(F_m) = m, \qquad \log S(F_m) = O(\log m),$$

contradicting the inequality required for smoothness. Thus the assumption that the graph has no bridges cannot be dropped from our smoothness result when degrees are unbounded.

1.3 Proof overview

We outline the lower bound $2^{\Omega(\mathrm{tw}(G))}$ for connected graphs and the stronger bound $2^{\Omega(\mathrm{tw}(L(G)))}$ for 2-edge-connected graphs. Together with the known constructive upper bounds and standard arguments for combining proofs across bridges, these bounds yield automatability and polynomial equivalence between regular and unrestricted resolution on Tseitin formulas.

1.3.1 The lower bound in terms of $\mathrm{tw}(G)$

Random restrictions. Our main tool is a random restriction argument of the type used to lift resolution width lower bounds to size lower bounds. Ben-Sasson [4] credits this approach to Alekhnovich and Razborov.

We first illustrate it in a simpler setting. Let $e(G)$ denote the minimum size of a balanced edge cut of G, where each side contains between one third and two thirds of the vertices. The measure argument of Ben-Sasson and Wigderson [5] shows that every resolution refutation of $\mathrm{T}(G,c)$ contains a clause of width at least $e(G)$.

Suppose that G' is obtained from G by doubling every edge. Independently for each parallel pair, leave one uniformly chosen edge free and assign a random bit to the other. The restricted formula is a Tseitin contradiction on a copy of G, so its refutation must still contain a clause of width at least $e(G)$. However, every original clause of width w survives with probability $2^{-\Omega(w)}$. Since some clause of width at least $e(G)$ must survive, the original proof must contain $2^{\Omega(e(G))}$ clauses.

We now extend this argument to 3-edge-connected graphs (without doubling their edges). To do so, we construct a distribution on spanning trees T such that, for every $B \subseteq E(G)$,

$$\Pr[B \subseteq T] \le p^{|B|}$$

for an absolute constant $p < 1$. We discuss its construction below.

Sample a tree T and leave its edges free. Independently for every other edge, leave it free with probability $1/2$, and otherwise assign a random bit. The resulting free-edge graph is connected because it contains T.

Put $b := e(G)/4$ and call clauses of width at least b *wide*. The defining property of the distribution on spanning trees guarantees that every wide clause survives with probability $2^{-\Omega(b)}$. We distinguish two cases.

If the probability that at least one wide clause survives is at least $1/2$, the union bound immediately implies that the proof contains $2^{\Omega(b)} = 2^{\Omega(e(G))}$ clauses.

Otherwise, with probability greater than $1/2$, all wide clauses disappear. Here we use the more precise conclusion of the Ben-Sasson–Wigderson argument: every refutation contains a clause whose edge variables contain a balanced cut. Thus some narrow clause of the original proof must contain a balanced cut of the free-edge graph after restriction. We show that each fixed narrow clause has this property with probability $2^{-\Omega(b)}$. Another union bound therefore gives the same exponential lower bound.

Finally, the maximum of $e(K)$ over subgraphs $K \subseteq G$ is $\Omega(\mathrm{tw}(G))$; see Section 2.2. A weighted version of the balanced-cut argument lets us measure balance relative to the vertices of such a subgraph while sampling restrictions on the whole graph. This strengthens the lower bound to $2^{\Omega(\mathrm{tw}(G))}$ for 3-edge-connected graphs.

The spanning-tree distribution for 3-edge-connected graphs. The Tutte–Nash-Williams theorem [27, 24] implies that every $2k$-edge-connected graph has k pairwise edge-disjoint spanning trees; see also Kaiser's elementary proof [22]. Doubling the edges of a 3-edge-connected graph gives a 6-edge-connected graph and hence three edge-disjoint spanning trees. Projecting these trees back to G gives spanning trees T_1, T_2, T_3 such that every edge belongs to at most two of them.

We bring these trees together by random exchanges. Choose unequal trees T_i, T_j and an edge $e \in T_i \setminus T_j$. The path in T_j between the endpoints of e contains an edge f crossing the cut of $T_i - e$. Both $T_i - e + f$ and $T_j - f + e$ are therefore spanning trees. With equal probabilities, make one of these two replacements, and repeat until all three trees coincide. The process terminates almost surely, and its common final tree satisfies the required inclusion bound with $p = 2/3$. This construction is closely related to swap rounding [9]; the details appear in Section 4.

Passing to arbitrary connected graphs. De Colnet and Mengel [12, Lemma 12] showed that every graph of treewidth at least three has a 3-vertex-connected topological minor with the same treewidth. Here K is a topological minor of G if a subdivision of K occurs as a subgraph of G. Since 3-vertex-connectivity implies 3-edge-connectivity, the preceding lower bound applies to this minor. The fact that resolution refutations of Tseitin formulas transfer to connected topological minors was established in [15, Lemma 30]. This gives the $2^{\Omega(\mathrm{tw}(G))}$ lower bound for all connected graphs.

1.3.2 The lower bound in terms of $\mathrm{tw}(L(G))$

For bounded-degree graphs, $\mathrm{tw}(G) + 1$ and $\mathrm{tw}(L(G)) + 1$ differ by at most a constant factor. It therefore suffices to replace G by a bounded-degree graph H such that $\mathrm{tw}(L(H)) + 1 \geq c\big(\mathrm{tw}(L(G)) + 1\big)$ for an absolute constant $c > 0$.

Replace every vertex v of degree d_v by a bounded-degree expander X_v on d_v vertices, one port for each incident edge, and reconnect the original edges through their corresponding ports. Give the vertices of X_v charges summing to $c(v)$. The resulting graph H has bounded degree. Expansion prevents a small number of deleted internal edges from separating many ports from the largest remaining component. Using the separator characterization of treewidth, we obtain

$$\mathrm{tw}(H) + 1 = \Omega(\mathrm{tw}(L(G)) + 1).$$

Our preceding result consequently gives a lower bound for $\mathrm{T}(H, c_H)$ in terms of $\mathrm{tw}(L(G))$.

To transfer this bound back to G, observe that the parity equations inside X_v imply the original equation at v: summing them cancels all internal variables. Implicational completeness of resolution therefore lets us replace each used original axiom by a derivation from the gadget axioms. The cost of this simulation is exponential only in $\Delta(G)$.

For arbitrary connected graphs, this cost can be bounded using the input size, giving

$$S(\mathrm{T}(G, c)) + |\,\mathrm{T}(G, c)| \geq 2^{\Omega(\mathrm{tw}(L(G)))}.$$

When G is 2-edge-connected, we additionally show that every refutation must already contain exponentially many clauses in $\Delta(G)$. The simulation cost can then be bounded using the proof size alone, yielding

$$S(\mathrm{T}(G, c)) \geq 2^{\Omega(\mathrm{tw}(L(G)))}.$$

2 Preliminaries

2.1 Graphs, treewidth, and line graphs

All graphs are finite, undirected, and loopless; parallel edges are allowed and counted separately. A graph is *simple* if it does not contain parallel edges. We write $H \subseteq G$ to mean that H is a *subgraph* of G: $V(H) \subseteq V(G)$ and $E(H)$ is a subset of the edges of G whose endpoints lie in $V(H)$. The subgraph need not be induced or connected. For $W \subseteq V(G)$, the *induced subgraph* $G[W]$ contains all edges of G with both endpoints in W. For $U \subseteq V(G)$, write $G - U$ for the graph obtained from G by deleting the vertices in U and all edges incident to them. For $A \subseteq E(G)$, write $G - A$ for the graph obtained by deleting the edges in A and retaining all vertices. For a single edge e, we also write $G - e$. We write $\Delta(G)$ for the maximum degree, counting multiplicities.

k-connected and k-edge-connected graphs. Let $k \geq 1$. A graph G is *k-connected* if $|V(G)| \geq k + 1$ and $G - U$ is connected for every set $U \subseteq V(G)$ with $|U| < k$. A graph with at least two vertices is *k-edge-connected* if it remains connected after deleting any set of fewer than k edges. Every k-connected graph is also k-edge-connected.

An edge is a *bridge* if deleting it increases the number of connected components. A connected graph with at least two vertices has no bridges if and only if it is 2-edge-connected.

Tree decompositions. A *tree decomposition* of G consists of a tree T and *bags* $B_t \subseteq V(G)$, one for each $t \in V(T)$, such that every vertex of G belongs to a bag, the endpoints of every edge of G occur together in a bag, and for each $v \in V(G)$, the nodes $t \in T$ for which $v \in B_t$ induce a connected subtree of T. Its *width* is $\max_t |B_t| - 1$. The *treewidth* $\mathrm{tw}(G)$ is the minimum width of a tree decomposition. Parallel edges do not affect treewidth. We set $\mathrm{tw}(\varnothing) = -1$, where $\varnothing$ denotes the graph with no vertices.

Minors and topological minors. A graph H is a *minor* of G if it can be obtained from G by deleting vertices, deleting edges, and contracting edges. Contracting an edge identifies its endpoints; resulting loops are deleted, while parallel edges are retained. In particular, every subgraph of G is a minor of G.

A *subdivision* of H is obtained by replacing each edge of H by a path of positive length, using new internal vertices that are distinct for different paths. A graph H is a *topological minor* of G if some subdivision of H is a subgraph of G. Every topological minor is also a minor.

We will use the following standard fact; see, for example, Diestel [13, Lemma 12.4.1].

Proposition 2.1. *If H is a minor of G, then*

$$\mathrm{tw}(H) \leq \mathrm{tw}(G).$$

Line graphs. The *line graph* $L(G)$ is the simple graph with vertex set $E(G)$; two distinct vertices of $L(G)$ are adjacent when the corresponding edges share an endpoint in G. In particular, distinct parallel edges of G give distinct, adjacent vertices of $L(G)$.

Proposition 2.2 (Harvey and Wood [17])**.** *For every graph G with at least one edge and maximum degree Δ,*

$$\max\{\Delta, \tfrac{1}{2}(\mathrm{tw}(G)+1)\} \le \mathrm{tw}(L(G))+1 \le \Delta(\mathrm{tw}(G)+1).$$

In particular, $\mathrm{tw}(G)+1$ *and* $\mathrm{tw}(L(G))+1$ *agree up to a constant factor when the maximum degree is bounded.*

2.2 Balanced cuts and treewidth

Balanced edge cuts. For $U \subseteq V(G)$, define the *cut* induced by U, denoted by $\delta_G(U)$, as the set of edges with exactly one endpoint in U. When $n = |V(G)| \ge 2$, the cut is *balanced* if $n/3 \le |U| \le 2n/3$. Define

$$e(G) = \min_{n/3 \le |U| \le 2n/3} |\delta_G(U)|.$$

Here e stands for *edge cut*. This parameter is also called *balanced edge expansion*; note that we count crossing edges without normalizing by $|U|$. Set $e(G) = 0$ when $|V(G)| \le 1$. More generally, for $W \subseteq V(G)$ with $|W| \ge 2$, a cut is *W-balanced* if $|W|/3 \le |U \cap W| \le 2|W|/3$, and we put

$$e_W(G) = \min_{|W|/3 \le |U\cap W| \le 2|W|/3} |\delta_G(U)|.$$

Thus $e_{V(G)}(G) = e(G)$.

Balanced vertex separators. A *separation* of a graph H is a pair (P, Q) with $P \cup Q = V(H)$ and no edges between $P \setminus Q$ and $Q \setminus P$. Its *separator* is $P \cap Q$, and its *order* is $|P \cap Q|$. It is *balanced* if both exclusive sides $P \setminus Q$ and $Q \setminus P$ have size at most $2|V(H)|/3$. Define

$$v(H) = \min\{\, |P \cap Q| : (P, Q) \text{ is a balanced separation of } H \,\}.$$

Here v stands for *vertex cut*: we count vertices in a separator rather than edges in a cut. In particular, $v(\varnothing) = 0$, and a singleton has $v(H) = 1$.

Lemma 2.3 (Comparing edge cuts and vertex separators)**.** *For every graph H with at least two vertices,*

$$v(H) \le e(H) \le \Delta(H)v(H).$$

Proof. Let U define a minimum balanced edge cut, and let $Z \subseteq U$ be its endpoints on the U side. Then $(U, (V(H) \setminus U) \cup Z)$ is a balanced separation of order $|Z| \le |\delta_H(U)| = e(H)$, proving the first inequality.

Conversely, take a minimum balanced vertex separation, write its exclusive sides as A, B and its separator as Z, and put $n = |V(H)|$. If $|A|$ or $|B|$ is at least $n/3$, use that side as U. Otherwise, add vertices of Z to A until $|U| = \lceil n/3 \rceil$. There are enough vertices available because $|A \cup Z| = n - |B| > 2n/3$. In either case U is balanced, and every crossing edge meets Z, since there are no edges between A and B. Hence $e(H) \le |\delta_H(U)| \le \Delta(H)|Z| = \Delta(H)v(H)$. □

Maximum over subgraphs. A star denotes the maximum of the corresponding parameter over all subgraphs:

$$v^*(G) = \max_{H \subseteq G} v(H), \qquad e^*(G) = \max_{H \subseteq G} e(H).$$

Thus $v^*(G)$ is the usual *separation number*, often denoted by $\mathrm{sn}(G)$, while $e^*(G)$ is the largest minimum balanced edge cut found in a subgraph. The star takes a maximum over subgraphs; inside each subgraph we still take the minimum over balanced separations or cuts. For both definitions, it is sufficient to take the maximum over induced subgraphs: adding the missing edges on a fixed vertex set cannot decrease either $v(H)$ or $e(H)$.

The following two standard results relate treewidth to vertex separators in both directions.

Lemma 2.4 (Treewidth and balanced separators)**.** *1. If* $\mathrm{tw}(H) \le k$*, then for every set* $A \subseteq V(H)$ *there is* $Z \subseteq V(H)$ *with* $|Z| \le k+1$ *such that every component* C *of* $H - Z$ *satisfies* $|C \cap A| \le |A|/2$*. Z may contain vertices outside A.*

2. If every subgraph of H has a balanced vertex separation of order at most s, then $\mathrm{tw}(H) \le 15s$*.*

Part 1 is the standard separator property of treewidth; see [6, Lemma 2.1]. Part 2 is the theorem of Dvořák and Norin [14, Theorem 1].

Corollary 2.5. *For every graph G,*

$$v^*(G) \le \mathrm{tw}(G) + 1, \qquad \mathrm{tw}(G) \le 15v^*(G).$$

Proof. For the first inequality, fix a nonempty subgraph $H \subseteq G$. Apply Lemma 2.4(1) to the graph H and $A = V(H)$. By Proposition 2.1, $\mathrm{tw}(H) \le \mathrm{tw}(G)$, hence, the resulting set Z has size at most $\mathrm{tw}(G)+1$, and every component of $H - Z$ has size at most $|V(H)|/2$.

Group these components into two sides of size at most $2|V(H)|/3$. To do so, use one component of size at least $|V(H)|/3$ if one exists. Otherwise, add components until reaching $|V(H)|/3$, obtaining a side of size less than $2|V(H)|/3$. If all remaining vertices together number fewer than $|V(H)|/3$, use them as one side and leave the other empty. Putting Z in both sides gives a balanced separation. Thus $v(H) \le \mathrm{tw}(G)+1$; taking the maximum proves the first inequality. The empty subgraph contributes zero.

The second inequality follows directly from Lemma 2.4(2), because every subgraph has a balanced separation of order at most $v^*(G)$. □

Proposition 2.6. *For every graph G with at least one edge,*

$$\mathrm{tw}(G) \le 15v^*(G) \le 15e^*(G).$$

If G has maximum degree Δ*, also*

$$e^*(G) \le \Delta(\mathrm{tw}(G)+1).$$

Proof. Lemma 2.3, applied to subgraphs, gives $v^*(G) \le e^*(G) \le \Delta v^*(G)$. For the first inequality, singletons cause no exception: they have $v(H) = 1 \le e^*(G)$, since considering one edge gives $e^*(G) \ge 1$. Combine these inequalities with Corollary 2.5. □

We will also use the two directions of Lemma 2.4 in different graphs: first obtain a small separator in a graph of known treewidth, then transfer it to the target graph and apply part 2. For this last step the bound must hold for *every subgraph* of the target, not just for the target graph itself.

2.3 Resolution

A *literal* is a Boolean variable or its negation. A *clause* is a disjunction of literals, identified with their set; its support $\mathrm{supp}(C)$ is the set of variables occurring in it. The empty clause $\bot$ is false. A clause containing both x and $\neg x$ is *tautological*. A CNF formula is a conjunction of clauses, identified with a set of clauses.

For a CNF formula F and a clause C, write $F \models C$ if every assignment satisfying F also satisfies C.

The resolution and weakening rules are

$$\frac{C \vee x \qquad D \vee \neg x}{C \vee D}, \qquad \frac{C}{C \vee D}.$$

The variable x in the resolution rule is called the *resolving variable*. A *resolution derivation* from a CNF F is a sequence of clauses, each either an input clause of F or obtained from earlier clauses by one of these rules. A *refutation* ends with $\bot$. Its size is the number of lines, including input clauses actually used; unused inputs need not be listed. We allow weakening throughout, so our lower bounds also hold without weakening. A refutation is *regular* if no variable is used as a resolving variable more than once along any path in the derivation DAG. Write $S(F)$ and $S_{\mathrm{reg}}(F)$ for the minimum lengths of unrestricted and regular resolution refutations of F, respectively.

The following propositions are well-known.

Proposition 2.7 (Soundness of resolution)**.** *If a CNF F has a resolution refutation, then F is unsatisfiable.*

Proposition 2.8 (Implicational completeness)**.** *Let F be a CNF and C a nontautological clause, with all variables of F and C in a set of size r. If $F \models C$, then resolution derives C from F in at most 3^r lines. In particular, if F is unsatisfiable, then F has a refutation (derivation of $\bot$).*

Proposition 2.9 (Eliminating weakening)**.** *Every resolution refutation can be transformed into a refutation of the same formula without weakening and with no more proof lines. If the original refutation is regular, the resulting refutation can also be chosen to be regular.*

Restrictions. A *restriction* ρ is a partial assignment of Boolean values to variables. In $F|_\rho$, satisfied clauses are deleted, and falsified literals are removed from the remaining clauses. We use $C|_\rho$ for the analogous simplification of a clause, writing $\top$ if it is satisfied.

The following proposition is well-known.

Proposition 2.10 (Resolution under restrictions)**.** *If F has a resolution refutation with S lines, then $F|_\rho$ has a resolution refutation with at most S lines. Each line of the new refutation can be chosen to be the restriction of an unsatisfied original line, or a subclause of that restriction.*

2.4 Tseitin formulas

Let $G = (V, E)$ be any graph, not necessarily connected, and let $c\colon V \to \{0, 1\}$ be any *charge function*. For $U \subseteq V$, write $c(U) = \bigoplus_{v \in U} c(v)$; thus all sums of charges are taken modulo two. Associate a Boolean variable x_e with each edge; parallel edges have distinct variables. The *Tseitin formula* $\mathrm{T}(G, c)$ is the CNF encoding of

$$P_v : \quad \bigoplus_{e \ni v} x_e = c(v) \qquad (v \in V).$$

For a vertex of degree $d \geq 1$, its CNF consists of the 2^{d-1} clauses forbidding the incident assignments of the wrong parity. At an isolated vertex the equation is $0 = c(v)$: its encoding is an empty conjunction if $c(v) = 0$, and the empty clause if $c(v) = 1$.

We call an unsatisfiable Tseitin formula a *Tseitin contradiction*. For a Tseitin contradiction $\mathrm{T}(G, c)$, write

$$S(G, c) := S(\mathrm{T}(G, c)), \qquad S_{\mathrm{reg}}(G, c) := S_{\mathrm{reg}}(\mathrm{T}(G, c)).$$

The following statements are well-known.

Proposition 2.11 (Satisfiability criterion)**.** *The formula $\mathrm{T}(G, c)$ is satisfiable if and only if $c(V(H)) = 0$ for every connected component H of G.*

Corollary 2.12 (Connected contradictions and a single violated equation)**.** *If G is connected and $c(V) = 1$, then $\mathrm{T}(G, c)$ is unsatisfiable. For every vertex r, however, there is an assignment satisfying all vertex equations except P_r. Consequently every proper subsystem of the vertex equations is satisfiable.*

Proposition 2.13 (Restricting a Tseitin formula)**.** *Let ρ assign the variables of a set $D \subseteq E(G)$. Retain all vertices and delete the assigned edges, defining*

$$G_\rho = (V, E \setminus D), \qquad c_\rho(v) = c(v) \oplus \bigoplus_{\substack{e \in D \\ e \ni v}} \rho(x_e).$$

Then, as CNFs after simplification,

$$\mathrm{T}(G, c)|_\rho = \mathrm{T}(G_\rho, c_\rho).$$

The total charge in every original connected component is unchanged. In particular, if $c(V) = 1$, then $c_\rho(V) = 1$ and the restricted formula is still unsatisfiable. If G_ρ is connected, it is again a Tseitin contradiction on a connected graph.

For a single edge $e = uv$, setting $x_e = b$ deletes e and replaces $c(u), c(v)$ by $c(u) \oplus b, c(v) \oplus b$. Thus assigning zero leaves the charges unchanged, and assigning one flips both endpoint charges. If G is connected with odd total charge and e is a bridge, exactly one of the two resulting components has odd charge.

2.5 Proof search

The *primal graph* of a CNF formula has its variables as vertices, with two distinct variables adjacent if they occur together in a clause. For a Tseitin formula $\mathrm{T}(G, c)$, the primal graph is the line graph $L(G)$.

Proposition 2.14 (Constructive regular-resolution upper bound)**.** *There is a deterministic algorithm that, given an unsatisfiable CNF formula F of encoding length ℓ, constructs a regular resolution refutation of F in time*

$$\ell^{O(1)}\, 2^{O(\mathrm{tw}(P(F)))},$$

where $P(F)$ is the primal graph of F.

This follows from the proof-construction algorithm of Broering and Lokam [8, Theorem 3], combined with the treewidth approximation algorithm of Bodlaender et al. [6]. Alternatively, one can use the branchwidth-based algorithm of Alekhnovich and Razborov [2, Theorem 2.3], together with the standard comparison with primal treewidth.

Corollary 2.15. *For every connected graph G and every odd charge vector c, a regular resolution refutation of $\mathrm{T}(G, c)$ can be constructed in time*

$$|V(G)|^{O(1)}\, 2^{O(\mathrm{tw}(L(G)))}.$$

In particular,

$$S_{\mathrm{reg}}(G, c) \le |V(G)|^{O(1)}\, 2^{O(\mathrm{tw}(L(G)))}.$$

Proof. Put $n = |V(G)|$ and $k = \mathrm{tw}(L(G))$. The edges incident to each vertex form a clique in $L(G)$, so $\Delta(G) \le k + 1$. Hence

$$|\,\mathrm{T}(G, c)| \le n\, 2^k.$$

Every clause has width at most $k + 1$, so the formula has at most $N = n(k + 1)2^k$ literal occurrences and can be encoded in $O(N \log(N + 1)) = n^{O(1)} 2^{O(k)}$ bits. Since its primal graph is $L(G)$, the claim follows from Proposition 2.14. □

3 Main lower bound argument

We first show that every resolution refutation of a Tseitin contradiction contains a clause whose support contains a balanced cut. Assuming a suitable distribution on spanning trees, we then use random restrictions to obtain a lower bound exponential in the minimum size of a balanced edge cut.

Throughout, $G = (V, E)$ is a graph without loops; parallel edges are allowed. We assume that $\mathrm{T}(G, c)$ is unsatisfiable.

3.1 A clause containing a balanced cut

For clauses over Tseitin variables, we identify their support with the corresponding set of edges of G. We say that C *contains a cut* if its support contains every edge of that cut.

Theorem 3.1 (Ben-Sasson and Wigderson [5])**.** *If G is connected and $|V| \ge 2$, every resolution refutation of $\mathrm{T}(G, c)$ contains a clause whose variables contain all edges of a balanced cut of G.*

We prove the following extension of Theorem 3.1 using a weighted version of the measure argument of Ben-Sasson and Wigderson [5].

Lemma 3.2 (Balanced edge cut relative to a vertex set)**.** *Let $G = (V, E)$ be connected and let $W \subseteq V$ satisfy $|W| \ge 2$. Then every resolution refutation of $\mathrm{T}(G, c)$ contains some clause C such that $\delta_G(U) \subseteq \mathrm{supp}(C)$ for some set $U \subseteq V$ with $|W|/3 \le |U \cap W| \le 2|W|/3$.*

Proof. Fix a resolution refutation of $\mathrm{T}(G, c)$. We define a measure on its clauses as follows. For $A \subseteq V$, write $A \models C$ if every assignment satisfying P_v for all $v \in A$ also satisfies C. For a clause C, let

$$\mu(C) = \min\{\, |A \cap W| : A \subseteq V,\ A \models C \,\}.$$

We call any set A attaining this minimum a *witness* for C.

The measure has the following properties.

- Every axiom has measure at most one. Indeed, each axiom belongs to the CNF encoding of a single vertex equation P_v, so it is implied by the singleton set $\{v\}$.
- The empty clause has measure $|W|$. Indeed, the only set of vertices implying the empty clause is V: by Corollary 2.12, every proper subsystem of the vertex equations is satisfiable.
- If D_3 is obtained by resolving D_1 and D_2, then

  $$\mu(D_3) \leq \mu(D_1) + \mu(D_2).$$

  The union of witnesses for the two parents implies both parents and hence their resolvent.
- If D is a weakening of B, then $\mu(D) \leq \mu(B)$, since every set of vertex equations implying B also implies D.

Consider the first clause in a topological ordering of the refutation whose measure exceeds $2|W|/3$. It cannot be an axiom or follow by weakening, so it must be obtained by resolution. Both parents have measure at most $2|W|/3$, and subadditivity implies that one parent C satisfies

$$|W|/3 < \mu(C) \leq 2|W|/3.$$

Choose a witness U for C with the fewest vertices. Suppose that an edge $e \in \delta_G(U)$ has its endpoint v in U and that x_e does not occur in C. By the choice of U, the set $U \setminus \{v\}$ does not imply C: otherwise it would also attain $\mu(C)$ and would be a smaller witness. Thus there is an assignment σ satisfying all equations indexed by $U \setminus \{v\}$ and falsifying C.

Because e has exactly one endpoint in U, flipping x_e changes the truth value of P_v and leaves all other equations indexed by U unchanged. Hence either σ or the assignment obtained from it by flipping x_e satisfies every equation indexed by U. Both assignments falsify C, since x_e does not occur in C. This contradicts $U \models C$.

Therefore every edge in $\delta_G(U)$ has its variable in C, so

$$\delta_G(U) \subseteq \mathrm{supp}(C).$$

□

3.2 The random restriction argument

Theorem 3.3. *Let $G = (V, E)$ be a connected graph with $|V| \geq 2$, and let $\mathrm{T}(G, c)$ be a Tseitin contradiction. Suppose there is a probability distribution on the spanning trees of G such that, for some constant $p \in (0, 1)$, a tree T sampled from this distribution satisfies*

$$\mathbb{P}[B \subseteq T] \leq p^{|B|} \qquad \text{for every } B \subseteq E.$$

Then, for every $W \subseteq V$ with $|W| \geq 2$,

$$S(G, c) \geq 2^{\Omega_p(e_W(G))}.$$

In particular, taking $W = V$ gives $S(G, c) \geq 2^{\Omega_p(e(G))}$.

Proof. We construct a random restriction ρ of the variables of $\mathrm{T}(G,c)$ as follows. Sample a spanning tree T from the given distribution and leave all its edge variables unassigned. Independently for each edge $e \notin T$, set $\rho(x_e) = *$ with probability $1/2$, and set $\rho(x_e) = 0$ or $\rho(x_e) = 1$ with probability $1/4$ each. Here $*$ denotes an unassigned variable.

Let H be the spanning subgraph of G consisting of the edges whose variables remain unassigned. Since H contains T, it is connected. Conditional on T, any fixed set of k edges outside T has all its variables assigned by ρ with probability 2^{-k}. By Proposition 2.13,

$$\mathrm{T}(G,c)|_\rho = \mathrm{T}(H, c_\rho),$$

where c_ρ has odd total charge.

Fix a minimum-length resolution refutation π of $\mathrm{T}(G,c)$. By Proposition 2.9, we may choose π without weakening steps. In particular, all its variables occur in $\mathrm{T}(G,c)$. Let $S = S(G,c)$ be the length of π. We say that a clause *survives* ρ if none of its literals is set to true. Thus its restriction consists of the literals whose variables remain unassigned and may be the empty clause. By Proposition 2.10, π restricts to a refutation of $\mathrm{T}(H, c_\rho)$ of length at most S. Each clause in this refutation is a subclause of a surviving clause of π.

Write $b = e_W(G)$. We call a clause *wide* if it contains more than $b/4$ distinct variables, and *narrow* otherwise. Let $\mathcal{E}$ be the event that at least one wide clause of π survives. We distinguish two cases according to whether $\mathbb{P}(\mathcal{E}) \geq 1/2$ or $\mathbb{P}(\mathcal{E}) < 1/2$.

Case 1: $\mathbb{P}(\mathcal{E}) \geq 1/2$.

Fix a clause $C \in \pi$ with support A. Conditional on T, the variables corresponding to edges in T remain unassigned. For each edge $e \in A \setminus T$, the probability of assigning a value that satisfies a literal of C is at least $1/4$. These choices are independent over edges outside T. Hence

$$\mathbb{P}[C \text{ survives} \mid T] \leq (3/4)^{|A \setminus T|}.$$

Averaging over the random tree T, we obtain

$$\begin{aligned}
\mathbb{P}[C \text{ survives}] = \mathbb{E}_T\big[\mathbb{P}[C \text{ survives} \mid T]\big] &\leq \mathbb{E}_T (3/4)^{|A\setminus T|} = \mathbb{E}_T \prod_{e \in A} \left(\frac{3}{4} + \frac{1}{4}\mathbf{1}_{e \in T}\right) \\
&= \mathbb{E}_T \sum_{B \subseteq A} \left(\frac{3}{4}\right)^{|A|-|B|} \left(\frac{1}{4}\right)^{|B|} \prod_{e \in B} \mathbf{1}_{e \in T} = \sum_{B \subseteq A} \left(\frac{3}{4}\right)^{|A|-|B|} \left(\frac{1}{4}\right)^{|B|} \mathbb{P}[B \subseteq T] \\
&\leq \sum_{B \subseteq A} \left(\frac{3}{4}\right)^{|A|-|B|} \left(\frac{p}{4}\right)^{|B|} = \sum_{j=0}^{|A|} \binom{|A|}{j} \left(\frac{3}{4}\right)^{|A|-j} \left(\frac{p}{4}\right)^{j} = \left(\frac{3+p}{4}\right)^{|A|}.
\end{aligned}$$

A union bound over the wide clauses yields

$$\frac{1}{2} \leq \mathbb{P}(\mathcal{E}) \leq S \left(\frac{3+p}{4}\right)^{b/4}.$$

Consequently,

$$S \geq \frac{1}{2} \left(\frac{4}{3+p}\right)^{b/4}.$$

Case 2: $\mathbb{P}(\mathcal{E}) < 1/2$. Then $\mathbb{P}(\overline{\mathcal{E}}) > 1/2$. On every outcome in $\overline{\mathcal{E}}$, all wide clauses are satisfied, so the restricted refutation uses only subclauses of narrow clauses from π. By Lemma 3.2, some such clause contains a W-balanced cut of H.

For a fixed set of edges A, let $\mathcal{F}_A$ be the event that there exists some W-balanced set $U \subseteq V$ such that $\delta_H(U) \subseteq A$. Writing $A_C = \mathrm{supp}(C)$ for the support of an original line C, the observation above gives

$$\overline{\mathcal{E}} \subseteq \bigcup_{C \text{ is a narrow clause from } \pi} \mathcal{F}_{A_C}.$$

We bound $\mathbb{P}(\mathcal{F}_A)$ for each fixed A.

Fix T. If $\delta_H(U) \subseteq A$, then every edge of H between U and its complement belongs to A. Since $T \subseteq H$, all tree edges crossing this cut are deleted in $T - A$. Consequently, no component of $T - A$ can meet both U and its complement: a path joining vertices on opposite sides would contain a surviving edge crossing the cut. Thus each component lies entirely inside or outside U, so U is a union of components of $T - A$. Deleting at most $|A|$ edges from a tree leaves at most $|A| + 1$ components. Hence there are at most $2^{|A|+1}$ possible unions.

For each W-balanced candidate U, the set $\delta_G(U) \setminus A$ contains at least $b - |A|$ edges. For $\delta_H(U) \subseteq A$ to hold, all these edges must be assigned by ρ. If any of these edges belongs to T, then $\delta_H(U) \subseteq A$ is impossible. Otherwise, conditional on T, each of these edges is assigned independently with probability $1/2$. In either case, the conditional probability that $\delta_H(U) \subseteq A$ is at most $2^{-(b-|A|)}$.

Taking a union bound over the at most $2^{|A|+1}$ candidate unions of components of $T - A$, we obtain

$$\mathbb{P}(\mathcal{F}_A \mid T) \leq 2^{|A|+1} 2^{-(b-|A|)} = 2^{2|A|+1-b}.$$

This bound holds for every fixed T, so averaging over T gives

$$\mathbb{P}(\mathcal{F}_A) \leq 2^{2|A|+1-b}.$$

For a narrow clause C, we have $|A_C| \leq b/4$, and hence

$$\mathbb{P}(\mathcal{F}_{A_C}) \leq 2^{1-b/2}.$$

There are at most S narrow clauses. By the event inclusion above and the union bound,

$$\frac{1}{2} < \mathbb{P}(\overline{\mathcal{E}}) \leq \sum_{C \text{ is a narrow clause from } \pi} \mathbb{P}(\mathcal{F}_{A_C}) \leq S\, 2^{1-b/2}.$$

Consequently,

$$S > 2^{b/2-2}.$$

In each case S is exponential in b. □

4 The spanning-tree distribution

Proposition 4.1. *Every 3-edge-connected graph $G = (V, E)$ admits a probability distribution on its spanning trees such that a tree T sampled from this distribution satisfies*

$$\mathbb{P}[B \subseteq T] \leq (2/3)^{|B|} \qquad \text{for every } B \subseteq E.$$

The proof of Proposition 4.1 uses the following special case of the Tutte–Nash-Williams theorem [27, 24]. For an elementary proof, see Kaiser [22].

Lemma 4.2 (Spanning-tree packing)**.** *For every integer $k \geq 1$, every $2k$-edge-connected graph, possibly with parallel edges, contains k pairwise edge-disjoint spanning trees.*

Proof of Proposition 4.1. Double each edge of G. The resulting graph is 6-edge-connected, and hence it has three edge-disjoint spanning trees.

Project the three disjoint trees to G, obtaining T_1, T_2, T_3. Each edge of G occurs in at most two of them.

Bringing the trees together by random exchanges. We define a random process whose state consists of three labelled trees T_1, T_2, T_3, initially the trees constructed above. As long as they do not all coincide, choose an unequal pair T_i, T_j and an edge $e \in T_i \setminus T_j$. The path in T_j between the endpoints of e contains an edge f crossing the cut defined by the two components of $T_i - e$. Then $f \notin T_i$, and both $T_i - e + f$ and $T_j - f + e$ are spanning trees. All these choices are made according to fixed rules.

Flip a fair coin. With probability $1/2$, replace T_i by $T_i - e + f$; otherwise, replace T_j by $T_j - f + e$. In either case, leave the other two trees unchanged. When all three trees coincide, stop and output their common tree. We will show below that the process terminates almost surely.

This is a three-copy variant of the exchange process underlying swap rounding [9, Sections IV and VI].

Claim 4.3. *The process terminates almost surely.*

Proof. Call an edge *agreed* if it belongs to all three current trees or to none of them. Once an edge is agreed, it remains so: an exchange only modifies edges on which the two selected trees disagree.

At each step, each of the two exchanged edges e, f becomes agreed in exactly one of the two coin outcomes. Indeed, the two outcomes make the edge either present in both selected trees or absent from both, and exactly one of these possibilities matches its membership in the third tree.

The number of agreed edges never decreases and, at every nonterminal state, increases with probability at least $1/2$. The probability of taking t consecutive steps without an increase is therefore at most 2^{-t}. Thus the waiting time until the next increase is finite almost surely. Since there can be at most $|E|$ strict increases, the process terminates almost surely. □

Lemma 4.4 (Frogs on a line)**.** *Consider k frogs, each initially at position 1 or 2. Positions 0 and 3 are absorbing: once a frog reaches either position, it never moves again. At each step, one of the following moves is chosen:*

1. *One frog at position 1 or 2 moves one unit to the left or right, with equal probabilities.*
2. *Two distinct frogs at positions 1 or 2 move one unit in opposite directions, with equal probabilities for the two possible orientations.*

The choice of the move and the frogs may depend on the entire previous history, but its orientation is determined by a fresh fair coin. Then the probability that all frogs eventually reach position 3 is at most $(2/3)^k$.

We apply Lemma 4.4 to a fixed set $B \subseteq E(G)$. The assertion is immediate when $B = \varnothing$. If some edge of B belongs to none of the three initial trees, it never enters any tree, so $\mathbb{P}[B \subseteq T] = 0$. We may therefore assume that every edge of B initially belongs to one or two trees.

Associate a frog with each edge $e \in B$, placing it at the position

$$a_e = \big|\{i \in \{1,2,3\} : e \in T_i\}\big|.$$

An exchange involving exactly one edge of B moves its frog one unit to the left or right, with equal probabilities. An exchange involving two edges of B moves their frogs one unit in opposite directions, again with equal probabilities for the two orientations. Exchanges involving no edge of B can be ignored. Moreover, an edge belonging to none or all three trees never participates in an exchange, so positions 0 and 3 are absorbing.

The process terminates almost surely with three copies of the output tree T. At that point, the frog corresponding to e is at position 3 exactly when $e \in T$. Thus Lemma 4.4 gives

$$\mathbb{P}[B \subseteq T] = \mathbb{P}[\text{all frogs reach position 3}] \le (2/3)^{|B|},$$

as required.

It remains to prove Lemma 4.4, which we do in the next subsection. □

4.1 Proof of the frog lemma

Proof of Lemma 4.4. Let $X_i(d)$ be the position of frog i after d steps, and put

$$P_d = \prod_{i=1}^{k} X_i(d).$$

After all frogs have reached absorbing positions, keep their positions unchanged.

A simultaneous move of two frogs never increases the product of their coordinates. Indeed, up to ordering, their initial positions are $(1,1)$, $(1,2)$, or $(2,2)$, and the corresponding products change as follows:

$$1 \longmapsto 0, \qquad 2 \longmapsto 0 \text{ or } 2, \qquad 4 \longmapsto 3.$$

Since all other coordinates remain unchanged and are nonnegative, such a move cannot increase P_d.

For a move of a single frog, let Q be the product of the other frogs' coordinates and let x be the position of the moving frog. The two equally likely new products are $(x-1)Q$ and $(x+1)Q$, whose arithmetic mean is xQ, the current product.

Thus, after any fixed history, the arithmetic mean of the products in the two possible outcomes of the next move is at most the current product. Averaging over all histories gives

$$\mathbb{E}P_{d+1} \leq \mathbb{E}P_d.$$

Since every frog starts at position 1 or 2, we obtain $\mathbb{E}P_d \leq P_0 \leq 2^k$ for every d.

Let p_d be the probability that all frogs have reached position 3 by step d. On this event $P_d = 3^k$, and on every other outcome $P_d \geq 0$. Hence

$$3^k p_d \leq \mathbb{E}P_d \leq 2^k,$$

so $p_d \leq (2/3)^k$. Finally, these events increase with d, and their union is the event that all frogs eventually reach position 3. Its probability is therefore

$$p = \lim_{d\to\infty} p_d \leq (2/3)^k.$$

□

5 Resolution lower bounds via $\mathrm{tw}(G)$

Theorem 5.1. *For every 3-edge-connected graph $G = (V, E)$ and odd charge vector c, $S(G, c) \geq 2^{\Omega(\mathrm{tw}(G))}$.*

Proof. Choose $W \subseteq V$ such that the induced subgraph $H := G[W]$ satisfies $e(H) = e^*(G)$. For every $U \subseteq V$, we have

$$\delta_H(U \cap W) \subseteq \delta_G(U).$$

If U is balanced relative to W, then $U \cap W$ defines a balanced cut in H, and hence

$$|\delta_G(U)| \geq |\delta_H(U \cap W)| \geq e(H).$$

Taking the minimum over all such U gives $e_W(G) \geq e(H) = e^*(G)$.

Applying Proposition 4.1 and Theorem 3.3 to G, we obtain

$$S(G, c) \geq 2^{\Omega(e^*(G))}.$$

Finally, Proposition 2.6 yields $S(G, c) \geq 2^{\Omega(\mathrm{tw}(G))}$. □

5.1 Removing the edge-connectivity assumption

Two results on topological minors extend the treewidth lower bound to arbitrary connected graphs.

Lemma 5.2 (Resolution under topological minors, [15, Lemma 30])**.** *Let K be a connected topological minor of a connected graph G. For any odd charges c on G and c' on K,*

$$S(K, c') \leq S(G, c).$$

Lemma 5.3 (A treewidth-preserving topological minor, [12, Lemma 12])**.** *If $\mathrm{tw}(G) \geq 3$, then G has a 3-vertex-connected topological minor K with $\mathrm{tw}(K) = \mathrm{tw}(G)$.*

Corollary 5.4. *For every connected graph G and every charge function c with odd total charge,*

$$S(G, c) \geq 2^{\Omega(\mathrm{tw}(G))}.$$

Proof. If $\mathrm{tw}(G) \geq 3$, take K from Lemma 5.3. It is 3-edge-connected, so Theorem 5.1 gives $S(K, c') \geq 2^{\Omega(\mathrm{tw}(K))}$ for any odd charges c'. Lemma 5.2 transfers the bound to G. If $\mathrm{tw}(G) \leq 2$ and G has an edge, use $S(G, c) \geq 2$ and decrease the absolute constant in the exponent. A singleton has treewidth zero. □

6 Lower bound in terms of treewidth of line graph

In this section, we derive a lower bound on the resolution complexity of $\mathrm{T}(G, c)$ in terms of the treewidth of its line graph $L(G)$. By Proposition 2.2,

$$\mathrm{tw}(L(G)) + 1 \leq \Delta(G)(\mathrm{tw}(G) + 1).$$

Thus, for graphs of bounded degree, the desired bound already follows from Corollary 5.4.

At unbounded degree the size convention matters. For a star with m edges, $L(G) = K_m$. The unit clauses at the leaves determine the values of all edge variables, and this assignment falsifies exactly one clause at the center. Resolving that clause with the unit clauses gives a refutation with $O(m)$ proof lines. So an exponential bound in $\mathrm{tw}(L(G))$ for the number of *used* lines is false on arbitrary graphs.

Theorem 6.1. *For every connected graph G and every odd charge function c,*

$$S(G, c) + |\,\mathrm{T}(G, c)| \geq 2^{\Omega(\mathrm{tw}(L(G)))}.$$

The expander gadgets. We use the following standard existence result for bounded-degree edge expanders; see, for example, Bollobás [7].

Lemma 6.2 (Expanders of every order)**.** *There are absolute constants $D \geq 1$ and $h > 0$ such that, for every integer $n \geq 1$, there is a connected graph X_n on exactly n vertices with maximum degree at most D satisfying*

$$|\delta_{X_n}(A)| \geq h \min\{|A|, n - |A|\} \qquad \text{for every } A \subseteq V(X_n).$$

Corollary 6.3. *Let X_n be a graph from Lemma 6.2. For every $A \subseteq E(X_n)$, a largest component R of $X_n - A$ satisfies*

$$|V(X_n) \setminus V(R)| \leq \frac{2|A|}{h}.$$

Proof. Every component U other than R has at most $n/2$ vertices. By the expansion property,

$$h|V(U)| \leq |\delta_{X_n}(V(U))|.$$

All edges in these boundaries belong to A, and summing over the components counts each edge at most twice. Thus

$$h|V(X_n) \setminus V(R)| \leq \sum_{U \neq R} |\delta_{X_n}(V(U))| \leq 2|A|.$$

□

The bounded-degree graph. For each vertex v of G, let $d_v = \deg_G(v)$ and replace v by a copy X_v of the graph X_{d_v} from Lemma 6.2. Identify its vertices with the edges incident to v; we call these vertices *ports*. Thus each original edge uv specifies one port in X_u and one in X_v. Join these two ports by an edge, retaining a distinct edge for each original edge of G.

Let H be the resulting graph. Since G and all the gadgets X_v are connected, H is connected. Each port has at most D neighbours within its gadget and is incident to exactly one original edge, so $\Delta(H) \leq D + 1$. Choose a charge function c_H on $V(H)$ such that

$$\bigoplus_{a \in V(X_v)} c_H(a) = c(v) \qquad \text{for every } v \in V(G).$$

For example, assign charge $c(v)$ to one port of X_v and charge zero to all its other ports.

Lemma 6.4 (Treewidth of the replacement)**.** *There is an absolute C such that* $\mathrm{tw}(L(G))+1 \le C(\mathrm{tw}(H)+1)$.

Proof. We identify each original edge of G with the corresponding external edge of H. Thus $E(G)$ is a subset of $E(H)$ and, equivalently, a common subset of the vertex sets of $L(G)$ and $L(H)$.

We first prove that, for an absolute constant C_1,

$$\mathrm{tw}(L(G)) + 1 \le C_1\big(\mathrm{tw}(L(H)) + 1\big).$$

Since H has bounded degree, Proposition 2.2 gives $\mathrm{tw}(L(H)) + 1 = O(\mathrm{tw}(H) + 1)$, so this comparison immediately implies the desired bound.

To compare the treewidths of the two line graphs, we proceed as follows:

1. Fix an arbitrary nonempty set $F \subseteq E(G)$ and find a small set $Q \subseteq V(L(H))$ such that every component of $L(H) - Q$ contains at most $|F|/2$ vertices of F.
2. Use Q to construct a set $Z \subseteq F$ and show that Z is the separator of a balanced separation of $L(G)[F]$.
3. Use the expansion property of the gadgets to bound $|Z|$ in terms of $|Q|$.
4. Apply the separator characterization of treewidth to obtain

$$\mathrm{tw}(L(G)) + 1 \le C_1\big(\mathrm{tw}(L(H)) + 1\big).$$

Put $k := \mathrm{tw}(L(H))$ and $C_0 := \max\{1, 2/h\}$.

Step 1: choose the set Q. Fix a nonempty set $F \subseteq E(G)$. By Lemma 2.4(1), there is a set

$$Q \subseteq V(L(H)) = E(H), \qquad |Q| \le k + 1,$$

such that every component of $L(H) - Q$ contains at most $|F|/2$ vertices of F. Viewed in H, deleting Q means deleting the corresponding edges.

Step 2: construct a separator in $L(G)[F]$. For each gadget X_v, let $A_v := Q \cap E(X_v)$ and let R_v be a largest component of $X_v - A_v$.

Let $R := \bigcup_{v \in V(G)} V(R_v)$. Define $Z \subseteq F$ to consist of the edges that belong to Q or have at least one endpoint outside R. We now show that Z is the separator of a balanced separation of $L(G)[F]$.

We first show that every component of $L(G)[F] - Z$ has at most $|F|/2$ vertices.

Every edge in $F \setminus Z$ lies outside Q and is therefore present in $H - Q$. Both of its ports lie in R. Suppose two such edges are adjacent in $L(G)$, so they share an endpoint v in G. Their ports in X_v both lie in R_v and are therefore connected by a path of internal edges avoiding Q. Together with the two original edges, this path gives a connection between their corresponding vertices in $L(H) - Q$.

Let $e_0, e_1, \ldots, e_t$ be a path in $L(G)[F] - Z$. By the preceding observation, each consecutive pair e_i, e_{i+1} lies in the same component of $L(H) - Q$. Hence all vertices of this path lie in a single component of $L(H) - Q$. Since any two vertices of a component of $L(G)[F] - Z$ are joined by a path, every such component is contained, as a set of original edges, in a single component of $L(H) - Q$. By the choice of Q, the latter contains at most $|F|/2$ vertices of F. Hence every component of $L(G)[F] - Z$ has at most $|F|/2$ vertices.

To obtain a balanced separation, partition these components into two groups whose vertex sets B_1, B_2 each have size at most $2|F|/3$. Such a grouping always exists: if a component has at least $|F|/3$ vertices, take it as B_1; otherwise, add components to B_1 until its size first reaches $|F|/3$, or take all components if their total size is smaller. In each case, both B_1 and the remaining set B_2 have size at most $2|F|/3$.

There are no edges of $L(G)[F]$ between B_1 and B_2, since each is a union of components of $L(G)[F] - Z$. Thus $(B_1 \cup Z, B_2 \cup Z)$ is a balanced separation of $L(G)[F]$ with separator Z.

Step 3: bound $|Z|$. By Corollary 6.3,

$$|V(X_v) \setminus V(R_v)| \le \frac{2|A_v|}{h}.$$

Each port serves exactly one original edge. Therefore, with $C_0 = \max\{1, 2/h\}$,

$$\begin{aligned}|Z| &\le |Q \cap E(G)| + \sum_v |V(X_v) \setminus V(R_v)| \\ &\le |Q \cap E(G)| + \frac{2}{h}\sum_v |A_v| = |Q \cap E(G)| + \frac{2}{h}\sum_v |Q \cap E(X_v)| \\ &\le C_0\left(|Q \cap E(G)| + \sum_v |Q \cap E(X_v)|\right) = C_0|Q| \le C_0(k+1).\end{aligned}$$

The last equality follows because the external edges and the internal edges of the gadgets partition $E(H)$.

Step 4: deduce the treewidth bound. The construction works for every nonempty $F \subseteq E(G)$. Moreover, a balanced separation of $L(G)[F]$ remains a balanced separation after deleting edges. Thus every subgraph of $L(G)$ has a balanced separation of order at most $C_0(k+1)$. By Lemma 2.4(2),

$$\mathrm{tw}(L(G)) \le 15C_0(k+1).$$

It follows that

$$\mathrm{tw}(L(G)) + 1 \le (15C_0 + 1)\big(\mathrm{tw}(L(H)) + 1\big).$$

Finally, $\Delta(H) \le D + 1$, so Proposition 2.2 gives

$$\mathrm{tw}(L(H)) + 1 \le (D+1)(\mathrm{tw}(H) + 1).$$

Combining the last two inequalities proves the lemma with $C = (15C_0 + 1)(D + 1)$, an absolute constant. □

Proof of Theorem 6.1. Put $S = S(G, c)$, $N = |\,\mathrm{T}(G, c)|$, and $\Delta = \Delta(G)$. Let (H, c_H) be the bounded-degree replacement constructed above. The graph H is connected, and its total charge equals that of G, so it is odd.

Simulating an original refutation on H. For each vertex v of G, let F_v be the conjunction of the Tseitin clauses at the vertices of the gadget X_v. Summing the corresponding parity equations modulo two cancels every internal edge variable, since it occurs twice. The remaining variables correspond precisely to the original edges incident to v, and the gadget charges sum to $c(v)$. Thus F_v implies the original parity equation at v, and hence every original axiom C at v.

There are d_v external edges incident to X_v and at most $Dd_v/2$ internal edges. Therefore F_v and C together involve $O(d_v)$ variables. By Proposition 2.8, C has a derivation from F_v of size $2^{O(d_v)}$.

Take a refutation of $\mathrm{T}(G, c)$ of size S. Replace each used axiom at v by such a derivation from the clauses of F_v. The resulting refutation of $\mathrm{T}(H, c_H)$ has size at most $S\,2^{O(\Delta)}$. Consequently, writing $S_H = S(H, c_H)$, we have

$$S_H \le S\,2^{O(\Delta)}.$$

Applying the treewidth lower bound. By Corollary 5.4,

$$\mathrm{tw}(H) + 1 = O\big(\log_2(S_H + 1)\big).$$

Combining this with Lemma 6.4 and the simulation above gives

$$\mathrm{tw}(L(G)) + 1 \le C(\mathrm{tw}(H) + 1) = O\big(\log_2(S_H + 1)\big) = O\big(\log_2(S + 1) + \Delta\big).$$

Absorbing the maximum degree into the input size. A vertex of degree Δ contributes $2^{\Delta-1}$ distinct input clauses. Hence

$$N \geq 2^{\Delta-1}, \qquad \text{so} \qquad \Delta \leq 1 + \log_2 N.$$

It follows that

$$\mathrm{tw}(L(G)) + 1 = O\big(\log_2(S+1) + 1 + \log_2 N\big) = O\big(\log_2(S+N)\big).$$

Rearranging yields

$$S(G,c) + |\,\mathrm{T}(G,c)| \geq 2^{\Omega(\mathrm{tw}(L(G)))}.$$

□

7 Automatability

We now combine the lower bound of Theorem 6.1 with the constructive upper bound of Proposition 2.14.

Let $\mathcal{C}$ be a family of CNF formulas, and let $S_{\mathcal{P}}(F)$ denote the minimum size of a $\mathcal{P}$-refutation of an unsatisfiable formula F. A proof system $\mathcal{P}$ is *automatable on* $\mathcal{C}$ if there is a deterministic algorithm that, given any unsatisfiable $F \in \mathcal{C}$, constructs a $\mathcal{P}$-refutation of F in time

$$\big(\ell(F) + S_{\mathcal{P}}(F)\big)^{O(1)},$$

where $\ell(F)$ is the encoding length of F. The algorithm is given only F, not a refutation or its minimum size.

Theorem 7.1. *Both unrestricted and regular resolution are automatable on the class of Tseitin formulas. More precisely, there is a deterministic algorithm that, given a Tseitin contradiction F, constructs a regular resolution refutation in time*

$$\big(\ell(F) + S(F)\big)^{O(1)},$$

where $S(F)$ is the minimum size of an unrestricted resolution refutation.

Proof. If F contains the empty clause, return it as a refutation. We may therefore assume that every clause is nonempty.

Connected graphs. First suppose that $F = \mathrm{T}(G,c)$ with G connected. Write $\ell = \ell(F)$, $S = S(F)$, and $k = \mathrm{tw}(L(G))$. The primal graph of F is $L(G)$, so Proposition 2.14 constructs a regular resolution refutation in time

$$\ell^{O(1)} 2^{ak}$$

for some absolute constant $a > 0$.

By Theorem 6.1, there is an absolute constant $b > 0$ such that

$$2^{bk} \leq S + |F| \leq S + \ell.$$

Hence

$$2^{ak} = \big(2^{bk}\big)^{a/b} \leq (S+\ell)^{a/b}.$$

The running time is therefore at most

$$\ell^{O(1)} (S+\ell)^{a/b} \leq (S+\ell)^{O(1)},$$

as required.

Disconnected graphs. Partition F into variable-disjoint subformulas $F_1, \dots, F_r$ corresponding to the connected components of its primal graph. This partition is computable in polynomial time, and each F_i is a Tseitin formula on a connected graph.

Some unsatisfiable component F_i satisfies

$$S(F_i) \le S(F).$$

Indeed, by Proposition 2.9, we may choose a minimum-length refutation of F without weakening. Resolution cannot combine clauses from distinct variable sets, so the subderivation of the empty clause uses axioms from a single component.

Run the connected-case proof searches on all F_i in parallel and return the first refutation produced. For the component identified above, the search takes time

$$\big(\ell(F_i) + S(F_i)\big)^{O(1)} \le \big(\ell(F) + S(F)\big)^{O(1)}.$$

Sharing computation time among the $r \le \ell(F)$ searches introduces only a polynomial factor.

The output is a regular resolution refutation and hence also an unrestricted resolution refutation. Finally, $S(F) \le S_{\mathrm{reg}}(F)$, so the same running-time bound is polynomial in $\ell(F) + S_{\mathrm{reg}}(F)$. This proves automatability for both systems. □

8 Polynomial equivalence of regular and unrestricted resolution on Tseitin formulas

In this section, we show that regular resolution polynomially simulates unrestricted resolution on Tseitin formulas. The lower bound in Theorem 6.1 applies to the sum of the input size and the length of a shortest refutation. Since a shortest refutation may be smaller than the input formula, we need to refine this bound to obtain a polynomial comparison in terms of refutation length alone.

Corollary 8.1. *For every 2-edge-connected graph G and every odd charge vector c,*

$$S(G, c) \ge 2^{\Omega(\mathrm{tw}(L(G)))}.$$

Proof. Put $S = S(G, c)$, $n = |V(G)|$, and $\Delta = \Delta(G)$. By Corollary 2.12, every refutation uses an axiom from each vertex, so $n \le S$. We next show that $S \ge 2^{\Delta/2}$.

Fix a vertex v of degree Δ, and let $K_1, \dots, K_r$ be the components of $G - v$. Each K_i is joined to v by at least two edges, since G has no bridges. Hence $r \le \Delta/2$.

Consider an assignment α to the edges incident to v such that

$$\bigoplus_{e \in \delta_G(V(K_i))} \alpha(e) = c(V(K_i)) \qquad \text{for every } i.$$

These conditions ensure that each K_i has even residual total charge. By Proposition 2.11, α therefore extends to an assignment satisfying all vertex equations outside v.

There are $2^{\Delta - r}$ such assignments α. Each extension falsifies exactly one input clause, at v, and distinct choices of α give distinct clauses. Every refutation must use all these clauses, since deleting any one of them makes the remaining input satisfiable. Therefore

$$S \ge 2^{\Delta - r} \ge 2^{\Delta/2}.$$

Consequently, $\Delta \le 2 \log_2 S$, and

$$|\,\mathrm{T}(G, c)| \le n\, 2^{\Delta} \le S^3.$$

Theorem 6.1 now gives

$$2^{\Omega(\mathrm{tw}(L(G)))} \le S + |\,\mathrm{T}(G, c)| \le S + S^3 \le 2S^3,$$

which implies the claimed bound. □

Theorem 8.2. *There is an absolute constant C such that every unsatisfiable Tseitin formula F satisfies*

$$S(F) \le S_{\mathrm{reg}}(F) \le S(F)^C.$$

Proof. The first inequality is immediate. First suppose that $F = \mathrm{T}(G, c)$ with G connected. The singleton case is trivial, so assume $n = |V(G)| \ge 2$.

Let $\mathcal{X}$ be the components obtained by deleting all bridges of G, and put

$$k = \max\left\{1, \max_{X \in \mathcal{X}} \big(\mathrm{tw}(L(X)) + 1\big)\right\}.$$

Contracting these components gives a tree, which we call the *bridge tree*. We show that

$$\max\{n, 2^{\Omega(k)}\} \le S(F) \le S_{\mathrm{reg}}(F) \le n^{O(1)} 2^{O(k)}.$$

Lower bounds. Every refutation uses an axiom from each vertex, since every proper subsystem of the vertex equations is satisfiable by Corollary 2.12. Thus $S(F) \ge n$.

Fix $X \in \mathcal{X}$. Root the bridge tree at X and assign each bridge the total charge on its side away from the root. Every other component then has even residual total charge and can be satisfied, leaving an odd-charge Tseitin formula on X. Restriction does not increase refutation size, so Corollary 8.1, applied to each nontrivial X, gives

$$S(F) \ge 2^{\Omega(\mathrm{tw}(L(X)))}.$$

Consequently, $k = O(\log S(F))$.

Regular upper bound. Process the bridge tree from its leaves towards a root. For a leaf component X with parent bridge b, summing its residual vertex equations determines a value β_b for x_b. Assigning $x_b = 1 - \beta_b$ gives an odd-charge Tseitin formula on X.

By Corollary 2.15, this formula has a regular refutation of size $|V(X)|^{O(1)} 2^{O(k)}$. For a singleton component, the refutation consists of the empty axiom. Let ℓ_b be the literal asserting $x_b = \beta_b$. Adding ℓ_b to every proof line turns this refutation into a derivation of the unit clause ℓ_b[1] from the clauses before assigning x_b, using weakening on input clauses where necessary. The resulting derivation is regular and never uses x_b as a resolving variable.

We next remove X and b from the graph and adjust the charge at the other endpoint of b by β_b. Whenever a subsequent derivation requires a clause obtained by eliminating the false literal $\neg\ell_b$ from a clause $D \vee \neg\ell_b$, derive D by resolving that clause with the unit clause ℓ_b on x_b. Clauses not containing x_b remain available unchanged. Only clauses needed in subsequent derivations are constructed.

Continue towards the root. At the root, the residual formula has odd total charge, so the same upper bound supplies a refutation. Each required local axiom is obtained from an original axiom by resolving against previously derived unit clauses for bridge variables, using at most $n - 1$ resolutions.

The combined proof is regular. Each local refutation is regular and uses only internal edge variables as resolving variables. Each bridge variable is used as a resolving variable at most once along a proof path, when its derived unit clause is applied in the parent component. Thus no variable is used as a resolving variable twice along a proof path.

Reusing the derivations of the bridge unit clauses as DAG premises, the total size is at most

$$n^{O(1)} 2^{O(k)}.$$

Since $n \le S(F)$ and $k = O(\log S(F))$, this is $S(F)^{O(1)}$.

[1] A unit clause is a clause consisting of a single literal.

Disconnected graph. Finally, suppose that G is disconnected. By Proposition 2.9, choose a minimum-length refutation of F without weakening. Resolution cannot combine clauses from different variable-disjoint components. Hence

$$S(F) = \min_{i:\, F_i \text{ unsatisfiable}} S(F_i),$$

where F_i are the component formulas. Choose a component attaining this minimum. The connected case gives

$$S_{\mathrm{reg}}(F) \le S_{\mathrm{reg}}(F_i) \le S(F_i)^C = S(F)^C,$$

for a sufficiently large absolute constant C. □

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